Part IV — Topics in Geometric Group Theory

Definitions

Based on lectures by H. Wilton

Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This graduate course will provide an introduction to the basic ideas of the subject.

Suppose $\Gamma$ is a discrete group of isometries of a metric space $X$. We focus on the theorems we can prove about $\Gamma$ by imposing geometric conditions on $X$. These conditions are motivated by curvature conditions in differential geometry, but apply to general metric spaces and are much easier to state. First we study the case when $X$ is Gromov-hyperbolic, which corresponds to negative curvature. Then we study the case when $X$ is CAT(0), which corresponds to non-positive curvature. In order for this theory to be useful, we need a rich supply of negatively and non-positively curved spaces. We develop the theory of non-positively curved cube complexes, which provide many examples of CAT(0) spaces and have been the source of some dramatic developments in low-dimensional topology over the last twenty years.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasimorphisms, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Part 3. We will cover the basic theory of CAT(0) spaces, working up to the Cartan–Hadamard theorem and Gromov’s Link Condition. These two results together enable us to check whether the universal cover of a complex admits a CAT(1) metric.

Part 4. We will introduce cube complexes, in which Gromov’s link condition becomes purely combinatorial. If there is time, we will discuss Haglund–Wise’s special cube complexes, which combine the good geometric properties of CAT(0) spaces with some strong algebraic and topological properties.
IV Geometric Group Theory (Definitions)

**Pre-requisites**

Part IB Geometry and Part II Algebraic topology are required.
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1 Cayley graphs and the word metric

1.1 The word metric

**Definition (Word length).** Let \( \Gamma \) be a group and \( S \) a finite generating set. If \( \gamma \in \Gamma \), the word length of \( \gamma \) is
\[
\ell_S(\gamma) = \min \{ n : \gamma = s_1^{\pm 1} \cdots s_n^{\pm 1} \text{ for some } s_i \in S \}.
\]

**Definition (Word metric).** Let \( \Gamma \) be a group and \( S \) a finite generating set. The word metric on \( \Gamma \) is given by
\[
d_S(\gamma_1, \gamma_2) = \ell_s(\gamma_1^{-1} \gamma_2).
\]

**Definition (Cayley graph).** The Cayley graph \( \text{Cay}_S(\Gamma) \) is defined as follows:
- \( V(\text{Cay}_S(\Gamma)) = \Gamma \)
- For each \( \gamma \in \Gamma \) and \( s \in S \), we draw an edge from \( \gamma \) to \( \gamma s \).

If we want it to be a directed and labelled graph, we can label each edge by the generator \( s \).

**Definition (Quasi-isometry).** Let \( \lambda \geq 1 \) and \( c \geq 0 \). A function between metric spaces \( f : X \to Y \) is a \((\lambda, c)\)-quasi-isometric embedding if for all \( x_1, x_2 \in X \)
\[
\frac{1}{\lambda} d_X(x_1, x_2) - c \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + c.
\]

If, in addition, there is a \( C \) such that for all \( y \in Y \), there exists \( x \in X \) such that
\[
d_Y(y, f(x)) \leq C,
\]
we say \( f \) is a quasi-isometry, and \( X \) is quasi-isometric to \( Y \). We write \( X \approx Y \).

**Definition (Geodesic).** Let \( X \) be a metric space. A geodesic in \( X \) is an isometric embedding of a closed interval \( \gamma : [a, b] \to X \).

**Definition (Geodesic metric space).** A metric space \( X \) is called geodesic if every pair of points \( x, y \in X \) is joined by a geodesic denoted by \([x, y]\).

**Definition (Proper metric space).** A metric space is proper if closed balls in \( X \) are compact.

**Definition (Proper discontinuous action).** An action \( \Gamma \) on a topological space \( X \) is proper discontinuous if for every compact set \( K \), the set
\[
\{ g \in \Gamma : gK \cap K \neq \emptyset \}
\]
is finite.

**Definition (Cocompact action).** An action \( \Gamma \) on a topological space \( X \) is cocompact if the quotient \( \Gamma \setminus X \) is compact.
1.2 Free groups

Definition (Free group). Let $S$ be a (usually finite) set. A group $F(S)$ with a map $S \to F(S)$ is called the free group on $S$ if it satisfies the following universal property: for any set map $S \to G$, there is a unique group homomorphism $F(S) \to G$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S & \longrightarrow & F(S) \\
& & \downarrow \downarrow \\
& & G
\end{array}
$$

Usually, if $|S| = r$, we just write $F(S) = F_r$.

1.3 Finitely-presented groups

Definition (Finitely-presentable group). A finitely-presentable group is a group $\Gamma$ such that there are finite $S$ and $R$ such that $\Gamma \cong \langle S \mid R \rangle$.

A finitely-presented group is a group $\Gamma$ equipped $S$ and $R$ such that $\Gamma \cong \langle S \mid R \rangle$.

1.4 The word problem

Definition (Null-homotopic). We say $w \in S^*$ is null-homotopic if $w = 1$ in $\Gamma$.

Definition (Algebraic area). Let $w \in S^*$ be null-homotopic. Its algebraic area is

$$
\text{Area}_{a,P}(w) = \min \left\{ d : w = \prod_{i=1}^{d} g_i r_i^{\pm 1} g_i^{-1} \right\}.
$$

Definition (Dehn function). Then Dehn function is the function $\delta_P : \mathbb{N} \to \mathbb{N}$ mapping

$$
n \mapsto \max \{ \text{Area}_{a,P}(w) \mid |w|_S \leq n, w \text{ is null-homotopic} \}.
$$

Definition ($\preceq$). We write $f \preceq g$ iff for some $C > 0$, we have

$$
f(x) \leq Cg(Cx + C) + Cx + C.
$$

for all $x$.

We write $f \simeq g$ if $f \preceq g$ and $g \preceq f$. 
2 Van Kampen diagrams

Definition (Singular disc diagram). A *(singular) disc diagram* is a compact, contractible subcomplex of \( \mathbb{R}^2 \), e.g.

![Disc Diagram](image)

We usually endow the disc diagram \( D \) with a base point \( p \), and define

\[
\text{Area}_p(D) = \text{number of 2-cells of } D
\]

and

\[
\text{Diam}_p(D) = \text{length of the longest embedded path in } D^{(1)}, \text{ starting at } p.
\]

If we orient \( \mathbb{R}^2 \), then \( D \) has a well-defined boundary cycle. A disc diagram is *labelled* if each (oriented) edge is labelled by an element \( s \in S^\pm \). The set of *face labels* is the set of boundary cycles of the 2-cells.

If all the face labels are elements of \( R^\pm \), then we say \( D \) is a *diagram* over \( \langle S \mid R \rangle \).

Definition (van Kampen diagram). If \( w \in \langle \langle R \rangle \rangle \), and \( w \) is the boundary cycle of a singular disc diagram \( D \) over the presentation \( \langle S \mid R \rangle \), we say that \( D \) is a *van Kampen diagram* for \( w \).

Definition (Filling disc). Let \( (M,g) \) be a closed Riemannian manifold. Let \( \gamma: S^1 \to M \) be a smooth null-homotopic loop. A filling disk for \( \gamma \) in \( M \) is a smooth map \( f: D^2 \to M \) such that the diagram

\[
\begin{array}{c}
S^1 \\
\downarrow \\
D^2 \\
\uparrow \\
\gamma \\
\Downarrow \\
M
\end{array}
\]

Definition (FArea).

\[
\text{FArea}(\gamma) = \inf \{ \text{Area}(f) \mid f: D^2 \to M \text{ is a filling disc for } \gamma \}.
\]

Definition (Isoperimetric function). The isoperimetric function of \( (M,g) \) is

\[
F_i U^M : [0, \infty) \to [0, \infty) \\
el \mapsto \sup \{ \text{FArea}(\gamma) : \gamma: S^1 \to M \text{ is a smooth null-homotopic loop, } l(\gamma) \leq l \}
\]
3 Bass–Serre theory

3.1 Graphs of spaces

**Definition** (Homotopy pushout). Let $X, Y, Z$ be spaces, and $\partial^- : Z \to X$ and $\partial^+ : Z \to Y$ be maps. We define

$$X \amalg_Z Y = (X \amalg Y \amalg Z \times [-1, 1])/\sim,$$

where we identify $\partial^\pm(z) \sim (z, \pm 1)$ for all $z \in Z$.

**Definition** (Graph of spaces). A **graph of spaces** $\mathcal{X}$ consists of the following data

- A connected graph $\Xi$.
- For each vertex $v \in V(\Xi)$, a path-connected space $X_v$.
- For each edge $e \in E(\Xi)$, a path-connected space $X_e$.
- For each edge $e \in E(\Xi)$ attached to $v^\pm \in V(\Xi)$, we have $\pi_1$-injective maps $\partial^\pm_e : X_e \to X_{v^\pm}$.

The **realization** of $\mathcal{X}$ is

$$|\mathcal{X}| = X = \bigsqcup_{v \in V(\Xi)} X_v \amalg \bigsqcup_{e \in E(\Xi)} (X_e \times [-1, 1]) / (\forall e \in E(\Xi), \forall x \in X_e, (x, \pm 1) \sim \partial^\pm_e(x)).$$

**Definition** (Graph of groups). A **graph of groups** $\mathcal{G}$ consists of

- A graph $\Gamma$
- Groups $G_v$ for all $v \in V(\Gamma)$
- Groups $G_e$ for all $e \in E(\Gamma)$
- For each edge $e$ with vertices $v^\pm(e)$, injective group homomorphisms

$$\partial^\pm_e : G_e \to G_{v^\pm(e)}.$$

**Definition** (Aspherical space). A space $X$ is **aspherical** if $\hat{X}$ is contractible. By Whitehead’s theorem and the lifting criterion, this is true iff $\pi_n(X) = 0$ for all $n \geq 2$.

3.2 The Bass–Serre tree
4 Hyperbolic groups

4.1 Definitions and examples

Definition (Geodesic triangle). A geodesic triangle $\Delta$ is a choice of three points $x, y, z$ and geodesics $[x, y], [y, z], [z, x]$.

Definition ($\delta$-slim triangle). We say $\Delta$ is $\delta$-slim if every side of $\Delta$ is contained in the union of the $\delta$-neighbourhoods of the other two sides.

Definition (Hyperbolic space). A metric space is (Gromov) hyperbolic if there exists $\delta \geq 0$ such that every geodesic triangle in $X$ is $\delta$-slim. In this case, we say it is $\delta$-hyperbolic.

4.2 Quasi-geodesics and hyperbolicity

Definition (Quasi-geodesic). A $(\lambda, \varepsilon)$-quasi-geodesic for $\lambda \geq 1, \varepsilon \geq 0$ is a $(\lambda, \varepsilon)$-quasi-isometric embedding $I \to X$, where $I \subseteq \mathbb{R}$ is a closed interval.

Definition (Hyperbolic group). A group $\Gamma$ is hyperbolic if it acts properly discontinuously and cocompactly by isometries on a proper, geodesic hyperbolic metric space. Equivalently, if it is finitely-generated and with a hyperbolic Cayley graph.

Definition (Length of path). Let $c : [a, b] \to X$ be a continuous path. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a dissection $D$ of $[a, b]$. Define

$$\ell(c) = \sup_D \sum_{i=1}^n d(c(t_{i-1}), c(t_i)).$$

Definition (Rectifiable path). We say a path $c$ is rectifiable if $\ell(c) < \infty$.

Definition (virtually-P). Let $P$ be a property of groups. Then we say $G$ is virtually-$P$ if there is a finite index subgroup $G_0 \leq G$ such that $G_0$ is $P$.

Definition ($k$-local geodesic). Let $X$ be a geodesic metric space, and $k > 0$. A path $c : [a, b] \to X$ is a $k$-local geodesic if

$$d(c(t), c(t')) = |t - t'|$$

for all $t, t' \in [a, b]$ with $|t - t'| \leq k$.

4.3 Dehn functions of hyperbolic groups

Definition (Dehn presentation). A finite presentation $\langle S \mid R \rangle$ for a group $\Gamma$ is called Dehn if for every null-homotopic reduced word $w \in S^*$, there is (a cyclic conjugate of) a relator $r \in R$ such that $r = u^{-1}v$ with $\ell_S(u) < \ell_S(v)$, and $w = u_1v_2$ (without cancellation).

Definition (Geodesic ray). Let $X$ be a $\delta$-hyperbolic geodesic metric space. A geodesic ray is an isometric embedding $r : [0, \infty) \to X$. 

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5 CAT(0) spaces and groups

5.1 Some basic motivations

Definition (Group (co)homology). The (co)homology of a group $\Gamma$ is the (co)homology of $K(\Gamma,1)$.

5.2 CAT(κ) spaces

Definition (Triangle). A triangle with vertices $\{p,q,r\} \subseteq X$ is a choice

$$\Delta(p,q,r) = [p,q] \cup [q,r] \cup [r,p].$$

Definition (CAT(κ) space). We say a space $(X,d)$ is CAT(κ) if for any geodesic triangle $\Delta \subseteq X$ of diameter $\leq 2D_\kappa$, any $p,q \in \Delta$ and any comparison points $\bar{p},\bar{q} \in \bar{\Delta}$,

$$d(p,q) \leq d_{ML}(\bar{p},\bar{q}).$$

If $X$ is locally CAT(κ), then $K$ is said to be of curvature at most $\kappa$.

In particular, a locally CAT(0) space is called a non-positively curved space.

Definition (CAT(0) group). A group is CAT(0) if it acts properly discontinuously and cocompactly by isometries on a proper CAT(0) space.

5.3 Length metrics

Definition (Length space). A metric space $X$ is called a length space if for all $x,y \in X$, we have

$$d(x,y) = \inf_{\gamma: x \to y} \ell(\gamma).$$

5.4 Alexandrov’s lemma

5.5 Cartan–Hadamard theorem

5.6 Gromov’s link condition

Definition (Euclidean cell complex). A locally finite cell complex $X$ is Euclidean if every cell is isometric to a convex polyhedron in Euclidean space and the attaching maps are isometries from the lower-dimensional cell to a face of the new cell.

Definition (Link). Let $X$ be a Euclidean complex, and let $v$ be a vertex of $X$, and let $0 < \varepsilon \ll$ shortest 1-cell. Then the link of $v$ is

$$Lk(v) = S_v(\varepsilon) = \{x \in X : d(x,v) = \varepsilon\}.$$ 

Definition (Residually finite group). A group $G$ is residually finite if for every $g \in G \setminus \{1\}$, there is a homomorphism $\varphi : G \to$ finite group such that $\varphi(g) \neq 0$. 

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5.7 Cube complexes

Definition (Cube complex). A Euclidean complex is a cube complex if every cell is isometric to a cube (of any dimension).

Definition (Flag simplicial complex). A simplicial complex $X$ is flag if for all $n \geq 2$, each copy of $\partial \Delta^n$ in $X$ is in fact the boundary of a copy of $\Delta^n$ in $X$.

Definition (right-angled Artin group). Let $N$ be a simplicial graph, i.e. a graph where the vertices determine the edges, i.e. a graph as a graph theorist would consider. Then

$$A_N = \langle V(N) \mid [u, v] = 1 \text{ for all } (u, v) \in E(N) \rangle$$

is the right-angled Artin group, or graph group of $N$.

Definition (Salvetti complex). Given a simplicial group $N$, the Salvetti complex $S_N$ is the cube complex defined as follows:

- Set $S_N^{(2)}$ is the presentation complex for $A_N$.
- For any immersion of the 2-skeleton of a $d$-dimensional cube, we glue in an $d$-dimensional cube to $S_N^{(2)}$.

Alternatively, we have a natural inclusion $S_N^{(2)} \subseteq (S^1)^{V(N)}$, and $S_N$ is the largest subcomplex whose 2-skeleton coincides with $S_N^{(2)}$.

Definition (Double). The double $D(K)$ of a simplicial complex $K$ is defined as follows:

- The vertices are $\{v_1^+, \ldots, v_n^+, v_1^-, \ldots, v_n^-\}$, where $\{v_1, \ldots, v_n\}$ are the vertices of $K$.
- The simplices are those of the form $\langle v_i^\pm, \ldots, v_k^\pm \rangle$, where $\langle v_i, \ldots, v_k \rangle \in K$.

Definition (Flag complex). The flag complex of $N$, written $\bar{N}$, is the only flag simplicial complex with 1-skeleton $N$.

5.8 Special cube complexes

Definition (Special cube complex). A cube complex is special if its hyperplanes do not exhibit any of the following four pathologies:

- One-sidedness
- Self-intersection
- Direct self-osculation
- Inter-osculation

Definition (Virtually special group). A group $\Gamma$ is virtually special if there exists a finite index subgroup $\Gamma_0 \leq \Gamma$ such that $\Gamma_0 \cong \pi_1 X$, where $X$ is a compact special cube complex.