Part IV — Topics in Number Theory

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The "Langlands programme" is a far-ranging series of conjectures describing the connections between automorphic forms on the one hand, and algebraic number theory and arithmetic algebraic geometry on the other. In these lectures we will give an introduction to some aspects of this programme.

Pre-requisites

The course will follow on naturally from the Michaelmas term courses *Algebraic Number Theory* and *Modular Forms and L-Functions*, and knowledge of them will be assumed. Some knowledge of algebraic geometry will be required in places.

Contents

0 Introduction

In this course, we shall first give an outline of class field theory. We then look at abelian L-functions (Hecke, Tate). We then talk about *non*-abelian L-functions, and in particular the Weil–Deligne group and local L- and ε-factors.

We then talk about local Langlands for GL_n a bit, and do a bit of global theory and automorphic forms at the end.

The aim is not to prove everything, because that will take 3 courses instead of one, but we are going to make precise definitions and statements of everything.

1 Class field theory

1.1 Preliminaries

Class field theory is the study of abelian extensions of local or global fields. Before we can do class field theory, we must first know *Galois* theory.

Notation. Let K be a field. We will write $\bar{K}$ for a separable closure of K, and $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$. We have

$$\Gamma_K = \lim_{L/K \text{ finite separable}} \mathrm{Gal}(L/K),$$

which is a *profinite group*. The associated topology is the *Krull topology*.

Galois theory tells us

Theorem (Galois theory)**.** There are bijections

$$\left\{\begin{array}{c}\text{closed subgroups of}\\ \Gamma_K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subfields}\\ K \subseteq L \subseteq \bar{K}\end{array}\right\}$$
$$\left\{\begin{array}{c}\text{open subgroups of}\\ \Gamma_K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite subfields}\\ K \subseteq L \subseteq \bar{K}\end{array}\right\}$$

Notation. We write K^{ab} for the maximal abelian subextension of $\bar{K}$, and then

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) = \Gamma_K^{\mathrm{ab}} = \frac{\Gamma_K}{\overline{[\Gamma_K, \Gamma_K]}}.$$

It is crucial to note that while $\bar{K}$ is unique, it is only unique up to non-canonical isomorphism. Indeed, it has many automorphisms, given by elements of $\mathrm{Gal}(\bar{K}/K)$. Thus, Γ_K is well-defined up to conjugation only. On the other hand, the abelianization Γ_K^{ab} *is* well-defined. This will be important in later naturality statements.

Definition (Non-Archimedean local field)**.** A *non-Archimedean local field* is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

We can also define Archimedean local fields, but they are slightly less interesting.

Definition (Archimedean local field)**.** An Archimedean local field is a field that is $\mathbb{R}$ or $\mathbb{C}$.

If F is a non-Archimedean local field, then it has a canonical normalized valuation

$$v = v_F : F^\times \twoheadrightarrow \mathbb{Z}.$$

Definition (Valuation ring)**.** The *valuation ring* of a non-Archimedean local field F is

$$\mathcal{O} = \mathcal{O}_F = \{x \in F : v(x) \geq 0\}.$$

Any element $\pi = \pi_f \in \mathcal{O}_F$ with $v(\pi) = 1$ is called a *uniformizer*. This generates the maximal ideal

$$\mathfrak{m} = \mathfrak{m}_F = \{x \in \mathcal{O}_F : v(x) \geq 1\}.$$

Definition (Residue field)**.** The *residue field* of a non-Archimedean local field F is

$$k = k_F = \mathcal{O}_F/\mathfrak{m}_F.$$

This is a finite field of order $q = p^r$.

A particularly well-understood subfield of F^{ab} is the *maximal unramified extension* F^{ur}. We have

$$\mathrm{Gal}(F^{\mathrm{ur}}/F) = \mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}} = \lim_{n\geq 1} \mathbb{Z}/n\mathbb{Z}.$$

and this is completely determined by the behaviour of the residue field. The rest of Γ_F is called the *inertia group*.

Definition (Inertia group)**.** The *inertia group* I_F is defined to be

$$I_F = \mathrm{Gal}(\bar{F}/F^{\mathrm{ur}}) \subseteq \Gamma_F.$$

We also define

Definition (Wild inertia group)**.** The *wild inertia group* P_F is the maximal pro-p-subgroup of I_F.

Returning to the maximal unramified extension, note that saying $\mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$ requires picking an isomorphism, and this is equivalent to picking an element of $\hat{\mathbb{Z}}$ to be the "1". Naively, we might pick the following:

Definition (Arithmetic Frobenius)**.** The *arithmetic Frobenius* $\varphi_q \in \mathrm{Gal}(\bar{k}/k)$ (where $|k| = q$) is defined to be

$$\varphi_q(x) = x^q.$$

Identifying this with $1 \in \hat{\mathbb{Z}}$ leads to infinite confusion, and we shall not do so. Instead, we define

Definition (Geometric Frobenius)**.** The *geometric Frobenius* is

$$\mathrm{Frob}_q = \varphi_q^{-1} \in \mathrm{Gal}(\bar{k}/k).$$

We shall identify $\mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$ by setting the *geometric* Frobenius to be 1.

The reason this is called the geometric Frobenius is that if we have a scheme over a finite field k, then there are two ways the Frobenius can act on it — either as a Galois action, or as a pullback along the morphism $(-)^q : k \to k$. The latter corresponds to the geometric Frobenius.

We now turn to understand the inertia groups. The point of introducing the wild inertia group is to single out the "p-phenomena", which we would like to avoid. To understand I_F better, let n be a natural number prime to p. As usual, we write

$$\mu_n(\bar{k}) = \{\zeta \in \bar{k} : \zeta^n = 1\}.$$

We also pick an nth root of π in $\bar{F}$, say π_n. By definition, this has $\pi_n^n = \pi$.

Definition (Tame mod n character)**.** The *tame mod n character* is the map $t(n) : I_F = \mathrm{Gal}(\bar{F}/F^{ur}) \to \mu_n(\bar{k})$ given by

$$\gamma \mapsto \gamma(\pi_n)/\pi_n \pmod{\pi}.$$

Note that since γ fixes $\pi = \pi_n^n$, we indeed have

$$\left(\frac{\gamma(\pi_n)}{\pi_n}\right)^n = \frac{\gamma(\pi_n^n)}{\pi_n^n} = 1.$$

Moreover, this doesn't depend on the choice of π_n. Any other choice differs by an nth root of unity, but the nth root of unity lies in F^{ur} since n is prime to p. So γ fixes it and so it cancels out in the fraction. For the same reason, if γ moves π_n at all, then this is visible down in $\bar{k}$, since γ would have multiplied π_n by an nth root of unity, and these nth roots are present in $\bar{k}$.

Now that everything is canonically well-defined, we can take the limit over all n to obtain a map

$$\hat{t} : I_F \to \lim_{(n,p)=1} \mu_n(\bar{k}) = \prod_{\ell \neq p} \lim_{m \geq 1} \mu_{\ell^m}(\bar{k}) \equiv \prod_{\ell \neq p} \mathbb{Z}_\ell(1)(\bar{k}).$$

This $\mathbb{Z}_\ell(1)(\bar{k})$ is the *Tate module* of $\bar{k}^\times$. This is isomorphic to $\mathbb{Z}_\ell$, but not canonically.

Theorem. $\ker \hat{t} = P_F$.

Thus, it follows that *maximal tamely ramified extension* of F, i.e. the fixed field of P_F is

$$\bigcup_{(n,p)=1} F^{ur}(\sqrt[n]{\pi}).$$

Note that $t(n)$ extends to a map $\Gamma_F \to \mu_n$ given by the same formula, but this now depends on the choice of π_n, and further, it is not a homomorphism, because

$$t(n)(\gamma\delta) = \frac{\gamma\delta(\pi_n)}{\pi_n} = \frac{\gamma(\pi_n)}{\pi_n}\gamma\left(\frac{\delta(\pi_n)}{\pi_n}\right) = t(n)(\gamma) \cdot \gamma(t(n)(\delta)).$$

So this formula just says that $t(n)$ is a 1-cocycle. Of course, picking another π_n will modify $t(n)$ by a coboundary.

1.2 Local class field theory

Local class field theory is a (collection of) theorems that describe abelian extensions of a local field. The key takeaway is that finite abelian extensions of F correspond to open finite index subgroups of $F^\times$, but the theorem says a bit more than that:

Theorem (Local class field theory)**.**

(i) Let F be a local field. Then there is a continuous homomorphism, the *local Artin map*

$$\mathrm{Art}_F : F^\times \to \Gamma_F^{\mathrm{ab}}$$

with dense image characterized by the properties

(a) The following diagram commutes:

$$\begin{tikzcd} F^\times \arrow[r, "\mathrm{Art}_F"] \arrow[d, two heads, "v_F"] & \Gamma_F^{\mathrm{ab}} \arrow[r, two heads] & \Gamma_F/I_F \arrow[d, "\sim"] \\ \mathbb{Z} \arrow[rr, hook] & & \hat{\mathbb{Z}} \end{tikzcd}$$

(b) If F'/F is finite, then the following diagram commutes:

$$\begin{tikzcd} (F')^\times \arrow[r, "\mathrm{Art}_{F'}"] \arrow[d, "N_{F'/F}"] & \Gamma_{F'}^{\mathrm{ab}} = \mathrm{Gal}(F'^{\mathrm{ab}}/F') \arrow[d, "\text{restriction}"] \\ F^\times \arrow[r, "\mathrm{Art}_F"] & \Gamma_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F) \end{tikzcd}$$

(ii) Moreover, the *existence theorem* says Art_F^{-1} induces a bijection

$$\left\{\begin{array}{c}\text{open finite index}\\ \text{subgroups of } F^\times\end{array}\right\} \longleftrightarrow \left\{\text{ open subgroups of } \Gamma_F^{\mathrm{ab}} \right\}$$

Of course, open subgroups of Γ_F^{ab} further corresponds to finite abelian extensions of F.

(iii) Further, Art_F induces an isomorphism

$$\mathcal{O}_F^\times \xrightarrow{\sim} \mathrm{im}(I_F \to \Gamma_F^{\mathrm{ab}})$$

and this maps $(1+\pi\mathcal{O}_F)^\times$ to the image of P_F. Of course, the quotient $\mathcal{O}_F^\times/(1+\pi\mathcal{O}_F)^\times \cong k^\times = \mu_\infty(k)$.

(iv) Finally, this is functorial, namely if we have an isomorphism $\alpha : F \xrightarrow{\sim} F'$ and extend it to $\bar{\alpha} : \bar{F} \xrightarrow{\sim} \bar{F}'$, then this induces isomorphisms between the Galois groups $\alpha_* : \Gamma_F \xrightarrow{\sim} \Gamma_{F'}$ (up to conjugacy), and $\alpha_*^{\mathrm{ab}} \circ \mathrm{Art}_F = \mathrm{Art}_{F'} \circ \alpha_*^{\mathrm{ab}}$. □

On the level of finite Galois extensions E/F, we can rephrase the first part of the theorem as giving a map

$$\mathrm{Art}_{E/F} : \frac{F^\times}{N_{E/F}(E^\times)} \to \mathrm{Gal}(E/F)^{\mathrm{ab}}$$

which is now an isomorphism (since a dense subgroup of a discrete group is the whole thing!).

We can write down these maps explicitly in certain special cases. We will not justify the following example:

Example. If $F = \mathbb{Q}_p$, then

$$F^{\mathrm{ab}} = \mathbb{Q}_p(\mu_\infty) = \bigcup \mathbb{Q}_p(\mu_n) = \mathbb{Q}_p^{ur}(\mu_{p^\infty}).$$

Moreover, if we write $x \in \mathbb{Q}_p^\times$ as $p^n y$ with $y \in \mathbb{Z}_p^\times$, then

$$\mathrm{Art}_{\mathbb{Q}}(x)|_{\mathbb{Q}_p^{ur}} = \mathrm{Frob}_p^n, \quad \mathrm{Art}_{\mathbb{Q}}(x)|_{\mathbb{Q}_p(\mu_{p^\infty})} = (\zeta_{p^n} \mapsto \zeta_{p^n}^{y \bmod p^n}).$$

If we had the arithmetic Frobenius instead, then we would have a $-y$ in the power there, which is less pleasant.

The cases of the Archimedean local fields are easy to write down and prove directly!

Example. If $F = \mathbb{C}$, then $\Gamma_F = \Gamma_F^{ab} = 1$ is trivial, and the Artin map is similarly trivial. There are no non-trivial open finite index subgroups of $\mathbb{C}^\times$, just as there are no non-trivial open subgroups of the trivial group.

Example. If $F = \mathbb{R}$, then $\bar{\mathbb{R}} = \mathbb{C}$ and $\Gamma_F = \Gamma_F^{ab} = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$. The Artin map is given by the sign map. The unique open finite index subgroup of $\mathbb{R}^\times$ is $\mathbb{R}^\times_{>0}$, and this corresponds to the finite Galois extension $\mathbb{C}/\mathbb{R}$.

As stated in the theorem, the Artin map has dense image, but is not surjective (in general). To fix this problem, it is convenient to introduce the *Weil group*.

Definition (Weil group)**.** Let F be a non-Archimedean local field. Then the *Weil group* of F is the topological group W_F defined as follows:

– As a group, it is

$$W_F = \{\gamma \in \Gamma_F \mid \gamma|_{F^{ur}} = \mathrm{Frob}_q^n \text{ for some } n \in \mathbb{Z}\}.$$

Recall that $\mathrm{Gal}(F^{ur}/F) = \hat{\mathbb{Z}}$, and we are requiring $\gamma|_{F^{ur}}$ to be in $\mathbb{Z}$. In particular, $I_F \subseteq W_F$.

– The topology is defined by the property that I_F is an *open* subgroup with the profinite topology. Equivalently, W_F is a fiber product of topological groups

$$\begin{tikzcd} W_F \arrow[r, hook] \arrow[d, two heads] & \Gamma_F \arrow[d, two heads] \\ \mathbb{Z} \arrow[r, hook] & \hat{\mathbb{Z}} \end{tikzcd}$$

where $\mathbb{Z}$ has the discrete topology.

Note that W_F is not profinite. It is totally disconnected but not compact.

This seems like a slightly artificial definition, but this is cooked up precisely so that

Proposition. Art_F induces an *isomorphism* of topological groups

$$\mathrm{Art}_F^W : F^\times \to W_F^{\mathrm{ab}}.$$

This maps $\mathcal{O}_F^\times$ isomorphically onto the inertia subgroup of Γ_F^{ab}.

In the case of Archimedean local fields, we make the following definitions. They will seem rather ad hoc, but we will provide some justification later.

– The Weil group of $\mathbb{C}$ is defined to be $W_\mathbb{C} = \mathbb{C}^\times$, and the Artin map $\mathrm{Art}_\mathbb{R}^W$ is defined to be the identity.

– The Weil group of $\mathbb{R}$ is defined the non-abelian group

$$W_\mathbb{R} = \langle \mathbb{C}^\times, \sigma \mid \sigma^2 = -1 \in \mathbb{C}^\times, \sigma z \sigma^{-1} = \bar{z} \text{ for all } z \in \mathbb{C}^\times \rangle.$$

This is a (non)-split extension of $\mathbb{C}^\times$ by $\Gamma_\mathbb{R}$,

$$1 \to \mathbb{C}^\times \to W_\mathbb{R} \to \Gamma_\mathbb{R} \to 1,$$

where the last map sends $z \mapsto 0$ and $\sigma \mapsto 1$. This is in fact the unique non-split extension of $\Gamma_\mathbb{R}$ by $\mathbb{C}^\times$ where $\Gamma_\mathbb{R}$ acts on $\mathbb{C}^\times$ in a natural way.

The map $\mathrm{Art}_\mathbb{R}^W$ is better described by its inverse, which maps

$$\begin{aligned} (\mathrm{Art}_\mathbb{R}^W)^{-1} : W_\mathbb{R}^{\mathrm{ab}} &\xrightarrow{\sim} \mathbb{R}^\times \\ z &\longmapsto z\bar{z} \\ \sigma &\longmapsto -1 \end{aligned}$$

To understand these definitions, we need the notion of the relative Weil group.

Definition (Relative Weil group)**.** Let F be a non-Archimedean local field, and E/F Galois but not necessarily finite. We define

$$W_{E/F} = \{\gamma \in \mathrm{Gal}(E^{\mathrm{ab}}/F) : \gamma|_{F^{ur}} = \mathrm{Frob}_q^n, n \in \mathbb{Z}\} = \frac{W_F}{\overline{[W_E, W_E]}}.$$

with the quotient topology.

The $W_{\bar{F}/F} = W_F$, while $W_{F/F} = W_F^{\mathrm{ab}} = F^\times$ by local class field theory.

Now if E/F is a finite extension, then we have an exact sequence of Galois groups

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[r] & \mathrm{Gal}(E^{\mathrm{ab}}/F) \arrow[r] & \mathrm{Gal}(E/F) \arrow[r] & 1 \\
1 \arrow[r] & W_E^{\mathrm{ab}} \arrow[r] \arrow[u, hook] & W_{E/F} \arrow[r] \arrow[u, hook] & \mathrm{Gal}(E/F) \arrow[r] \arrow[u, equal] & 1.
\end{tikzcd}$$

By the Artin map, $W_E^{\mathrm{ab}} = E^\times$. So the relative Weil group is an extension of $\mathrm{Gal}(E/F)$ by $E^\times$. In the case of non-Archimedean fields, we have

$$\lim_{E} E^\times = \{1\},$$

where the field extensions are joined by the norm map. So $\bar{F}^\times$ is invisible in $W_F = \lim W_{E/F}$. The weirdness above comes from the fact that the separable closures of $\mathbb{R}$ and $\mathbb{C}$ are finite extensions.

We are, of course, not going to prove local class field theory in this course. However, we can say something about the proofs. There are a few ways of proving it:

– The cohomological method (see Artin–Tate, Cassels–Fröhlich), which only treats the first part, namely the existence of Art_K. We start off with a finite Galois extension E/F, and we want to construct an isomorphism

$$\mathrm{Art}_{E/F} : F^\times/N_{E/F}(E^\times) \to \mathrm{Gal}(E/F)^{\mathrm{ab}}.$$

Writing $G = \mathrm{Gal}(E/F)$, this uses the cohomological interpretation

$$F^\times/N_{E/F}(E^\times) = \hat{H}^0(G, E^\times),$$

where $\hat{H}$ is the Tate cohomology of finite groups. On the other hand, we have

$$G^{\mathrm{ab}} = H_1(G, \mathbb{Z}) = \hat{H}^{-2}(G, \mathbb{Z}).$$

The main step is to compute

$$H^2(G, E^\times) = \hat{H}^2(G, E^\times) \cong \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z} = H^2(\Gamma_F, \bar{F}^\times).$$

where $n = [E : F]$. The final group $H^2(\Gamma_F, \bar{F}^\times)$ is the *Brauer group* $\mathrm{Br}(F)$, and the subgroup is just the kernel of $\mathrm{Br}(F) \to \mathrm{Br}(E)$.

Once we have done this, we then define $\mathrm{Art}_{E/F}$ to be the cup product with the generator of $\hat{H}^2(G, E^\times)$, and this maps $\hat{H}^{-2}(G, \mathbb{Z}) \to \hat{H}^0(G, E^\times)$. The fact that this map is an isomorphism is rather formal.

The advantage of this method is that it generalizes to duality theorems about $H^*(G, M)$ for arbitrary M, but this map is not at all explicit, and is very much tied to abelian extensions.

- Formal group methods: We know that the maximal abelian extension of $\mathbb{Q}_p$ is obtained in two steps — we can write

$$\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p^{ur}(\mu_{p^\infty}) = \mathbb{Q}_p^{ur}(\text{torsion points in } \hat{\mathbb{G}}_m),$$

where $\hat{\mathbb{G}}_m$ is the formal multiplication group, which we can think of as $(1 + \mathfrak{m}_{\bar{\mathbb{Q}}_p})^\times$. This generalizes to any $F/\mathbb{Q}_p$ — we have

$$F^{\mathrm{ab}} = F^{ur}(\text{torsion points in } \hat{\mathbb{G}}_\pi),$$

where $\hat{\mathbb{G}}_\pi$ is the "Lubin–Tate formal group". This is described in Iwasawa's book, and also in a paper of Yoshida's. The original paper by Lubin and Tate is also very readable.

The advantage of this is that it is very explicit, and when done correctly, gives both the existence of the Artin map and the existence theorem. This also has a natural generalization to non-abelian extensions. However, it does not give duality theorems.

- Neukrich's method: Suppose E/F is abelian and finite. If $g \in \mathrm{Gal}(E/F)$, we want to construct $\mathrm{Art}_{E/F}^{-1}(g) \in F^\times / N_{E/F}(E^\times)$. The point is that there is only one possibility, because $\langle g \rangle$ is a cyclic subgroup of $\mathrm{Gal}(E/F)$, and corresponds to some cyclic extension $\mathrm{Gal}(E/F')$. We have the following lemma:

Lemma. There is a finite K/F' such that $K \cap E = F'$, so $\mathrm{Gal}(KE/K) \cong \mathrm{Gal}(E/F') = \langle g \rangle$. Moreover, KE/K is unramified.

Let $g'|_E = g$, and suppose $g' = \mathrm{Frob}_{KE/K}^a$. If local class field theory is true, then we have to have

$$\mathrm{Art}_{KE/K}^{-1}(g') = \pi_K^a \pmod{N_{KE/K}(KE^\times)}.$$

Then by our compatibility conditions, this implies

$$\mathrm{Art}_{E/F}^{-1}(g) = N_{K/F}(\pi_K^a) \pmod{N_{E/F}(E^\times)}.$$

The problem is then to show that this does not depend on the choices, and then show that it is a homomorphism. These are in fact extremely complicated. Note that everything so far is just Galois theory. Solving these two problems is then where all the number theory goes in.

- When class field theory was first done, we first did *global* class field theory, and deduced the local case from that. No one does that anymore nowadays, since we now have purely local proofs. However, when we try to generalize to the Langlands programme, what we have so far all start with global theorems and then proceed to deduce local results.

1.3 Global class field theory

We now proceed to discuss global class field theory.

Definition (Global field)**.** A *global field* is a number field or $k(C)$ for a smooth projective absolutely irreducible curve $C/\mathbb{F}_q$, i.e. a finite extension of $\mathbb{F}_q(t)$.

A lot of what we can do can be simultaneously done for both types of global fields, but we are mostly only interested in the case of number fields, and our discussions will mostly focus on those.

Definition (Place)**.** Let K be a global field. Then a *place* is a valuation on K. If K is a number field, we say a valuation v is a *finite place* if it is the valuation at a prime $\mathfrak{p} \lhd \mathcal{O}_K$. A valuation v is an *infinite place* if comes from a complex or real embedding of K. We write Σ_K for the set of places of K, and Σ_K^∞ and $\Sigma_{K,\infty}$ for the sets of finite and infinite places respectively. We also write $v \nmid \infty$ if v is a finite place, and $v \mid \infty$ otherwise.

If K is a function field, then all places are are finite, and these correspond to closed points of the curve.

If $v \in \Sigma_K$ is a place, then there is a completion $K \hookrightarrow K_v$. If v is infinite, then K_v is $\mathbb{R}$ or $\mathbb{C}$, i.e. an Archimedean local field. Otherwise, K_v is a non-Archimedean local field.

Example. If $\mathbb{Q} = K$, then there is one infinite prime, which we write as ∞, given by the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$. If $v = p$, then we get the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ into the p-adic completion.

Notation. If v is a finite place, we write $\mathcal{O}_v \subseteq K_v$ for the valuation ring of the completion.

Any local field has a canonically normalized valuation, but there is no canonical absolute value. It is useful to fix the absolute value of our local fields. For doing class field theory, the right way to put an absolute value on K_v (and hence K) is by

$$|x|_v = q_v^{-v(x)},$$

where

$$q_v = \left|\frac{\mathcal{O}_v}{\pi_v \mathcal{O}_v}\right|$$

is the cardinality of the residue field at v. For example, if $K = \mathbb{Q}$ and $v = p$, then $q_v = p$, and $|p|_v = \frac{1}{p}$.

In the Archimedean case, if K_v is real, then we set $|x|$ to be the usual absolute value; if K_v is complex, then we take $|x|_v = x\bar{x} = |x|^2$.

The reason for choosing these normalizations is that we have the following product formula:

Proposition (Product formula)**.** If $x \in K^\times$, then

$$\prod_{v \in \Sigma_K} |x|_v = 1.$$

The proof is not difficult. First observe that it is true for $\mathbb{Q}$, and then show that this formula is "stable under finite extensions", which extends the result to all finite extensions of $\mathbb{Q}$.

Global class field theory is best stated in terms of adeles and ideles. We make the following definition:

Definition (Adele)**.** The *adeles* is defined to be the restricted product

$$\mathbb{A}_K = \prod_v{}' K_V = \left\{(x_v)_{v \in K_v} : x_v \in \mathcal{O}_v \text{ for all but finitely many } v \in \Sigma_K^\infty\right\}.$$

We can write this as $K_\infty \times \hat{K}$ or $\mathbb{A}_{K,\infty} \times \mathbb{A}_K^\infty$, where

$$K_\infty = \mathbb{A}_{K,\infty} = K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

consists of the product over the infinite places, and

$$\hat{K} = \mathbb{A}_K^\infty = \prod_{v \nmid \infty}{}' K_V = \bigcup_{S \subseteq \Sigma_K^\infty \text{ finite}} \prod_{v \in S} K_v \times \prod_{v \in \Sigma_K^\infty \setminus S} \mathcal{O}_v.$$

This contains $\hat{\mathcal{O}}_K = \prod_{v \nmid \infty} \mathcal{O}_K$. In the case of a number field, $\hat{\mathcal{O}}_K$ is the profinite completion of $\mathcal{O}_K$. More precisely, if K is a number field then

$$\hat{\mathcal{O}}_K = \lim_{\mathfrak{a}} \mathcal{O}_K/\mathfrak{a} = \lim \mathcal{O}_K/N\mathcal{O}_K = \mathcal{O}_K \otimes_{\mathbb{Z}} \hat{\mathbb{Z}},$$

where the last equality follows from the fact that $\mathcal{O}_K$ is a finite $\mathbb{Z}$-module.

Definition (Idele)**.** The *ideles* is the restricted product

$$J_K = \mathbb{A}_K^\times = \prod_v{}' K_v^\times = \left\{(x_v)_v \in \prod K_v^\times : x_v \in \mathcal{O}_v^\times \text{ for almost all } v\right\}.$$

These objects come with natural topologies. On $\mathbb{A}_K$, we take $K_\infty \times \hat{\mathcal{O}}_K$ to be an open subgroup with the product topology. Once we have done this, there is a unique structure of a topological ring for which this holds. On J_K, we take $K_\infty^\times \times \hat{\mathcal{O}}_K^\times$ to be open with the product topology. Note that this is not the subspace topology under the inclusion $J_K \hookrightarrow \mathbb{A}_K$. Instead, it is induced by the inclusion

$$\begin{aligned} J_K &\hookrightarrow \mathbb{A}_K \times \mathbb{A}_K \\ x &\mapsto (x, x^{-1}). \end{aligned}$$

It is a basic fact that $K^\times \subseteq J_K$ is a *discrete* subgroup.

Definition (Idele class group)**.** The *idele class group* is then

$$C_K = J_K/K^\times.$$

The idele class group plays an important role in global class field theory. Note that J_K comes with a natural absolute value

$$\begin{aligned} |\cdot|_A : J_K &\to \mathbb{R}_{>0}^\times \\ (x_v) &\mapsto \prod_{v \in \Sigma_K} |x_v|_v. \end{aligned}$$

The product formula implies that $|K^\times|_{\mathbb{A}} = \{1\}$. So this in fact a map $C_K \to \mathbb{R}_{>0}^\times$. Moreover, we have

Theorem. The map $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}^{\times}_{>0}$ has compact kernel.

This seemingly innocent theorem is actually quite powerful. For example, we will later construct a continuous surjection $C_K \to \mathrm{Cl}(K)$ to the ideal class group of K. In particular, this implies $\mathrm{Cl}(K)$ is finite! In fact, the theorem is equivalent to the finiteness of the class group and Dirichlet's unit theorem.

Example. In the case $K = \mathbb{Q}$, we have, by definition,

$$J_{\mathbb{Q}} = \mathbb{R}^{\times} \times \prod_p{}' \mathbb{Q}_p^{\times}.$$

Suppose we have an idele $x = (x_v)$. It is easy to see that there exists a unique rational number $y \in \mathbb{Q}^{\times}$ such that $\mathrm{sgn}(y) = \mathrm{sgn}(x_\infty)$ and $v_p(y) = v_p(x_p)$. So we have

$$J_{\mathbb{Q}} = \mathbb{Q}^{\times} \times \left(\mathbb{R}^{\times}_{>0} \times \prod_p \mathbb{Z}_p^{\times}\right).$$

Here we think of $\mathbb{Q}^{\times}$ as being embedded diagonally into $J_{\mathbb{Q}}$, and as we have previously mentioned, $\mathbb{Q}^{\times}$ is discrete. From this description, we can read out a description of the idele class group

$$C_{\mathbb{Q}} = \mathbb{R}^{\times}_{>0} \times \hat{\mathbb{Z}}^{\times},$$

and $\hat{\mathbb{Z}}^{\times}$ is the kernel $|\cdot|_{\mathbb{A}}$.

From the decomposition above, we see that $C_{\mathbb{Q}}$ has a maximal connected subgroup $\mathbb{R}^{\times}_{>0}$. In fact, this is the intersection of all open subgroups containing 1. For a general K, we write C_K^0 for the maximal connected subgroup, and then

$$\pi_0(C_K) = C_K / C_K^0,$$

In the case of $K = \mathbb{Q}$, we can naturally identify

$$\pi_0(C_{\mathbb{Q}}) = \hat{\mathbb{Z}}^{\times} = \lim_n (\mathbb{Z}/n\mathbb{Z})^{\times} = \lim_n \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\zeta_\infty)/\mathbb{Q}).$$

The field $\mathbb{Q}(\zeta_\infty)$ is not just any other field. The Kronecker–Weber theorem says this is in fact the maximal abelian extension of $\mathbb{Q}$. Global class field theory is a generalization of this isomorphism to all fields.

Just like in local class field theory, global class field theory involves a certain Artin map. In the local case, we just pulled them out of a hat. To construct the global Artin map, we simply have to put these maps together to form a global Artin map.

Let L/K be a finite Galois extension, v a place of K, and $w \mid v$ a place of L extending v. For finite places, this means the corresponding primes divide; for infinite places, this means the embedding w is an extension of v. We then have the *decomposition group*

$$\mathrm{Gal}(L_w/K_v) \subseteq \mathrm{Gal}(L/K).$$

If L/K is abelian, since any two places lying above v are conjugate, this depends only on v. In this case, we can now define the *global Artin map*

$$\begin{aligned} \mathrm{Art}_{L/K} : J_K &\longrightarrow \mathrm{Gal}(L/K) \\ (x_v)_v &\longmapsto \prod_v \mathrm{Art}_{L_w/K_v}(x_v), \end{aligned}$$

where we pick one w for each v. To see this is well-defined, note that if $x_v \in \mathcal{O}_v^\times$ and L/K is unramified at $v \nmid \infty$, then $\mathrm{Art}_{K_w/K_v}(x_v) = 1$. So the product is in fact finite.

By the compatibility of the Artin maps, we can passing on to the limit over all extensions L/K, and get a continuous map

$$\mathrm{Art}_K : J_K \to \Gamma_K^{\mathrm{ab}}.$$

Theorem (Artin reciprocity law)**.** $\mathrm{Art}_K(K^\times) = \{1\}$, so induces a map $C_K \to \Gamma_K^{\mathrm{ab}}$. Moreover,

(i) If $\mathrm{char}(K) = p > 0$, then Art_K is injective, and induces an isomorphism $\mathrm{Art}_K : C_k \overset{\sim}{\to} W_K^{\mathrm{ab}}$, where W_K is defined as follows: since K is a finite extension of $\mathbb{F}_q(T)$, and wlog assume $\bar{\mathbb{F}}_q \cap K = \mathbb{F}_q \equiv k$. Then W_K is defined as the pullback

$$\begin{tikzcd} W_K \arrow[r, hook] \arrow[d] & \Gamma_K = \mathrm{Gal}(\bar{K}/K) \arrow[d, "\mathrm{restr.}"] \\ \mathbb{Z} \arrow[r, hook] & \hat{\mathbb{Z}} \cong \mathrm{Gal}(\bar{k}/k) \end{tikzcd}$$

(ii) If $\mathrm{char}(K) = 0$, we have an isomorphism

$$\mathrm{Art}_K : \pi_0(C_K) = \frac{C_K}{C_K^0} \overset{\sim}{\to} \Gamma_K^{\mathrm{ab}}.$$

Moreover, if L/K is finite, then we have a commutative diagram

$$\begin{tikzcd} C_L \arrow[r, "\mathrm{Art}_L"] \arrow[d, "N_{L/K}"] & \Gamma_L^{\mathrm{ab}} \arrow[d, "\mathrm{restr.}"] \\ C_K \arrow[r, "\mathrm{Art}_K"] & \Gamma_K^{\mathrm{ab}} \end{tikzcd}$$

If this is in fact Galois, then this induces an isomorphism

$$\mathrm{Art}_{L/K} : \frac{J_K}{K^\times N_{L/K}(J_L)} \overset{\sim}{\to} \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

Finally, this is functorial, namely if $\sigma : K \overset{\sim}{\to} K'$ is an isomorphism, then we have a commutative square

$$\begin{tikzcd} C_K \arrow[r, "\mathrm{Art}_K"] \arrow[d, "\sigma"] & \Gamma_K^{\mathrm{ab}} \arrow[d] \\ C_{K'} \arrow[r, "\mathrm{Art}_{K'}"] & \Gamma_{K'}^{\mathrm{ab}} \end{tikzcd}$$

Observe that naturality and functoriality are immediate consequences of the corresponding results for local class field theory.

As a consequence of the isomorphism, we have a correspondence

$$\left\{\begin{array}{c}\text{finite abelian extensions}\\ L/K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite index open subgroups}\\ \text{of } J_K \text{ containing } K^\times\end{array}\right\}$$

$$L \longmapsto \ker(\mathrm{Art}_{L/K} : J_K \to \mathrm{Gal}(L/K))$$

Note that there exists finite index subgroups that are not open!

Recall that in local class field theory, if we decompose $K_v = \langle \pi \rangle \times \mathcal{O}_v^\times$, then the local Artin map sends $\mathcal{O}_v^\times$ to (the image of) the inertia group. Thus an extension L_w/K_v is unramified iff the local Artin map kills of $\mathcal{O}_v^\times$. Globally, this tells us

Proposition. If L/K is an abelian extension of global fields, which corresponds to the open subgroup $U \subseteq J_K$ under the Artin map, then L/K is unramified at a finite $v \nmid \infty$ iff $\mathcal{O}_v^\times \subseteq U$.

We can extend this to the infinite places if we make the appropriate definitions. Since $\mathbb{C}$ cannot be further extended, there is nothing to say for complex places.

Definition (Ramification)**.** If $v \mid \infty$ is a real place of K, and L/K is a finite abelian extension, then we say v is ramified if for some (hence all) places w of L above v, w is complex.

The terminology is not completely standard. In this case, Neukrich would say v is inert instead.

With this definition, L/K is unramified at a real place v iff $K_v^\times = \mathbb{R}^\times \subseteq U$. Note that since U is open, it automatically contains $\mathbb{R}_{>0}^\times$.

We can similarly read off splitting information.

Proposition. If v is finite and unramified, then v splits completely iff $K_v^\times \subseteq U$.

Proof. v splits completely iff $L_w = K_v$ for all $w \mid v$, iff $\mathrm{Art}_{L_w/K_v}(K_v^\times) = \{1\}$. □

Example. We will use global class field theory to compute all S_3 extensions $L/\mathbb{Q}$ which are unramified outside 5 and 7.

If we didn't have global class field theory, then to solve this problem we have to find all cubics whose discriminant are divisible by 5 and 7 only, and there is a cubic diophantine problem to solve.

While S_3 is not an abelian group, it is solvable. So we can break our potential extension as a chain

$$\begin{array}{c} L \\ \big| \, 3 \\ K \\ \big| \, 2 = \langle \sigma \rangle \\ \mathbb{Q} \end{array}$$

Since $L/\mathbb{Q}$ is unramified outside 5 and 7, we know that K must be one of $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-35})$. We then consider each case in turn, and then see what are the possibilities for L. We shall only do the case $K = \mathbb{Q}(\sqrt{-7})$ here. If we perform similar computations for the other cases, we find that the other choices of K do not work.

So fix $K = \mathbb{Q}(\sqrt{-7})$. We want L/K to be cyclic of degree 3, and σ must act non-trivially on L (otherwise we get an abelian extension).

Thus, by global class field theory, we want to find a subgroup $U \leq J_K$ of index 3 such that $\mathcal{O}_v^\times \subseteq U$ for all $v \nmid 35$. We also need $\sigma(U) = U$, or else the composite extension would not even be Galois, and σ has to acts as -1 on $J_K/U \cong \mathbb{Z}/3\mathbb{Z}$ to get a non-abelian extension.

We know $K = \mathbb{Q}(\sqrt{-7})$ has has class number 1, and the units are ± 1. So we know
$$\frac{C_K}{C_K^0} = \frac{\hat{\mathcal{O}}_K^\times}{\{\pm 1\}}.$$
By assumption, we know U contains $\prod_{v \nmid 35} \mathcal{O}_v^\times$. So we have to look at the places that divide 35. In $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, the prime 5 is inert and 7 is ramified.

Since 5 is inert, we know $K_5/\mathbb{Q}_5$ is an unramified quadratic extension. So we can write
$$\mathcal{O}_{(5)}^\times = \mathbb{F}_{25}^\times \times (1 + 5\mathcal{O}_{(5)})^\times.$$
The second factor is a pro-5 group, and so it must be contained in U for the quotient to have order 3. On $\mathbb{F}_{25}^\times$, σ acts as the Frobenius $\sigma(x) = x^5$. Since $\mathbb{F}_{25}^\times$ is cyclic of order 24, there is a unique index 3 subgroup, cyclic of order 6. This gives an index 3 subgroup $U_5 \subseteq \mathcal{O}_{(5)}^\times$. Moreover, on here, σ acts by $x \mapsto x^5 = x^{-1}$. Thus, we can take
$$U = \prod_{v \neq (5)} \mathcal{O}_v^\times \times U_5,$$
and this gives an S_3 extension of $\mathbb{Q}$ that is unramified outside 5 and 7. It is an exercise to explicitly identify this extension.

We turn to the prime $7 = -\sqrt{-7}^2$. Since this is ramified, we have
$$\mathcal{O}_{(\sqrt{-7})}^\times = \mathbb{F}_7^\times \times \left(1 + (\sqrt{-7})\mathcal{O}_{\sqrt{-7}}\right)^\times,$$
and again the second factor is a pro-7 group. Moreover σ acts trivially on $\mathbb{F}_7^\times$. So U must contain $\mathcal{O}_{(\sqrt{-7})}^\times$. So what we found above is the unique such extension.

We previously explicitly described $C_\mathbb{Q}$ as $\mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}$. It would be nice to have a similar description of C_K for an arbitrary K. The connected component will come from the infinite places $K_\infty^\times \prod_{v \mid \infty} K_v^\times$. The connected component is given by
$$K_\infty^{\times, 0} = (\mathbb{R}_{>0}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2},$$
where there are r_1 real places and r_2 complex ones. Thus, we find that

Proposition.
$$C_K / C_K^0 = \frac{\{\pm 1\}^{r_1} \times \hat{K}^\times}{\overline{K^\times}}.$$

There is a natural map from the ideles to a more familiar group, called the content homomorphism.

Definition (Content homomorphism)**.** The *content homomorphism* is the map
$$\begin{aligned} c : J_K &\to \text{ fractional ideals of } K \\ (x_v)_v &\mapsto \prod_{v \nmid \infty} \mathfrak{p}_v^{v(x_v)}, \end{aligned}$$
where $\mathfrak{p}_v$ is the prime ideal corresponding to v. We ignore the infinite places completely.

Observe that $c(K^\times)$ is the set of all principal ideals by definition. Moreover, the kernel of the content map is $K_\infty^\times \times \hat{\mathcal{O}}_K^\times$, by definition. So we have a short exact sequence

$$1 \to \frac{\{\pm 1\}^{r_1} \times \hat{\mathcal{O}}_K^\times}{\overline{\mathcal{O}_K^\times}} \to C_K/C_K^0 \to \mathrm{Cl}(K) \to 1.$$

If $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-D})$, then $\overline{\mathcal{O}_K^\times} = \mathcal{O}_K^\times$ is finite, and in particular is closed. But in general, it will not be closed, and taking the closure is indeed needed.

Returning to the case $K = \mathbb{Q}$, our favorite abelian extensions are those of the form $L = \mathbb{Q}(\zeta_N)$ with $N > 1$. This comes with an Artin map

$$\hat{\mathbb{Z}}^\times \cong C_{\mathbb{Q}}/C_{\mathbb{Q}}^0 \to \mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times.$$

By local class field theory for $\mathbb{Q}_p$, we see that with our normalizations, this is just the quotient map, whose kernel is

$$(1 + N\hat{\mathbb{Z}})^\times = \prod_{p \nmid N} \mathbb{Z}_p^\times \times \prod_{p \mid N} (1 + N\mathbb{Z}_p)^\times \subseteq \prod \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times.$$

Note that if we used the arithmetic Frobenius, then we would get the *inverse* of the quotient map.

These subgroups of $\hat{\mathbb{Z}}^\times$ are rather special ones. First of all $(1 + N\hat{\mathbb{Z}})^\times$ form a neighbourhood of the identity in $\hat{\mathbb{Z}}^\times$. Thus, any open subgroup contains a subgroup of this form. Equivalently, every abelian extension of $\mathbb{Q}$ is contained in $\mathbb{Q}(\zeta_N)$ for some N. This is the *Kronecker–Weber theorem*.

For a general number field K, we want to write down an explicit basis for open subgroups of 1 in $\pi_0(C_k)$.

Definition (Modulus)**.** A *modulus* is a finite formal sum

$$\mathfrak{m} = \sum_{v \in \Sigma_k} m_v \cdot (v)$$

of places of K, where $m_v \geq 0$ are integers.

Given a modulus $\mathfrak{m}$, we define the subgroup

$$U_{\mathfrak{m}} = \prod_{v \mid \infty, m_v > 0} K_v^{\times, 0} \times \prod_{v \mid \infty, m_v = 0} K_v^\times \times \prod_{v \nmid \infty, m_v > 0} (1 + \mathfrak{p}_v^{m_v} \mathcal{O}_v)^\times \times \prod_{v \nmid \infty, m_v = 0} \mathcal{O}_v^\times \subseteq J_K.$$

Then essentially by definition of the topology of J_K, any open subgroup of J_K containing $K_\infty^{\times, 0}$ contains some $U_{\mathfrak{m}}$.

In our previous example, our moduli are all of the form

Definition ($\mathfrak{a}(\infty)$)**.** If $\mathfrak{a} \lhd \mathcal{O}_K$ is an ideal, we write $\mathfrak{a}(\infty)$ for the modulus with $m_v = v(\mathfrak{a})$ for all $v \nmid \infty$, and $m_v = 1$ for all $v \mid \infty$.

If $k = \mathbb{Q}$ and $\mathfrak{m} = (N)(\infty)$, then we simply get $U_{\mathfrak{m}} = \mathbb{R}_{>0}^\times \times (1 + N\hat{\mathbb{Z}})^\times$, and so

$$\frac{J_{\mathbb{Q}}}{\mathbb{Q}^\times U_{\mathfrak{m}}} = (\mathbb{Z}/n\mathbb{Z})^\times,$$

corresponding to the abelian extension $\mathbb{Q}(\zeta_N)$.

In general, we define

Definition (Ray class field)**.** If L/K is abelian with $\mathrm{Gal}(L/K) \cong J_K/K^\times U_\mathfrak{m}$ under the Artin map, we call L the *ray class field* of K modulo $\mathfrak{m}$.

Definition (Conductor)**.** If L corresponds to $U \subseteq J_K$, then $U \supseteq K^\times U_\mathfrak{m}$ for some $\mathfrak{m}$. The minimal such $\mathfrak{m}$ is the *conductor* of L/K.

1.4 Ideal-theoretic description of global class field theory

Originally, class field theory was discovered using ideals, and the ideal-theoretic formulation is at times more convenient.

Let $\mathfrak{m}$ be a modulus, and let S be the set of finite v such that $m_v > 0$. Let I_S be the group of fractional ideals prime to S. Consider

$$P_\mathfrak{m} = \{(x) \in I_S : x \equiv 1 \bmod \mathfrak{m}\}.$$

To be precise, we require that for all $v \in S$, we have $v(x-1) \geq m_v$, and for all infinite v real with $m_v > 0$, then $\tau(x) > 0$ for $\tau : K \to \mathbb{R}$ the corresponding to v. In other words, $x \in K^\times \cap U_\mathfrak{m}$.

Note that if $\mathfrak{m}$ is trivial, then $I_S/P_\mathfrak{m}$ is the ideal class group. Thus, it makes sense to define

Definition (Ray class group)**.** Let $\mathfrak{m}$ be a modulus. The *generalized ideal class group*, or *ray class group* modulo $\mathfrak{m}$ is

$$\mathrm{Cl}_\mathfrak{m}(K) = I_S/P_\mathfrak{m}.$$

One can show that this is always a finite group.

Proposition. There is a canonical isomorphism

$$\frac{J_K}{K^\times U_\mathfrak{m}} \overset{\sim}{\to} \mathrm{Cl}_\mathfrak{m}(K)$$

such that for $v \notin S \cup \Sigma_{K,\infty}$, the composition

$$K_v^\times \hookrightarrow J_K \to \mathrm{Cl}_\mathfrak{m}(K)$$

sends $x \mapsto \mathfrak{p}_v^{-v(x)}$.

Thus, in particular, the Galois group $\mathrm{Gal}(L/K)$ of the ray class field modulo $\mathfrak{m}$ is $\mathrm{Cl}_\mathfrak{m}(K)$. Concretely, if $\mathfrak{p} \notin S$ is an ideal, then $[\mathfrak{p}] \in \mathrm{Cl}_\mathfrak{m}(K)$ corresponds to $\sigma_\mathfrak{p} \in \mathrm{Gal}(L/K)$, the arithmetic Frobenius. This was Artin's original reciprocity law.

When $\mathfrak{m} = 0$, then this map is the inverse of the map given by content. However, in general, it is not simply (the inverse of) the prime-to-S content map, even for ideles whose content is prime to S. According to Frölich, this is the "*fundamental mistake of class field theory*".

Proof sketch. Let $J_K(S) \subseteq J_K$ be given by

$$J_K(S) = \prod_{v \notin S \cup \Sigma_{K,\infty}} K_v^\times.$$

Here we do have the inverse of the content map

$$\begin{aligned} c^{-1} : J_K(S) &\twoheadrightarrow I_S \\ (x_v) &\mapsto \prod \mathfrak{p}_v^{-v(x_v)} \end{aligned}$$

We want to extend it to an isomorphism. Observe that

$$J_K(S) \cap U_\mathfrak{m} = \prod_{v \notin S \cup \Sigma_{K,\infty}} \mathcal{O}_v^\times,$$

which is precisely the kernel of the map c^{-1}. So c^{-1} extends uniquely to a homomorphism

$$\frac{J_K(S)U_\mathfrak{m}}{U_\mathfrak{m}} \cong \frac{J_K(S)}{J_K(S) \cap U_\mathfrak{m}} \to I_S.$$

We then use that $K^\times J_K(S)U_\mathfrak{m} = J_K$ (weak approximation), and

$$K^\times \cap V_\mathfrak{m} = \{x \equiv 1 \bmod \mathfrak{m},\ x \in K^*\},$$

where

$$V_\mathfrak{m} = J_K(S)U_\mathfrak{m} = \{(x_v) \in J_K \mid \text{ for all } v \text{ with } m_v > 0,\, x_v \in U_\mathfrak{m}\}. \qquad \square$$

2 L-functions

Recall that Dirichlet characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ give rise to Dirichlet L-functions. Explicitly, we extend χ to a function on $\mathbb{Z}$ by setting $\chi(a) = 0$ if $[a] \notin (\mathbb{Z}/N\mathbb{Z})^\times$, and then the Dirichlet L-function is defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}.$$

As one can see from the n and p appearing, Dirichlet L-functions are "about $\mathbb{Q}$", and we would like to have a version of L-functions that are for arbitrary number fields K. If $\chi = 1$, we already know what this should be, namely the ζ-function

$$\zeta_K(s) = \sum_{\mathfrak{a} \triangleleft \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p} \triangleleft \mathcal{O}_K} \frac{1}{1 - N(\mathfrak{p})^{-s}},$$

where the first sum is over all ideals of $\mathcal{O}_K$, and the second product is over all *prime* ideals of $\mathcal{O}_K$.

In general, the replacement of Dirichlet characters is a *Hecke character*.

2.1 Hecke characters

Definition (Hecke character)**.** A *Hecke character* is a continuous (not necessarily unitary) homomorphism

$$\chi : J_K/K^\times \to \mathbb{C}^\times.$$

These are also known as *quasi-characters* in some places, where character means unitary. However, we shall adopt the convention that characters need not be unitary. The German term *Größencharakter* (or suitable variations) are also used.

In this section, we will seek to understand Hecke characters better, and see how Dirichlet characters arise as a special case where $K = \mathbb{Q}$. Doing so is useful if we want to write down actual Hecke characters. The theory of L-functions will be deferred to the next chapter.

We begin with the following result:

Proposition. Let G be a profinite group, and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ continuous. Then $\ker \rho$ is open.

Of course, the kernel is always closed.

Proof. It suffices to show that $\ker \rho$ contains an open subgroup. We use the fact that $\mathrm{GL}_n(\mathbb{C})$ has "no small subgroups", i.e. there is an open neighbourhood U of $1 \in \mathrm{GL}_n(\mathbb{C})$ such that U contains no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ (exercise!). For example, if $n = 1$, then we can take U to be the right half plane.

Then for such U, we know $\rho^{-1}(U)$ is open. So it contains an open subgroup V. Then $\rho(V)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ contained in U, hence is trivial. So $V \subseteq \ker(\rho)$. □

While the multiplicative group of a local field is not profinite, it is close enough, and we similarly have

Exercise. Let F be a local field. Then any continuous homomorphism $F^\times \to \mathbb{C}^\times$ has an open kernel, i.e. $\chi(1 + \mathfrak{p}_F^N) = 1$ for some $N \gg 0$.

Definition (Unramified character)**.** If F is a local field, a character $\chi : F \to \mathbb{C}^\times$ is *unramified* if

$$\chi|_{\mathcal{O}_F^\times} = 1.$$

If $F \cong \mathbb{R}$, we say $\chi : F^\times \to \mathbb{C}^\times$ is unramified if $\chi(-1) = 1$.

Using the decomposition $F \cong \mathcal{O}_F \times \langle \pi_F \rangle$ (for the local case), we see that

Proposition. χ is unramified iff $\chi(x) = |x|_F^s$ for some $s \in \mathbb{C}$. □

We now return to global fields. We will think of Hecke characters as continuous maps $J_K \to \mathbb{C}^\times$ that factor through $J_K/K^\times$, since it is easier to reason about J_K than the quotient. We can begin by discussing arbitrary characters $\chi : J_K \to \mathbb{C}^\times$.

Proposition. The set of continuous homomorphisms $\chi : J_K = \prod'_v K_v^\times \to \mathbb{C}^\times$ bijects with the set of all families $(\chi_v)_{v \in \Sigma_k}$, $\chi_v : K_v^\times \to \mathbb{C}^\times$ such that χ_v is unramified for almost all (i.e. all but finitely many) v, with the bijection given by $\chi \mapsto (\chi_v)$, $\chi_v = \chi|_{K_v^\times}$.

Proof. Let $\chi : J_K \to \mathbb{C}^\times$ be a character. Since $\hat{\mathcal{O}}_K \subseteq J_K$ is profinite, we know $\ker \chi|_{\hat{\mathcal{O}}_K^\times}$ is an open subgroup. Thus, it contains $\mathcal{O}_v^\times$ for all but finitely many v. So we have a map from the LHS to the RHS.

In the other direction, suppose we are given a family $(\chi_v)_v$. We attempt to define a character $\chi : J_K \to \mathbb{C}^\times$ by

$$\chi(x_v) = \prod \chi_v(x_v).$$

By assumption, $\chi_v(x_v) = 1$ for all but finitely many v. So this is well-defined. These two operations are clearly inverses to each other. □

In general, we can write χ as

$$\chi = \chi_\infty \chi^\infty, \quad \chi^\infty = \prod_{v \nmid \infty} \chi_v : K^{\times,\infty} \to \mathbb{C}^\times, \quad \chi_\infty = \prod_{v \mid \infty} \chi_v : K_\infty^\times \to \mathbb{C}^\times.$$

Lemma. Let χ be a Hecke character. Then the following are equivalent:

(i) χ has finite image.

(ii) $\chi_\infty(K_\infty^{\times,0}) = 1$.

(iii) $\chi_\infty^2 = 1$.

(iv) $\chi(C_K^0) = 1$.

(v) χ factors through $\mathrm{Cl}_\mathfrak{m}(K)$ for some modulus $\mathfrak{m}$.

In this case, we say χ is a *ray class character*.

Proof. Since $\chi_\infty(K_\infty^{\times,0})$ is either 1 or infinite, we know (i) ⇒ (ii). It is clear that (ii) ⇔ (iii), and these easily imply (iv). Since C_K/C_K^0 is profinite, if (iii) holds, then χ factors through C_K/C_K^0 and has open kernel, hence the kernel contains $U_\mathfrak{m}$ for some modulus $\mathfrak{m}$. So χ factors through $\mathrm{Cl}_\mathfrak{m}(K)$. Finally, since $\mathrm{Cl}_\mathfrak{m}(K)$ is finite, (v) ⇒ (i) is clear. □

Using this, we are able to classify all Hecke characters when $K = \mathbb{Q}$.

Example. The idele norm $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}^\times_{>0}$ is a character not of finite order. In the case $K = \mathbb{Q}$, we have $C_{\mathbb{Q}} = \mathbb{R}^\times_{>0} \times \hat{\mathbb{Z}}^\times$. The idele norm is then the projection onto $\mathbb{R}^\times_{>0}$.

Thus, if $\chi; C_{\mathbb{Q}} \to \mathbb{C}^\times$ is a Hecke character, then the restriction to $\mathbb{R}^\times_{>0}$ is of the form $x \mapsto x^s$ for some s. If we write

$$\chi(x) = |x|^s_{\mathbb{A}} \cdot \chi'(x)$$

for some χ', then χ' vanishes on $\mathbb{R}^\times_{>0}$, hence factors through $\hat{\mathbb{Z}}^\times$ and has finite order. Thus, it factors as

$$\chi' : C_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

for some N. In other words, it is a Dirichlet character.

For other fields, there can be more interesting Hecke characters.

For a general field K, we have finite order characters as we just saw. They correspond to characters on I_S which are trivial on $P_{\mathfrak{m}}$. In fact, we can describe all Hecke characters in terms of ideals.

There is an alternative way to think about Hecke characters. We can think of Dirichlet characters as (partial) functions $\mathbb{Z} \to \mathbb{C}^\times$ that satisfy certain multiplicativity properties. In general, a Hecke character can be thought of as a function on a set of ideals of $\mathcal{O}_K$.

Pick a modulus $\mathfrak{m}$ such that χ^∞ is trivial on $\hat{K}^\times \cap U_{\mathfrak{m}}$. Let S be the set of finite v such that $\mathfrak{m}_v$ is positive, and let I_S be the set of fractional ideals prime to S. We then define a homomorphism

$$\begin{aligned} \Theta : I_S &\to \mathbb{C}^\times \\ \mathfrak{p}_v &\mapsto \chi_v(\pi_v)^{-1} \end{aligned}$$

One would not expect Θ to remember much information about the infinite part χ_∞. However, once we know Θ and χ_∞ (and $\mathfrak{m}$), it is not difficult to see that we can recover all of χ.

On the other hand, an arbitrary pair (Θ, χ_∞) doesn't necessarily come from a Hecke character. Indeed, the fact that χ vanishes on $K^\times$ implies there is some compatibility condition between Θ and χ_∞.

Suppose $x \in K^\times$ is such that $x \equiv 1 \bmod \mathfrak{m}$. Then $(x) \in I_S$, and we have

$$1 = \chi(x) = \chi_\infty(x) \prod_{v \notin S \text{ finite}} \chi_v(x) = \chi_\infty(x) \prod_{\text{finite } v \notin S} \chi_v(\pi_v)^{v(x)}.$$

Writing $P_{\mathfrak{m}}$ for the set of principal ideals generated by these x, as we previously did, we see that for all $x \in P_{\mathfrak{m}}$,

$$\chi_\infty(x) = \Theta(x).$$

One can check that given (Θ, χ_∞) (and $\mathfrak{m}$) satisfying this compatibility condition, there is a unique Hecke character that gives rise to this pair. This was Hecke's original definition of a Hecke character, which is more similar to the definition of a Dirichlet character.

Example. Take $K = \mathbb{Q}(i)$, and fix an embedding into $\mathbb{C}$. Since $\mathrm{Cl}(K) = 1$, we have

$$C_K = \frac{\mathbb{C}^\times \times \hat{\mathcal{O}}_K^\times}{\mu_4 = \{\pm 1, \pm i\}}.$$

Let v_2 be the place over 2, corresponding to the prime $(1+i)\mathcal{O}_K$. Then K_{v_2} is a ramified extension over $\mathbb{Q}_2$ of degree 2. Moreover,

$$\left(\frac{\mathcal{O}_K}{(1+i)^3\mathcal{O}_K}\right)^\times = \mu_4 = \{\pm 1, \pm i\}.$$

So we have a decomposition

$$\mathcal{O}_{v_2}^\times = (1 + (1+i)^3\mathcal{O}_{v_2}) \times \mu_4.$$

Thus, there is a natural projection

$$C_K \twoheadrightarrow \frac{\mathbb{C}^\times \times \mathcal{O}_{v_2}^\times}{\mu_4} \cong \mathbb{C}^\times \times (1 + (1+i)^3\mathcal{O}_{v_2}) \twoheadrightarrow \mathbb{C}^\times.$$

This gives a Hecke character with $\chi_\infty(z) = z$, and is trivial on

$$\prod_{v \notin \{v_2, \infty\}} \mathcal{O}_v^\times \times (1 + (1+i)^3\mathcal{O}_{v_2}),$$

This has modulus

$$\mathfrak{m} = 3(v_2).$$

In ideal-theoretic terms, if $\mathfrak{p} \neq (1+i)$ is a prime ideal of K, then $\mathfrak{p} = (\pi_\mathfrak{p})$ for a unique $\pi_\mathfrak{p} \in \mathcal{O}_K$ with $\pi_\mathfrak{p} \equiv 1 \bmod (1+i)^3$. Then Θ sends $\mathfrak{p}$ to $\pi_\mathfrak{p}$.

This is an example of an algebraic Hecke character.

Definition (Algebraic homomorphism)**.** A homomorphism $K^\times \to \mathbb{C}^\times$ is *algebraic* if there exists integers $n(\sigma)$ (for all $\sigma : K \hookrightarrow \mathbb{C}$) such that

$$\varphi(x) = \prod \sigma(x)^{n(\sigma)}.$$

The first thing to note is that if φ is algebraic, then $\varphi(K^\times)$ is contained in the Galois closure of K in $\mathbb{C}$. In particular, it takes values in the number field. Another equivalent definition is that it is algebraic in the sense of algebraic geometry, i.e. if $K = \bigoplus \mathbb{Q}e_i$ for $i = 1, \ldots, n$ as a vector space, then we can view K as the $\mathbb{Q}$-points of an n-dimensional affine group scheme. We can then define $R_{K/\mathbb{Q}}\mathbb{G}_m \subseteq \mathbb{A}$ to be the set on which X is invertible, and then an algebraic Hecke character is a homomorphism of algebraic groups $(T_K)/\mathbb{C} \to \mathbb{G}_m/\mathbb{C}$, where $T_K = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$.

If we have a real place v of K, then this corresponds to a real embedding $\sigma_v : K \to K_v \cong \mathbb{R}$, and if v is a complex place, we have a pair of embedding $\sigma_v, \bar{\sigma}_v : K \hookrightarrow K_v \simeq \mathbb{C}$, picking one of the pair to be σ_v. So φ extends to a homomorphism

$$\varphi : K_\infty^\times \to \mathbb{C}^\times$$

given by

$$\varphi(x_v) = \prod_{v \text{ real}} x_v^{n(\sigma_v)} \prod_{v \text{ complex}} x_v^{n(\sigma_v)} \bar{x}_v^{n(\bar{\sigma}_v)}$$

Definition (Algebraic Hecke character)**.** A Hecke character $\chi = \chi_\infty \chi^\infty : J_K/K^\times \to \mathbb{C}^\times$ is *algebraic* if there exists an algebraic homomorphism $\varphi : K^\times \to \mathbb{C}^\times$ such that $\varphi(x) = \chi_\infty(x)$ for all $x \in K_\infty^{\times,0}$, i.e. $\chi_\infty = \varphi \prod_{v \text{ real}} \mathrm{sgn}_v^{e_v}$ for $e_v \in \{0, 1\}$.

We say φ (or the tuple $(n(\sigma))_\sigma$) is the *infinite type* of χ.

Example. The adelic norm $|\cdot|_{\mathbb{A}} : J_K \to \mathbb{C}^\times$ has

$$\chi_\infty = \prod |\cdot|_v,$$

and so χ is algebraic, and the associated φ is just $N_{K/\mathbb{Q}} : K^\times \to \mathbb{Q}^\times \subseteq \mathbb{C}^\times$, with $(n_\sigma) = (1, \ldots, 1)$.

Exercise. Let $K = \mathbb{Q}(i)$, and χ from the previous example, whose associated character of ideals was $\Theta : \mathfrak{p} \mapsto \pi_\mathfrak{p}$, where $\pi_\mathfrak{p} \equiv 1 \bmod (2+2i)$. The infinity type is the inclusion $K^\times \hookrightarrow \mathbb{C}^\times$, i.e. it has type $(1, 0)$.

Observe that the image of an algebraic homomorphism $\varphi : K^\times \to \mathbb{C}^\times$ lies in the normal closure of K. More generally,

Proposition. If χ is an algebraic Hecke character, then χ^∞ takes values in some number field. We write $E = E(\chi)$ for the smallest such field.

Of course, we cannot expect χ to take algebraic values, since J_K contains copies of $\mathbb{R}$ and $\mathbb{C}$.

Proof. Observe that $\chi^\infty(\hat{\mathcal{O}}_K^\times)$ is finite subgroup, so is μ_n for some n. Let $x \in K^\times$, totally positive. Then

$$\chi^\infty(x) = \chi_\infty(x)^{-1} = \varphi(x)^{-1} \in K^{cl},$$

where K^{cl} is the Galois closure. Then since $K_{>0}^\times \times \hat{\mathcal{O}}_K^\times \to \hat{K}^\times$ has finite cokernel (by the finiteness of the class group), so

$$\chi^\infty(\hat{K}^\times) = \coprod_{i=1}^d z_i \chi^\infty(K_{>0}^\times \hat{\mathcal{O}}_K^\times),$$

where $z_i^d \in \chi^\infty(K_{>0}^\times \hat{\mathcal{O}}_K^\times)$, and is therefore contained inside a finite extension of the image of $K_{>0}^\times \times \hat{\mathcal{O}}_K^\times$. $\square$

Hecke characters of finite order (i.e. algebraic Hecke characters with infinity type $(0, \ldots, 0)$) are in bijection with continuous homomorphisms $\Gamma_K \to \mathbb{C}^\times$, necessarily of finite order. What we show now is how to associate to a general algebraic Hecke character χ a continuous homomorphism $\psi_\ell : \Gamma_K \to E(\chi)_\lambda^\times \supseteq \mathbb{Q}_\ell^\times$, where λ is a place of $E(\chi)$ over ℓ. This is continuous for the ℓ-adic topology on E_λ. In general, this will not be of finite order. Thus, algebraic Hecke characters correspond to ℓ-adic Galois representations.

The construction works as follows: since $\chi^\infty(x) = \varphi(x)^{-1}$, we can restrict the infinity type φ to a homomorphism $\varphi : K^\times \to E^\times$. We define $\tilde{\chi} : J_K \to E^\times$ as follows: if $x = x_\infty x^\infty \in K_\infty^\times K^{\infty,\times} \in J_K$, then we set

$$\tilde{\chi}(x) = \chi(x)\varphi(x_\infty)^{-1}.$$

Notice that this is not trivial on $K^\times$ in general. Then $\tilde{\chi}_\infty$ takes values in $\{\pm 1\}$. Thus, $\tilde{\chi}$ takes values in $E^\times$. Thus, we know that $\tilde{\chi}$ has open kernel, i.e. it is continuous for the discrete topology on $E^\times$, and $\tilde{\chi}|_{K^\times} = \varphi^{-1}$.

Conversely, if $\tilde{\chi} : K^\times \to E^\times$ is a continuous homomorphism for the discrete topology on $E^\times$, and $\tilde{\chi}|_{K^\times}$ is an algebraic homomorphism, then it comes from an algebraic Hecke character in this way.

Let λ be a finite place of E over ℓ, a rational prime. Recall that $\varphi : K^\times \to E^\times$ is an algebraic homomorphism, i.e.

$$\varphi\left(\sum x_i e_i\right) = f(\mathbf{x}), \quad f \in E(X_1, \ldots, X_n).$$

We can extend this to $K_\ell^\times = (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times = \prod_{v \mid \ell} K_v^\times$ to get a homomorphism

$$\varphi_\lambda : K_\ell^\times \to E_\lambda^\times$$

This is still algebraic, so it is certainly continuous for the ℓ-adic topology.

Now consider the character $\psi_\lambda : J_K \to E_\lambda^\times$, where now

$$\psi_\lambda((x_v)) = \tilde{\chi}(x)\varphi_\lambda((x_v)_{v \mid \ell}).$$

This is then continuous for the ℓ-adic topology on $E_\lambda^\times$, and moreover, we see that $\psi_\lambda(K^\times) = \{1\}$ as $\tilde{\chi}|_{K^\times} = \varphi^{-1}$ while $\varphi_\lambda|_{K^\times} = \varphi$. Since $\tilde{\chi}(K_\infty^{\times,0}) = \{1\}$, we know that ψ_λ it is in fact defined on $C_K/C_K^0 \cong \Gamma_K^{ab}$.

Obviously, ψ_λ determines $\tilde{\chi}$ and hence χ.

Fact. An ℓ-adic character $\psi : C_K/C_K^0 \to E_\lambda^\times$ comes from an algebraic Hecke character in this way if and only if the associated Galois representation is *Hodge–Tate*, which is a condition on the restriction to the decomposition groups $\mathrm{Gal}(\bar{K}_v/K_v)$ for the primes $v \mid \ell$.

Example. Let $K = \mathbb{Q}$ and $\chi = |\cdot|_{\mathbb{A}}$, then

$$\tilde{\chi} = \mathrm{sgn}(x_\infty) \prod_p |x_p|_p.$$

So

$$\psi_\ell((x_v)) = \mathrm{sgn}(x_\infty) \prod_{p \neq \ell} |x_p|_p \cdot |x_\ell|_\ell \cdot x_\ell.$$

Note that $|x_\ell|_\ell x_\ell \in \mathbb{Z}_\ell^\times$. We have

$$C_{\mathbb{Q}}/C_{\mathbb{Q}}^0 \cong \hat{\mathbb{Z}}^\times.$$

Under this isomorphism, the map $\hat{\mathbb{Z}}^\times \mathbb{Q}_\ell^\times$ is just the projection onto $\mathbb{Z}_\ell^\times$ followed by the inclusion, and by class field theory, $\psi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ is just the cyclotomic character of the field $\mathbb{Q}(\{\zeta_{\ell^n}\})$,

$$\sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\psi_\ell(\sigma) \bmod \ell^n}.$$

Example. Consider the elliptic curve $y^2 = x^3 - x$ with complex multiplication over $\mathbb{Q}(i)$. In other words, $\mathrm{End}(E/\mathbb{Q}(i)) = \mathbb{Z}[i]$, where we let i act by

$$i \cdot (x, y) \mapsto (-x, iy).$$

Its Tate module

$$T_\ell E = \lim \mathbb{E}[\ell^n]$$

is a $\mathbb{Z}_\ell[i]$-module. If $\lambda \mid \ell$, then we define

$$V_\lambda E = T_\ell E \otimes_{\mathbb{Z}_\ell[i]} K_\lambda.$$

Then Γ_K act by $\Gamma_K : \mathrm{Aut}_{K_\lambda} V_\lambda E = K_\lambda^\times$.

We now want to study the infinity types of an algebraic Hecke character.

Lemma. Let K be a number field, $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ be an algebraic homomorphism, and suppose $E/\mathbb{Q}$ is Galois. Then φ factors as

$$K^\times \overset{\text{norm}}{\longrightarrow} (K \cap E)^\times \overset{\phi'}{\longrightarrow} E^\times.$$

Note that since E is Galois, the intersection $K \cap E$ makes perfect sense.

Proof. By definition, we can write

$$\varphi(x) = \prod_{\sigma : K \hookrightarrow \mathbb{C}} \sigma(x)^{n(\sigma)}.$$

Then since $\varphi(x) \in E$, for all $x \in K^\times$ and $\tau \in \Gamma_E$, we have

$$\prod \tau\sigma(x)^{n(\sigma)} = \prod \sigma(x)^{n(\sigma)}.$$

In other words, we have

$$\prod_\sigma \sigma(x)^{n(\tau^{-1}\sigma)} = \prod_\sigma \sigma(x)^{n(\sigma)}.$$

Since the homomorphisms σ are independent, we must have $n(\tau\sigma) = n(\sigma)$ for all embeddings $\sigma : K \hookrightarrow \bar{\mathbb{Q}}$ and $\tau \in \Gamma_E$. This implies the theorem. □

Recall that if $\mathfrak{m}$ is a modulus, then we defined open subgroups $U_\mathfrak{m} \subseteq J_K$, consisting of the elements (x_v) such that if a real $v \mid \mathfrak{m}$, then $x_v > 0$, and if $v \mid \mathfrak{m}$ for a finite v, then $v(x_v - 1) \geq m_v$. We can write this as

$$U_\mathfrak{m} = U_{\mathfrak{m},\infty} \times U_\mathfrak{m}^\infty.$$

Proposition. Let $\varphi : K^\times \to \mathbb{C}^\times$ be an algebraic homomorphism. Then φ is the infinity type of an algebraic Hecke character χ iff $\varphi(\mathcal{O}_K^\times)$ is finite.

Proof. To prove the ($\Rightarrow$) direction, suppose $\chi = \chi_\infty\chi^\infty$ is an algebraic Hecke character with infinity type φ. Then $\chi^\infty(U_\mathfrak{m}^\infty) = 1$ for some $\mathfrak{m}$. Let $E_\mathfrak{m} = K^\times \cap U_\mathfrak{m} \subseteq \mathcal{O}_K^\times$, a subgroup of finite index. As $\chi^\infty(E_\mathfrak{m}) = 1 = \chi(E_\mathfrak{m})$, we know $\chi_\infty(E_\mathfrak{m}) = 1$. So $\varphi(\mathcal{O}_K^\times)$ is finite.

To prove ($\Leftarrow$), given φ with $\varphi(\mathcal{O}_K^\times)$ finite, we can find some $\mathfrak{m}$ such that $\varphi(E_\mathfrak{m}) = 1$. Then $(\varphi, 1) : K_\infty^\times \times U_\mathfrak{m}^\infty \to \mathbb{C}^\times$ is trivial on $E_\mathfrak{m}$. So we can extend this to a homomorphism

$$\frac{K_\infty^\times U_\mathfrak{m} K^\times}{K^\times} \cong \frac{K_\infty^\times U_\mathfrak{m}}{E_\mathfrak{m}} \to \mathbb{C}^\times,$$

since $E_\mathfrak{m} = K^\times \cap U_\mathfrak{m}$. But the LHS is a finite index subgroup of C_K. So the map extends to some χ. □

Here are some non-standard terminology:

Definition (Serre type)**.** A homomorphism $\varphi : K^\times \to \mathbb{C}^\times$ is of *Serre type* if it is algebraic and $\varphi(\mathcal{O}_K^\times)$ is finite.

These are precisely homomorphisms that occur as infinity types of algebraic Hecke characters.

Note that the unit theorem implies that

$$\mathcal{O}_K^\times \hookrightarrow K_\infty^{\times,1} = \{x \in K_\infty^\times : |x|_{\mathbb{A}} = 1\}$$

has compact cokernel. If $\varphi(\mathcal{O}_K^\times)$ is finite, then $\varphi(K_\infty^{\times,1})$ is compact. So it maps into $\mathrm{U}(1)$.

Example. Suppose K is totally real. Then

$$K_\infty^\times = (\mathbb{R}^\times)^{\{\sigma : K \hookrightarrow \mathbb{R}\}}.$$

Then we have

$$K_\infty^{\times,1} = \{(x_\sigma) : \prod x_\sigma = \pm 1\}.$$

Then $\varphi((x_\sigma)) = \prod x_\sigma^{n(\sigma)}$, so $|\varphi(K_\infty^{\times,1})| = 1$. In other words, all the n_σ are equal. Thus, φ is just a power of the norm map.

Thus, algebraic Hecke characters are all of the form

$$|\cdot|_{\mathbb{A}}^m \cdot (\text{finite order character}).$$

Another class of examples comes from CM fields.

Definition (CM field)**.** K is a CM field if K is a totally complex quadratic extension of a totally real number field K^+.

This CM refers to *complex multiplication.*

This is a rather restrictive condition, since this implies $\mathrm{Gal}(K/K^+) = \{1, c\} = \mathrm{Gal}(K_w/K_v^+)$ for every $w \mid v \mid \infty$. So c is equal to complex conjugation for *every* embedding $K \hookrightarrow \mathbb{C}$.

From this, it is easy to see that CM fields are all contained in $\mathbb{Q}^{\mathrm{CM}} \subseteq \bar{\mathbb{Q}} \subseteq \mathbb{C}$, given by the fixed field of the subgroup

$$\langle c\sigma c\sigma^{-1} : \sigma \in \Gamma_{\mathbb{Q}} \rangle \subseteq \Gamma_{\mathbb{Q}}.$$

For example, we see that the compositum of two CM fields is another CM field.

Exercise. Let K be a totally complex S_3-extension over $\mathbb{Q}$. Then K is not CM, but the quadratic subfields is complex and is equal to $K \cap \mathbb{Q}^{\mathrm{CM}}$.

Example. Let K be a CM field of degree $2r$. Then Dirichlet's unit theorem tells us

$$\operatorname{rk} \mathcal{O}_K^\times = r - 1 = \operatorname{rk} \mathcal{O}_{K^+}^\times.$$

So $\mathcal{O}_K^\times$ is a finite index subgroup of $\mathcal{O}_{K^+}^\times$. So $\varphi : K^\times \to \mathbb{C}^\times$ is of Serre type iff it is algebraic and its restriction to $K^{+,\times}$ is of Serre type. In other words, we need $n(\sigma) + n(\bar{\sigma})$ to be independent of σ.

Theorem. Suppose K is arbitrary, and $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ is algebraic, and we assume $E/\mathbb{Q}$ is Galois, containing the normal closure of K. Thus, we can write

$$\varphi(x) = \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{n(\sigma)}.$$

Then the following are equivalent:

(i) φ is of Serre type.

(ii) $\varphi = \psi \circ N_{K/F}$, where F is the maximal CM subfield and ψ is of Serre type.

(iii) For all $c' \in \mathrm{Gal}(E/\mathbb{Q})$ conjugate to complex conjugation c, the map $\sigma \mapsto n(\sigma) + n(c'\sigma)$ is constant.

(iv) (in the case $K \subseteq \mathbb{C}$ and $K/\mathbb{Q}$ is Galois with Galois group G) Let $\lambda = \sum n(\sigma)\sigma \in \mathbb{Z}[G]$. Then for all $\tau \in G$, we have

$$(\tau - 1)(c + 1)\lambda = 0 = (c + 1)(\tau - 1)\lambda.$$

Note that in (iii), the constant is necessarily

$$\frac{2}{[K : \mathbb{Q}]} \sum_\sigma n(\sigma).$$

So in particular, it is independent of c'.

Proof.

- (iii) $\Leftrightarrow$ (iv): This is just some formal symbol manipulation.
- (ii) $\Rightarrow$ (i): The norm takes units to units.
- (i) $\Rightarrow$ (iii): By the previous lecture, we know that if φ is of Serre type, then

  $$|\varphi(K_\infty^{\times,1})| = 1.$$

  Now if $(x_v) \in K_\infty^\times$, we have

  $$|\varphi((x_v))| = \prod_{\text{real } v} |x_v|^{n(\sigma_v)} \prod_{\text{complex } v} |x_v|^{n(\sigma_v) + n(\bar{\sigma}_v)} = \prod_v |x_v|_v^{\frac{1}{2}(n(\sigma_v) + n(\bar{\sigma}_v))}.$$

  Here the modulus without the subscript is the usual modulus. Then $|\varphi(K_\infty^{\times,1})| = 1$ implies $n(\sigma_v) + n(\bar{\sigma}_v)$ is constant. In other words, $n(\sigma) + n(c\sigma) = m$ is constant.

  But if $\tau \in \mathrm{Gal}(E/\mathbb{Q})$, and $\varphi' = \tau \circ \varphi$, $n'(\sigma) = n(\tau^{-1}\sigma)$, then this is also of Serre type. So

  $$m = n'(\sigma) + n'(c\sigma) = n(\tau^{-1}\sigma) + n(\tau^{-1}c\sigma) = n(\tau^{-1}\sigma) + n((\tau^{-1}c\tau)\tau^{-1}\sigma).$$

- (iii) $\Rightarrow$ (ii): Suppose $n(\sigma) + n(c'\sigma) = m$ for all σ and all $c' = \tau c \tau^{-1}$. Then we must have

  $$n(c'\sigma) = n(c\sigma)$$

for all σ. So
$$n(\sigma) = n(c\tau c\tau^{-1}\sigma)$$
So n is invariant under $H = [c, \mathrm{Gal}(E/\mathbb{Q})] \leq \mathrm{Gal}(E/\mathbb{Q})$, noting that c has order 2. So φ takes values in the fixed field $E^H = E \cap \mathbb{Q}^{\mathrm{CM}}$. By the proposition last time, this implies φ factors through $N_{K/F}$, where $F = E^H \cap K = K \cap \mathbb{Q}^{\mathrm{CM}}$. $\square$

Recall that a homomorphism $\varphi : K^\times \to \mathbb{C}^\times$ is algebraic iff it is a character of the commutative algebraic group $T_K = R_{K/\mathbb{Q}}\mathbb{G}_m$, so that $T_K(\mathbb{Q}) = K^\times$, i.e. there is an algebraic character $\varphi' : T_K/\mathbb{C} \to \mathbb{G}_m/\mathbb{C}$ such that φ' restricted to $T_K(\mathbb{Q})$ is φ.

Then φ is of Serre type iff φ is a character of ${}^K S^0 = T_K/\mathcal{E}_K^0$, where $\mathcal{E}_K$ is the Zariski closure of $\mathcal{O}_K^\times$ in T_K and $\mathcal{E}_K^0$ is the identity component, which is the same as the Zariski closure of $\Delta \subseteq \mathcal{O}_K^\times$, where Δ is a sufficiently small finite-index subgroup.

The group ${}^K S^0$ is called the *connected Serre group*. We have a commutative diagram (with exact rows)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K^\times & \longrightarrow & J_K & \longrightarrow & J_K/K^\times & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_0} & & \downarrow \\
1 & \longrightarrow & {}^K S^0 & \longrightarrow & {}^K S & \longrightarrow & \Gamma_K^{ab} & \longrightarrow & 1
\end{array}$$

This ${}^K S$ is a projective limit of algebraic groups over $\mathbb{Q}$. We have
$$\mathrm{Hom}({}^K S, \mathbb{C}^\times) = \mathrm{Hom}({}^K S, \mathbb{G}_m/\mathbb{C}) = \{\text{algebraic Hecke characters of } K\}$$
The infinity type is just the restriction to ${}^K S^0$.

Langlands created a larger group, the *Tamiyama group*, an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by ${}^K S^0$, which is useful for abelian varieties with CM and conjugations and Shimura varieties.

2.2 Abelian L-functions

We are now going to define L-functions for Hecke characters. Recall that amongst all other things, an L-function is a function in a complex variable. Here we are going to do things slightly differently. For any Hecke character χ, we will define $L(\chi)$, which will be a *number*. We then define
$$L(\chi, s) = L(|\cdot|_{\mathbb{A}}^s \chi).$$
We shall define $L(\chi)$ as an Euler product, and then later show it can be written as a sum.

Definition (Hecke L-function)**.** Let $\chi : C_K \to \mathbb{C}^\times$ be a Hecke character. For $v \in \Sigma_K$, we define local L-factors $L(\chi_v)$ as follows:

- If v is non-Archimedean and χ_v unramified, i.e. $\chi_v|_{\mathcal{O}_{K_v}^\times} = 1$, we set
$$L(\chi_v) = \frac{1}{1 - \chi_v(\pi_v)}.$$

– If v is non-Archimedean and χ_v is ramified, then we set
$$L(\chi_v) = 1.$$

– If v is a real place, then χ_v is of the form
$$\chi_v(x) = x^{-N}|x|_v^s,$$
where $N = 0, 1$. We write
$$L(\chi_v) = \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

– If v is a complex place, then χ_v is of the form
$$\chi_v(x) = \sigma(x)^{-N}|x|_v^s,$$
where σ is an embedding of K_v into $\mathbb{C}$ and $N \geq 0$. Then
$$L(\chi_v) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

We then define
$$L(\chi_v, s) = L(\chi_v \cdot |\cdot|_v^s).$$
So for finite unramified v, we have
$$L(\chi_v, s) = \frac{1}{1 - \chi_v(\pi_v)q_v^{-s}},$$
where $q_v = |\mathcal{O}_{K_v}/(\pi_v)|$.

Finally, we define
$$L(\chi, s) = \prod_{v \nmid \infty} L(\chi_v, s)$$
$$\Lambda(\chi, s) = \prod_{v} L(\chi_v, s).$$

Recall that the kernel of the idelic norm $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}_{>0}^\times$ is compact. It is then not hard to see that for every χ, there is some $t \in \mathbb{R}$ such that $\chi \cdot |\cdot|_{\mathbb{A}}^t$ is unitary. Thus, $L(\chi, s)$ converges absolutely on some right half-plane. Observe that
$$\Lambda(\chi|\cdot|_{\mathbb{A}}^t, s) = \Lambda(\chi, t + s).$$

Theorem (Hecke–Tate)**.**

(i) $\Lambda(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$, entire unless $\chi = |\cdot|_{\mathbb{A}}^t$ for some $t \in \mathbb{C}$, in which case there are simple poles at $s = 1 - t, -t$.

(ii) There is some function, the *global ε-factor*,
$$\varepsilon(\chi, s) = AB^s$$
for some $A \in \mathbb{C}^\times$ and $B \in \mathbb{R}_{>0}$ such that
$$\Lambda(\chi, s) = \varepsilon(\chi, s)\Lambda(\chi^{-1}, 1 - s).$$

(iii) There is a factorization
$$\varepsilon(\chi, s) = \prod_v \varepsilon_v(\chi_v, \mu_v, \psi_v, s),$$
where $\varepsilon_v = 1$ for almost all v, and ε_v depends only on χ_v and certain auxiliary data ψ_v, μ_v. These are the *local ε-factors*.

Traditionally, we write
$$L(\chi, s) = \prod_{\text{finite } v} L(\chi_v, s),$$
and then
$$\Lambda(\chi, s) = L(\chi, s) L_\infty(\chi, s).$$
However, Tate (and others, especially the automorphic people) use $L(\chi, s)$ for the product over all v.

At first, Hecke proved (i) and (ii) using global methods, using certain Θ functions. Later, Tate proved (i) to (iii) using local-global methods and especially Fourier analysis on K_v and $\mathbb{A}_K$. This generalizes considerably, e.g. to automorphic representations.

We can explain some ideas of Hecke's method. We have a decomposition
$$K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{C}^n,$$
and this has a norm $\|\cdot\|$ induces by the Euclidean metric on $\mathbb{R}^n$. Let $\Delta \subseteq \mathcal{O}_{K,+}^\times$ be a subgroup of totally positive units of finite index, which is $\cong \mathbb{Z}^{r_1+r_2-1}$. This has an embedding $\Delta = K_\infty^{\times,1}$, which extends to a continuous homomorphism $\Delta \otimes \mathbb{R} \to K_\infty^{\times,1}$. The key fact is

Proposition. Let $x \in K^\times$. Pick some invariant measure $\mathrm{d}u$ on $\Delta \otimes \mathbb{R}$. Then
$$\int_{\Delta \otimes \mathbb{R}} \frac{1}{\|ux\|^{2s}} \,\mathrm{d}u = \frac{\text{stuff}}{|N_{K/\mathbb{Q}}(x)|^{2s/n}},$$
where the stuff is some ratio of Γ factors and powers of π (and depends on s).

Exercise. Prove this when $K = \mathbb{Q}[\sqrt{d}]$ for $d > 0$. Then $\Delta = \langle \varepsilon \rangle$, and then
$$\begin{aligned} \text{LHS} &= \int_{-\infty}^{\infty} \frac{1}{|\varepsilon^t x + \varepsilon^{-t} x'|^{2s}} \,\mathrm{d}t \\ \text{RHS} &= \frac{\text{stuff}}{|xx'|^s}. \end{aligned}$$

The consequence of this is that if $\mathfrak{a} \subseteq K$ is a fractional ideal, then
$$\begin{aligned} \sum_{0 \neq x \in \mathfrak{a} \bmod \Delta} \frac{1}{|N_{K/\mathbb{Q}}(x)|^s} &= \text{stuff} \cdot \sum_{0 \neq x \in \mathfrak{a} \bmod \Delta} \int_{\Delta \otimes \mathbb{R}} \frac{1}{\|ux\|^{ns} \,\mathrm{d}u} \\ &= \text{stuff} \cdot \int_{\Delta \otimes \mathbb{R}/\Delta} \left(\sum_{0 \neq x \in \mathfrak{a}} \frac{1}{\|ux\|^{ns}} \right) \mathrm{d}u \end{aligned}$$

The integrand has a name, and is called the *Epstein ζ-function* of the lattice $(\mathfrak{a}, \|u \cdot\|^2)$. By the Poisson summation formula, we get an analytic continuity and functional equation for the epsilon ζ function. On the other hand, taking linear combinations of the left gives $L(\chi, s)$ for $\chi : \mathrm{Cl}(K) \to \mathbb{C}^\times$. For more general χ, we modify this with some extra factors. When the infinity type is non-trivial, this is actually quite subtle.

Note that if χ is unramified outside S and ramified at S, recall we had a homomorphism $\Theta : I_S \to \mathbb{C}^\times$ sending $\mathfrak{p}_v \mapsto \chi_v(\pi_v)^{-1}$. So

$$L(\chi, s) = \prod_{\text{finite } v \notin S} \left(\frac{1}{1 - \Theta(\mathfrak{p}_v)^{-1}(N\mathfrak{p}_v)^{-s}}\right) = \sum_{\mathfrak{a} \in \mathcal{O}_K \text{ prime to } S} \frac{\Theta(\mathfrak{a})^{-1}}{(N\mathfrak{a})^s}.$$

This was Hecke's original definition of the Hecke character.

If $K = \mathbb{Q}$ and $\chi : C_{\mathbb{Q}} \to \mathbb{C}^\times$ is of finite order, then it factors through $C_{\mathbb{Q}} \to C_{\mathbb{Q}}/C_{\mathbb{Q}}^0 \cong \hat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$, and so χ is just some Dirichlet character $\varphi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$. The associated L-functions are just Dirichlet L-functions. Indeed, if $p \nmid N$, then

$$\chi_p(p) = \chi(1, \ldots, 1, p, 1, \ldots) = \chi(p^{-1}, \ldots, p^{-1}, 1, p^{-1}, \ldots) = \varphi(p \bmod N)^{-1}.$$

In other words, $L(\chi, s)$ is the Dirichlet L-series of φ^{-1} (assuming N is chosen so that χ ramifies exactly at $v \mid N$).

Tate's method uses local ε-factors $\varepsilon(\chi_v, \mu_v, \psi_v, s)$, where $\psi_v : K_v \to \mathrm{U}(1)$ is a non-trivial additive character, e.g. for v finite,

$$K_V \xrightarrow{\mathrm{tr}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[v_p]/\mathbb{Z} \xhookrightarrow{e^{2\pi i x}} \mathbb{C}^\times,$$

which we needed because Fourier transforms take in additive measures, and μ_v is a Haar measure on K_v. The condition for (iii) to hold is

$$\prod \psi_v : \mathbb{A}_K \to \mathrm{U}(1)$$

is well-defined and trivial on $K \subseteq \mathbb{A}_K$, and $\mu_{\mathbb{A}} = \prod \mu_v$ is a well-defined measure on $\mathbb{A}_K$, i.e. $\mu_v(\mathcal{O}_v) = 1$ for all v and

$$\int_{\mathbb{A}_K/K} \mu_{\mathbb{A}} = 1.$$

There exists explicit formulae for these ε_v's. If χ_v is unramified, then it is just $A_v B_v^s$, and is usually 1; for ramified finite v, they are given by Gauss sums.

2.3 Non-abelian L-functions

Let K be a number field. Then we have a reciprocity isomorphism

$$\mathrm{Art}_K : C_K/C_K^0 \xrightarrow{\sim} \Gamma_K^{\mathrm{ab}}.$$

If $\chi : C_K \to C_K^0 \to \mathbb{C}^\times$ is a Hecke character of finite order, then we can view it as a map $\psi = \chi \circ \mathrm{Art}_K^{-1} : \Gamma_K \to \mathbb{C}^\times$. Then

$$L(\chi, s) = \prod_{\text{finite } v \text{ unramified}} \frac{1}{1 - \chi_v(\pi_v) q_v^{-s}}^{-1} = \prod \frac{1}{1 - \psi(\mathrm{Frob}_v) q_v^{-s}},$$

where $\mathrm{Frob}_v \in \Gamma_{K_v}/I_{K_v}$ is the geometric Frobenius, using that $\psi(I_{K_v}) = 1$. Artin generalized this to arbitrary complex representations of Γ_K.

Let $\rho : \Gamma_K \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Define

$$L(\rho, s) = \prod_{\text{finite } v} L(\rho_v, s),$$

where ρ_v is the restriction to the decomposition group at v, and depends only on the isomorphism class of ρ. We first define these local factors for non-Archimedean fields:

Definition. Let F be local and non-Archimedean. Let $\rho : W_F \to \mathrm{GL}_{\mathbb{C}}(V)$ be a representation. Then we define

$$L(\rho, s) = \det(1 - q^{-s}\rho(\mathrm{Frob}_F)|_{V^{I_F}})^{-1},$$

where V^{I_F} is the invariants under I_F.

Note that in this section, all representations will be finite-dimensional and continuous for the complex topology (so in the case of W_F, we require $\ker \sigma$ to be open).

Proposition.

(i) If
$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V') \to 0$$
is exact, then
$$L(\rho, s) = L(\rho', s) \cdot L(\rho'', s).$$

(ii) If E/F is finite separable, $\rho : W_E \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_{W_E}^{W_F} \rho : W_F \to \mathrm{GL}_{\mathbb{C}}(U)$, then
$$L(\rho, s) = L(\sigma, s).$$

Proof.

(i) Since ρ has open kernel, we know $\rho(I_F)$ is *finite*. So
$$0 \to (V')^{I_F} \to V^{I_F} \to (V')^{I_F} \to 0$$
is exact. Then the result follows from the multiplicativity of det.

(ii) We can write
$$U = \{\varphi : W_F \to V : \varphi(gx) = \rho(g)\varphi(x) \text{ for all } g \in W_E, x \in W_F\}.$$
where W_F acts by
$$\sigma(g)\varphi(x) = \varphi(xg).$$
Then we have
$$U^{I_F} = \{\varphi : W_F/I_F \to V : \cdots\}.$$
Then whenever $\varphi \in U^{I_F}$ and $g \in I_E$, then
$$\sigma(g)\varphi(x) = \varphi(xg) = \varphi((xgx^{-1})x) = \varphi(x).$$

So in fact φ takes values in V^{I_E}. Therefore

$$U^{I_F} = \mathrm{Ind}_{W_E/I_E}^{W_F/I_F} V^{I_E}.$$

Of course, $W_F/I_F \cong \mathbb{Z}$, which contains W_E/I_E as a subgroup. Moreover,

$$\mathrm{Frob}_F^d = \mathrm{Frob}_E,$$

where $d = [k_E : k_F]$. We note the following lemma:

Lemma. Let $G = \langle g \rangle \supseteq H = \langle h = g^d \rangle$, $\rho : H \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_H^G \rho$. Then

$$\det(1 - t^d \rho(h)) = \det(1 - t\sigma(g)).$$

Proof. Both sides are multiplicative for exact sequences of representations of H. So we can reduce to the case of $\dim V = 1$, where $\rho(h) = \lambda \in \mathbb{C}^\times$. We then check it explicitly. □

To complete the proof of (ii), take $g = \mathrm{Frob}_F$ and $t = q_F^{-s}$ so that $t^d = q_E^{-s}$. □

For Archimedean F, we define $L(\rho, s)$ in such a way to ensure that (i) and (ii) hold, and if $\dim V = 1$, then

$$L(\rho, s) = L(\chi, s),$$

where if $\rho : W_F^{\mathrm{ab}} \to \mathbb{C}^\times$, then χ Is the corresponding character of $F^\times$ under the Artin map.

If $F \simeq \mathbb{C}$, then this is rather easy, since every irreducible representation of $W_F \cong \mathbb{C}^\times$ is one-dimensional. We then just define for ρ 1-dimensional using $W_F^{\mathrm{ab}} \cong F^\times$ and extend to all ρ by (i). The Jordan–Hölder theorem tells us this is well-defined.

If $F \simeq \mathbb{R}$, then recall that

$$W_{\mathbb{R}} = \langle \mathbb{C}^\times, s : s^2 = -1 \in \mathbb{C}^\times, szs^{-1} = \bar{z} \rangle.$$

Contained in here is $W_{\mathbb{R}}^{(1)} = \langle \mathrm{U}(1), s \rangle$. Then

$$W_{\mathbb{R}} = W_{\mathbb{R}}^{(1)} \times \mathbb{R}_{>0}^\times.$$

It is then easy to see that the irreducible representations of $W_{\mathbb{R}}$ are

(i) 1-dimensional $\rho_{W_{\mathbb{R}}}$; or

(ii) 2-dimensional, $\sigma = \mathrm{Ind}_{\mathbb{C}}^{W_{\mathbb{R}}} \rho$, where $\rho \neq \rho^s : \mathbb{C}^\times \to \mathbb{C}^\times$.

In the first case, we define

$$L(\rho, s) = L(\chi, s)$$

using the Artin map, and in the second case, we define

$$L(\sigma, s) = L(\rho, s)$$

using (ii).

To see that the properties are satisfied, note that (i) is true by construction, and there is only one case to check for (ii), which is if $\rho = \rho^s$, i.e.

$$\rho(z) = (z\bar{z})^t.$$

Then $\operatorname{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}} \rho$ is reducible, and is a sum of characters of $W_\mathbb{R}^{\mathrm{ab}} \cong \mathbb{R}^\times$, namely $x \mapsto |x|^t$ and $x \mapsto \operatorname{sgn}(x)|x|^t = x^{-1}|x|^{t+1}$. Then (ii) follows from the identity

$$\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1) = \Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Now let K be global, and let $\rho : \Gamma_K \to \mathrm{GL}_\mathbb{C}(V)$. For each $v \in \Sigma_K$, choose $\bar{k}$ of $\bar{K}$ over v. Let $\Gamma_v \cong \Gamma_{K_v}$ be the decomposition group at $\bar{v}$. These contain I_V, and we have the geometric Frobenius $\mathrm{Frob}_v \in \Gamma_v/I_V$. We define $\rho_v = \rho|_{\Gamma_v}$, and then set

$$\begin{aligned} L(\rho, s) &= \prod_{v \nmid \infty} L(\rho_v, s) = \prod_{v \nmid \infty} \det(1 - q_v^{-s}\mathrm{Frob}_v|_{V^{I_v}})^{-1} \\ \Lambda(\rho, s) &= LL_\infty \\ L_\infty &= \prod_{v \mid \infty} L(\rho_v, s). \end{aligned}$$

This is well-defined as the decomposition groups $\bar{v} \mid v$ are conjugate. If $\dim V = 1$, then $\rho = \chi \circ \mathrm{Art}_K^{-1}$ for a finite-order Hecke character χ, and then

$$L(\rho, s) = L(\chi, s).$$

The facts we had for local factors extend to global statements

Proposition.

(i) $L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s)$.

(ii) If L/K is finite separable and $\rho : \Gamma_L \to \mathrm{GL}_\mathbb{C}(V)$ and $\sigma = \operatorname{Ind}_{\Gamma_L}^{\Gamma_K}(\rho)$, then

$$L(\rho, s) = L(\sigma, s).$$

The same are true for $\Lambda(\rho, s)$.

Proof. (i) is clear. For (ii), we saw that if $w \in \Sigma_L$ over $v \in \Sigma_K$ and consider the local extension L_w/K_v, then

$$L(\rho_w, s) = L(\operatorname{Ind}_{\Gamma_{L_w}}^{\Gamma_{K_v}} \rho_w).$$

In the global world, we have to take care of the splitting of primes. This boils down to the fact that

$$\left(\operatorname{Ind}_{\Gamma_L}^{\Gamma_K} \rho\right)\Big|_{\Gamma_{K_v}} = \bigoplus_{w \mid v} \operatorname{Ind}_{\Gamma_{L_w}}^{\Gamma_{K_v}}(\rho|_{\Gamma_{L_w}}). \tag{$*$}$$

We fix a valuation $\bar{v}$ of $\bar{K}$ over v. Write $\Gamma_{\bar{v}/v}$ for the decomposition group in Γ_K. Write $\bar{S}$ for the places of $\bar{K}$ over v, and S the places of L over v.

The Galois group acts transitively on $\bar{S}$, and we have

$$\bar{S} \cong \Gamma_K/\Gamma_{\bar{v}/v}.$$

We then have
$$S \cong \Gamma_L \backslash \Gamma_K / \Gamma_{\bar{v}/v},$$
which is compatible with the obvious map $\bar{S} \to S$.

For $\bar{w} = g\bar{v}$, we have
$$\Gamma_{\bar{w}/v} = g\Gamma_{\bar{v}/v} g^{-1}.$$
Conjugating by g^{-1}, we can identify this with $\Gamma_{\bar{v}/v}$. Similarly, if $w = \bar{w}|_L$, then this contains
$$\Gamma_{\bar{w}/w} = g\Gamma_{\bar{v}/v} g^{-1} \cap \Gamma_L,$$
and we can identify this with $\Gamma_{\bar{v}/v} \cap g^{-1}\Gamma_L g$.

There is a theorem, usually called Mackey's formula, which says if $H, K \subseteq G$ are two subgroups of finite index, and $\rho : H \to \mathrm{GL}_{\mathbb{C}}(V)$ is a representation of H. Then
$$(\operatorname{Ind}_H^G V)|_K \cong \bigoplus_{g \in H \backslash G / K} \operatorname{Ind}_{K \cap g^{-1} H g}^K ({}^{g^{-1}} V),$$
where ${}^{g^{-1}}V$ is the $K \cap g^{-1}Hg$-representation where $g^{-1}xg$ acts by $\rho(x)$. We then apply this to $G = \Gamma_K, H = \Gamma_L, K = \Gamma_{\bar{v}/v}$. □

Example. If ρ is trivial, then
$$L(\rho, s) = \prod_v (1 - q_v^{-s})^{-1} = \sum_{\mathfrak{a} \lhd \mathcal{O}_K} \frac{1}{N\mathfrak{a}^s} = \zeta_K(s).$$
This is just the Dedekind ζ-function of K.

Example. Let L/K be a finite Galois extension with Galois group G. Consider the *regular representation* $r_{L/K}$ on $\mathbb{C}[G]$. This decomposes as $\bigoplus \rho_i^{d_i}$, where $\{\rho_i\}$ run over the irreducible representations of G of dimension d_i. We also have
$$r_{L/K} = \operatorname{Ind}_{\Gamma_L}^{\Gamma_K}(1).$$
So by the induction formula, we have
$$\zeta_L(s) = L(r_{L/K}, s) = \prod_i L(\rho_i, s)^{d_i}.$$

Example. For example, if $L/K = \mathbb{Q}(\zeta_N)/\mathbb{Q}$, then
$$\zeta_{\mathbb{Q}(\zeta_N)}(s) = \prod_\chi L(\chi, s),$$
where the product runs over all primitive Dirichlet characters mod $M \mid N$. Since $\zeta_{\mathbb{Q}(\zeta_N)}, \zeta_{\mathbb{Q}}$ have simple poles at $s = 1$, we know that $L(\chi, 1) \neq 0$ if $\chi \neq \chi_0$.

Theorem (Brauer induction theorem)**.** Suppose $\rho : G \to \mathrm{GL}_N(\mathbb{C})$ is a representation of a finite group. Then there exists subgroups $H_j \subseteq G$ and homomorphisms $\chi_j : H_j \to \mathbb{C}^\times$ and integers $m_i \in \mathbb{Z}$ such that
$$\operatorname{tr} \rho = \sum_j m_j \operatorname{tr} \operatorname{Ind}_{H_j}^G \chi_j.$$
Note that the m_j need not be non-negative. So we cannot quite state this as a statement about representations.

Corollary. Let $\rho : \Gamma_K \to \mathrm{GL}_N(\mathbb{C})$. Then there exists finite separable L_j/K and $\chi_j : \Gamma_{L_j} \to \mathbb{C}^\times$ of finite order and $m_j \in \mathbb{Z}$ such that

$$L(\rho, s) = \prod_j L(\chi_j, s)^{m_j}.$$

In particular, $L(\rho, s)$ has meromorphic continuation to $\mathbb{C}$ and has a functional equation

$$\Lambda(\rho, s) = L \cdot L_\infty = \varepsilon(\rho, s) L(\tilde{\rho}, 1 - s)$$

where

$$\varepsilon(\rho, s) = AB^s = \prod \varepsilon(\chi_j, s)^{m_j},$$

and $\tilde{\rho}(g) =^t \rho(g^{-1})$.

Conjecture (Artin conjecture)**.** If ρ does not contain the trivial representation, then $\Lambda(\rho, s)$ is entire.

This is closely related to the global Langlands conjecture.

In general, there is more than one way to write ρ as an sum of virtual induced characters. But when we take the product of the ε factors, it is always well-defined. We also know that

$$\varepsilon(\chi_j, s) = \prod \varepsilon_v(\chi_{j,v}, s)$$

is a product of local factors. It turns out the local decomposition is not independent of the decomposition, so if we want to write

$$\varepsilon(\rho, s) = \prod_v \varepsilon_v(\rho_v, s),$$

we cannot just take $\varepsilon_v(\rho_v, s) = \prod \varepsilon_v(\chi_{j,v}, s)$, as this is not well-defined. However, Langlands proved that there exists a unique factorization of $\varepsilon(\rho, s)$ satisfying certain conditions.

We fix F a non-Archimedean local field, $\chi : F^\times \to \mathbb{C}^\times$ and local ε factors

$$\varepsilon(\chi, \psi, \mu),$$

where μ is a Haar measure on F and $\psi : F \to \mathrm{U}(1)$ is a non-trivial character. Let $n(\psi)$ be the least integer such that $\psi(\pi_F^n \mathcal{O}_F) = 1$. Then

$$\varepsilon(\chi, \psi, \mu) = \begin{cases} \mu(\mathcal{O}_F) & \chi \text{ unramified}, n(\psi) = 0 \\ \int_{F^\times} \chi^{-1} \cdot \psi \; \mathrm{d}\mu & \chi \text{ ramified} \end{cases}$$

Since χ and ψ are locally constant, the integral is actually sum, which turns out to be finite (this uses the fact that χ is ramified).

For $a \in F^\times$ and $b > 0$, we have

$$\varepsilon(\chi, \psi(ax), b\mu) = \chi(a) |a|^{-1} b \varepsilon(\chi, \psi, \mu).$$

Theorem (Langlands–Deligne)**.** There exists a unique system of local constants $\varepsilon(\rho, \psi, \mu)$ for $\rho : W_F \to \mathrm{GL}_{\mathbb{C}}(V)$ such that

(i) ε is multiplicative in exact sequences, so it is well-defined for virtual representations.

(ii) $\varepsilon(\rho, \psi, b\mu) = b^{\dim V}\varepsilon(\rho, \psi, \mu)$.

(iii) If E/F is finite separable, and ρ is a virtual representation of W_F of degree 0 and $\sigma = \operatorname{Ind}_{W_E}^{W_F} \rho$, then

$$\varepsilon(\sigma, \psi, \mu) = \varepsilon(\rho, \psi \circ \operatorname{tr}_{E/F}, \mu').$$

Note that this is independent of the choice of μ and μ', since "$\dim V = 0$".

(iv) If $\dim \rho = 1$, then $\varepsilon(\rho)$ is the usual abelian $\varepsilon(\chi)$.

3 ℓ-adic representations

In this section, we shall discuss ℓ-adic representations of the Galois group, which often naturally arise from geometric situations. At the end of the section, we will relate these to *complex* representations of the *Weil–Langlands group*, which will be what enters the Langlands correspondence.

Definition (ℓ-adic representation)**.** Let G be a topological group. An *ℓ-adic representation* consists of the following data:

- A finite extension $E/\mathbb{Q}_\ell$;
- An E-vector space V; and
- A continuous homomorphism $\rho : G \to \mathrm{GL}_E(V) \cong \mathrm{GL}_n(E)$.

In this section,we will always take $G = \Gamma_F$ or W_F, where $F/\mathbb{Q}_p$ is a finite extension with $p \neq \ell$.

Example. The *cyclotomic character* $\chi_{\mathrm{cycl}} : \Gamma_K \to \mathbb{Z}_\ell^\times \subseteq \mathbb{Q}_\ell^\times$ is defined by the relation

$$\zeta^{\chi_{\mathrm{cycl}}(\gamma)} = \gamma(\zeta)$$

for all $\zeta \in \bar{K}$ with $\zeta^{\ell^n} = 1$ and $\gamma \in \Gamma_K$. This is a one-dimensional ℓ-adic representation.

Example. Let E/K be an elliptic curve. We define the *Tate module* by

$$T_\ell E = \varprojlim_n E[\ell^n](\bar{K}), \quad V_\ell E = T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Then $V_\ell E$ is is a 2-dimensional ℓ-adic representation of Γ_K over $\mathbb{Q}_\ell$.

Example. More generally, if X/K is any algebraic variety, then

$$V = H^i_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Q}_\ell)$$

is an ℓ-adic representation of Γ_K.

We will actually focus on the representations of the Weil group W_F instead of the full Galois group G_F. The reason is that every representation of the Galois group restricts to one of the Weil group, and since the Weil group is dense, no information is lost when doing so. On the other hand, the Weil group can have more representations, and we seek to be slightly more general.

Another reason to talk about the Weil group is that local class field theory says there is an isomorphism

$$\mathrm{Art}_F : W_F^{ab} \cong F^\times.$$

So one-dimensional representations of W_F are the same as one-dimensional representations of $F^\times$.

For example, there is an absolute value map $F^\times \to \mathbb{Q}^\times$, inducing a representation $\omega : W_F \to \mathbb{Q}^\times$. Under the Artin map, this sends the geometric Frobenius to $\frac{1}{q}$. In fact, ω is the restriction of the cyclotomic character to W_F.

Recall that we previously defined the tame character. Pick a sequence $\pi_n \in \bar{F}$ by $\pi_0 = \pi$ and $\pi_{n+1}^\ell = \pi_n$. We defined, for any $\gamma \in \Gamma_F$,

$$t_\ell(\gamma) = \left(\frac{\gamma(\pi_n)}{\pi_n}\right)_n \in \varprojlim \mu_{\ell^n}(\bar{F}) = \mathbb{Z}_\ell(1).$$

When we restrict to the inertia group, this is a homomorphism, independent of the choice of (π_n), which we call the *tame character*. In fact, this map is Γ_F-equivariant, where Γ_F acts on I_F by conjugation. In general, this still defines a function $\Gamma_F \to \mathbb{Z}_\ell(1)$, which depends on the choice of π_n.

Example. Continuing the previous notation, where π is a uniformizer of F, we let T_n be the ℓ^n-torsion subgroup of $\bar{F}^\times/\langle\pi\rangle$. Then

$$T_n = \langle\zeta_n, \pi_n\rangle/\langle\pi_n^{\ell^n}\rangle \cong (\mathbb{Z}/\ell^n\mathbb{Z})^2.$$

The ℓth power map $T_n \to T_{n-1}$ is surjective, and we can form the inverse limit T, which is then isomorphic to $\mathbb{Z}_\ell^2$. This then gives a 2-dimensional ℓ-adic representation of Γ_F.

In terms of the basis $(\zeta_{\ell^n}), (\pi_n)$, the representation is given by

$$\gamma \mapsto \begin{pmatrix} \chi_{\mathrm{cycl}}(\gamma) & t_\ell(\gamma) \\ 0 & 1 \end{pmatrix}.$$

Notice that the image of I_F is $\begin{pmatrix} 1 & \mathbb{Z}_\ell \\ 0 & 1 \end{pmatrix}$. In particular, it is infinite. This cannot happen for one-dimensional representations.

Perhaps surprisingly, the category of ℓ-adic representations of W_F over E does not depend on the topology of E, but only the E as an abstract field. In particular, if we take $E = \bar{\mathbb{Q}}_\ell$, then after taking care of the slight annoyance that it is infinite over $\bar{\mathbb{Q}}_\ell$, the category of representations over $\bar{\mathbb{Q}}_\ell$ does not depend on ℓ!

To prove this, we make use of the following understanding of ℓ-adic representations.

Theorem (Grothendieck's monodromy theorem)**.** Fix an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$. In other words, fix a system (ζ_{ℓ^n}) such that $\zeta_{\ell^n}^\ell = \zeta_{\ell^{n-1}}$. We then view t_ℓ as a homomorphism $I_F \to \mathbb{Z}_\ell$ via this identification.

Let $\rho : W_F \to \mathrm{GL}(V)$ be an ℓ-adic representation over E. Then there exists an open subgroup $I' \subseteq I_F$ and a nilpotent $N \in \mathrm{End}_E V$ such that for all $\gamma \in I'$,

$$\rho(\gamma) = \exp(t_\ell(\gamma)N) = \sum_{j=0}^\infty \frac{(t_\ell(\gamma)N)^j}{j!}.$$

In particular, $\rho(I')$ unipotent and abelian.

In our previous example, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Proof. If $\rho(I_F)$ is finite, let $I' = \ker\rho \cap I_F$ and $N = 0$, and we are done.

Otherwise, first observe that G is any compact group and $\rho : G \to \mathrm{GL}(V)$ is an ℓ-adic representation, then V contains a G-invariant lattice, i.e. a finitely-generated $\mathcal{O}_E$-submodule of maximal rank. To see this, pick any lattice $L_0 \subseteq V$. Then $\rho(G)L_0$ is compact, so generates a lattice which is G-invariant.

Thus, pick a basis of an I_F-invariant lattice. Then $\rho : W_F \to \mathrm{GL}_n(E)$ restricts to a map $I_F \to \mathrm{GL}_n(\mathcal{O}_E)$.

We seek to understand this group $\mathrm{GL}_n(\mathcal{O}_E)$ better. We define a filtration on $\mathrm{GL}_n(\mathcal{O}_E)$ by

$$G_k = \{g \in \mathrm{GL}_n(\mathcal{O}_E) : g \equiv I \bmod \ell^k\},$$

which is an open subgroup of $\mathrm{GL}_n(\mathcal{O}_E)$. Note that for $k \geq 1$, there is an isomorphism

$$G_k/G_{k+1} \to M_n(\mathcal{O}_E/\ell\mathcal{O}_E),$$

sending $1 + \ell^k g$ to g. Since the latter is an ℓ-group, we know G_1 is a pro-ℓ group. Also, by definition, $(G_k)^\ell \subseteq G_{k+1}$.

Since $\rho^{-1}(G_2)$ is open, we can pick an open subgroup $I' \subseteq I_F$ such that $\rho(I') \subseteq G_2$. Recall that $t_\ell(I_F)$ is the maximal pro-ℓ quotient of I_F, because the tame characters give an isomorphism

$$I_F/P_F \cong \prod_{\ell \nmid p} \mathbb{Z}_\ell(1).$$

So $\rho|_{I'} : I' \to G_2$ factors as

$$I' \overset{t_\ell}{\twoheadrightarrow} t_\ell(I') = \ell^s \mathbb{Z}_\ell \xrightarrow{\nu} G_2\ ,$$

using the assumption that $\rho(I_F)$ is infinite.

Now for $r \geq s$, let $T_r = \nu(\ell^r) = T_s^{r-s} \in G_{r+2-s}$. For r sufficiently large,

$$N_r = \log(T_r) = \sum_{m \geq 1} (-1)^{m-1} \frac{(T_r - 1)^m}{m}$$

converges ℓ-locally, and then $T_r = \exp N_r$.

We claim that N_r is nilpotent. To see this, if we enlarge E, we may assume that all the eigenvalues of N_r are in E. For $\delta \in W_F$ and $\gamma \in I_F$, we know

$$t_\ell(\delta\gamma\delta^{-1}) = \omega(\delta) t_\ell(\gamma).$$

So

$$\rho(\delta\gamma\delta^{-1}) = \rho(\gamma)^{w(\sigma)}$$

for all $\gamma \in I'$. So

$$\rho(\sigma) N_r \rho(\delta^{-1}) = \omega(\delta) N_r.$$

Choose δ lifting φ_q, $w(\delta) = q$. Then if v is an eigenvector for N_r with eigenvalue λ, then $\rho(\delta)v$ is an eigenvector of eigenvalue $q^{-1}\lambda$. Since N_r has finitely many eigenvalues, but we can do this as many times as we like, it must be the case that $\lambda = 0$.

Then take

$$N = \frac{1}{\ell^r} N_r$$

for r sufficiently large, and this works. $\square$

There is a slight unpleasantness in this theorem that we fixed a choice of ℓ^n roots of unity. To avoid this, we can say there exists an $N : v(1) = V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1) \to V$ nilpotent such that for all $\gamma \in I'$, we have

$$\rho(\gamma) = \exp(t_\ell(\gamma)N).$$

Grothendieck's monodromy theorem motivates the definition of the Weil–Deligne groups, whose category of representations are isomorphic to the category of ℓ-adic representations. It is actually easier to state what the representations of the Weil–Deligne group are. One can then write down a definition of the Weil–Deligne group as a semi-direct product if they wish.

Definition (Weil–Deligne representation)**.** A *Weil–Deligne representation* of W_F over a field E of characteristic 0 is a pair (ρ, N), where

- $\rho : W_F \to \mathrm{GL}_E(V)$ is a finite-dimensional representation of W_F over E with open kernel; and
- $N \in \mathrm{End}_E(V)$ is nilpotent such that for all $\gamma \in W_F$, we have

$$\rho(\gamma)N\rho(\gamma)^{-1} = \omega(\gamma)N,$$

Note that giving N is the same as giving a unipotent $T = \exp N$, which is the same as giving an algebraic representation of $\mathbb{G}_a$. So a Weil–Deligne representation is a representation of a suitable semi-direct product $W_F \ltimes \mathbb{G}_a$.

Weil–Deligne representations form a symmetric monoidal category in the obvious way, with

$$(\rho, N) \otimes (\rho', N') = (\rho \otimes \rho', N \otimes 1 + 1 \otimes N).$$

There are similarly duals.

Theorem. Let $E/\mathbb{Q}_\ell$ be finite (and $\ell \neq p$). Then there exists an equivalence of (symmetric monoidal) categories

$$\left\{\begin{array}{c}\ell\text{-adic representations}\\ \text{of } W_F \text{ over } E\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Weil–Deligne representations}\\ \text{of } W_F \text{ over } E\end{array}\right\}$$

Note that the left-hand side is pretty topological, while the right-hand side is almost purely algebraic, apart from the requirement that ρ has open kernel. In particular, the topology of E is not used.

Proof. We have already fixed an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$. We also pick a lift $\Phi \in W_F$ of the geometric Frobenius. In other words, we are picking a splitting

$$W_F = \langle \Phi \rangle \ltimes I_F.$$

The equivalence will take an ℓ-adic representation ρ_ℓ to the Weil–Deligne representation (ρ, N) on the same vector space such that

$$\rho_\ell(\Phi^m\gamma) = \rho(\Phi^m\gamma)\exp t_\ell(\gamma)N \tag{$*$}$$

for all $m \in \mathbb{Z}$ and $\gamma \in I_F$.

To check that this "works", we first look at the right-to-left direction. Suppose we have a Weil–Deligne representation (ρ, N) on V. We then define $\rho_\ell : W_F \to \mathrm{Aut}_E(V)$ by $(*)$. Since ρ has open kernel, it is continuous. Since t_ℓ is also continuous, we know ρ_ℓ is continuous. To see that ρ_ℓ is a homomorphism, suppose

$$\Phi^m \gamma \cdot \Phi^m \delta = \Phi^{m+n} \gamma' \delta$$

where $\gamma, \delta \in I_F$ and

$$\gamma' = \Phi^{-n} \gamma \Phi^n.$$

Then

$$\begin{aligned} \exp t_\ell(\gamma) N \cdot \rho(\Phi^n \delta) &= \sum_{j \geq 0} \frac{1}{j!} t_\ell(\gamma)^j N^j \rho(\Phi^n \delta) \\ &= \sum_{j \geq 0} \frac{1}{j!} t_\ell(\gamma) q^{nj} \rho(\Phi^n \delta) N^j \\ &= \rho(\Phi^n \delta) \exp(q^n t_\ell(\gamma)). \end{aligned}$$

But

$$t_\ell(\gamma') = t_\ell(\Phi^{-n} \gamma \Phi^n) = \omega(\Phi^{-n}) t_\ell(\gamma) = q^n t_\ell(\gamma).$$

So we know that

$$\rho_\ell(\Phi^m \gamma) \rho_\ell(\Phi^n \delta) = \rho_\ell(\Phi^{m+n} \gamma' \delta).$$

Notice that if $\gamma \in I_F \cap \ker \rho$, then $\rho_\ell(\gamma) = \exp t_\ell(\gamma) N$. So N is the nilpotent endomorphism occurring in the Grothendieck theorem.

Conversely, given an ℓ-adic representation ρ_ℓ, let $N \in \mathrm{End}_E V$ be given by the monodromy theorem. We then define ρ by $(*)$. Then the same calculation shows that (ρ, N) is a Weil–Deligne representation, and if $I' \subseteq I_F$ is the open subgroup occurring in the theorem, then $\rho_\ell(\gamma) = \exp t_\ell(\gamma) N$ for all $\gamma \in I'$. So by $(*)$, we know $\rho(I') = \{1\}$, and so ρ has open kernel. $\square$

This equivalence depends on two choices — the isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$ and also on the choice of Φ. It is not hard to check that up to natural isomorphisms, the equivalence does not depend on the choices.

We can similarly talk about representations over $\bar{\mathbb{Q}}_\ell$, instead of some finite extension E. Note that if we have a continuous homomorphism $\rho : W_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, then there exists a finite $E/\mathbb{Q}_\ell$ such that ρ factors through $\mathrm{GL}_n(E)$.

Indeed, $\rho(I_F) \subseteq \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ is compact, since it is a continuous image of compact group. So it is a complete metric space. Moreover, the set of finite extensions of $E/\mathbb{Q}_\ell$ is countable (Krasner's lemma). So by the Baire category theorem, $\rho(I_F)$ is contained in some $\mathrm{GL}_n(E)$, and of course, $\rho(\Phi)$ is contained in some $\mathrm{GL}_n(E)$.

Recalling that a Weil–Deligne representation over E only depends on E as a field, and $\bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_{\ell'}$ for any ℓ, ℓ', we know that

Theorem. Let $\ell, \ell' \neq p$. Then the category of $\bar{\mathbb{Q}}_\ell$ representations of W_F is equivalent to the category of $\bar{\mathbb{Q}}_{\ell'}$ representations of W_F.

Conjecturally, ℓ-adic representations coming from algebraic geometry have semi-simple Frobenius. This notion is captured by the following proposition/definition.

Proposition. Suppose ρ_ℓ is an ℓ-adic representation corresponding to a Weil–Deligne representation (ρ, N). Then the following are equivalent:

(i) $\rho_\ell(\Phi)$ is semi-simple (where Φ is a lift of Frob_q).

(ii) $\rho_\ell(\gamma)$ is semi-simple for all $\gamma \in W_F \setminus I_F$.

(iii) ρ is semi-simple.

(iv) $\rho(\Phi)$ is semi-simple.

In this case, we say ρ_ℓ and (ρ, N) are *F-semisimple* (where F refers to *Frobenius*).

Proof. Recall that $W_F \cong \mathbb{Z} \rtimes I_F$, and $\rho(I_F)$ is finite. So that part is always semisimple, and thus (iii) and (iv) are equivalent.

Moreover, since $\rho_\ell(\Phi) = \rho(\Phi)$, we know (i) and (iii) are equivalent. Finally, $\rho_\ell(\Phi)$ is semi-simple iff $\rho_\ell(\Phi^n)$ is semi-simple for all Φ. Then this is equivalent to (ii) since the equivalence before does not depend on the choice of Φ. $\square$

Example. The Tate module of an elliptic curve over F is not semi-simple, since it has matrix
$$\rho_\ell(\gamma) = \begin{pmatrix} \omega(\gamma) & t_\ell(\gamma) \\ 0 & 1 \end{pmatrix}.$$
However, it is F-semisimple, since
$$\rho(\gamma) = \begin{pmatrix} \omega(\gamma) & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It turns out we can classify all the indecomposable and F-semisimple Weil–Deligne representations. In the case of vector spaces, if a matrix N acts nilpotently on a vector space V, then the Jordan normal form theorem says there is a basis of V in which N takes a particularly nice form, namely the only non-zero entries are the entries of the form $(i, i+1)$, and the entries are all either 0 or 1. In general, we have the following result:

Theorem (Jordan normal form)**.** If V is semi-simple, $N \in \mathrm{End}(V)$ is nilpotent with $N^{m+1} = 0$, then there exists subobjects $P_0, \ldots, P_m \subseteq V$ (not unique as subobjects, but unique up to isomorphism), such that $N^r : P_r \to N^r P_r$ is an isomorphism, and $N^{r+1} P_r = 0$, and
$$V = \bigoplus_{r=0}^{m} P_r \oplus N P_r \oplus \cdots \oplus N^r P_r = \bigoplus_{r=0}^{m} P_r \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[N]}{(N^{r+1})}.$$

For vector spaces, this is just the Jordan normal form for nilpotent matrices.

Proof. If we had the desired decomposition, then heuristically, we want to set P_0 to be the things killed by N but not in the image of N. Thus, using semisimplicity, we pick P_0 to be a splitting
$$\ker N = (\ker N \cap \operatorname{im} N) \oplus P_0.$$
Similarly, we can pick P_1 by
$$\ker N^2 = (\ker N + (\operatorname{im} N \cap \ker N^2)) \oplus P_1.$$
One then checks that this works. $\square$

We will apply this when V is a representation of $W_F \to \mathrm{GL}(V)$ and N is the nilpotent endomorphism of a Weil–Deligne representation. Recall that we had

$$\rho(\gamma) N \rho(\gamma)^{-1} = \omega(\gamma) N,$$

so N is a map $V \to V \otimes \omega^{-1}$, rather than an endomorphism of V. Thankfully, the above result still holds (note that $V \otimes \omega^{-1}$ is still the same vector space, but with a different action of the Weil–Deligne group).

Proposition. Let (ρ, N) be a Weil–Deligne representation.

(i) (ρ, N) is irreducible iff ρ is irreducible and $N = 0$.

(ii) (ρ, N) is indecomposable and F-semisimple iff

$$(\rho, N) = (\sigma, 0) \otimes \mathrm{sp}(n),$$

where σ is an irreducible representation of W_F and $\mathrm{sp}(n) \cong E^n$ is the representation

$$\rho = \mathrm{diag}(\omega^{n-1}, \ldots, \omega, 1), \quad N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Example. If

$$\rho = \begin{pmatrix} \omega & & \\ \omega & \omega & \\ & & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

then this is an indecomposable Weil–Deligne representation not of the above form.

Proof. (i) is obvious.

For (ii), we first prove ($\Leftarrow$). If $(\rho, N) = (\sigma, 0) \otimes \mathrm{sp}(n)$, then F-semisimplicity is clear, and we have to check that it is indecomposable. Observe that the kernel of N is still a representation of W_F. Writing $V^{N=0}$ for the kernel of N in V, we note that $V^{N=0} = \sigma \otimes \omega^{n-1}$, which is irreducible. Suppose that

$$(\rho, N) = U_1 \oplus U_2.$$

Then for each i, we must have $U_i^{N=0} = 0$ or $V^{N=0}$. We may wlog assume $U_1^{N=0} = 0$. Then this forces $U_1 = 0$. So we are done.

Conversely, if (ρ, N, V) is F-semisimple and indecomposable, then V is a representation of W_F which is semi-simple and $N : V \to V \otimes \omega^{-1}$. By Jordan normal form, we must have

$$V = U \oplus NU \oplus \cdots \oplus N^r U$$

with $N^{r+1} = 0$, and U is irreducible. So $V = (\sigma, 0) \otimes \mathrm{sp}(r + 1)$. □

Given this classification result, when working over complex representations, the representation theory of $\mathrm{SU}(2)$ lets us capture the N bit of F-semisimple Weil–Deligne representation via the following group:

Definition (Weil–Langlands group)**.** We define the *(Weil–)Langlands group* to be
$$\mathcal{L}_F = W_F \times \mathrm{SU}(2).$$

A representation of $\mathcal{L}_F$ is a continuous action on a finite-dimensional vector space (thus, the restriction to W_F has open kernel).

Theorem. There exists a bijection between F-semisimple Weil–Deligne representations over $\mathbb{C}$ and semi-simple representations of $\mathcal{L}_F$, compatible with tensor products, duals, dimension, etc. In this correspondence:

– The representations ρ of $\mathcal{L}_F$ that factor through W_F correspond to the Weil–Deligne representations $(\rho, 0)$.

– More generally, simple $\mathcal{L}_F$ representations $\sigma \otimes (\mathrm{Sym}^{n-1}\,\mathbb{C}^2)$ correspond to the Weil–Deligne representation $(\sigma \otimes \omega^{(-1+n)/2}, 0) \otimes \mathrm{sp}(n)$. $\square$

If one sits down and checks the theorem, then one sees that the twist in the second part is required to ensure compatibility with tensor products.

Of course, the (F-semisimple) Weil–Deligne representations over $\mathbb{C}$ are in bijection those over $\bar{\mathbb{Q}}_\ell$, using an isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$.

4 The Langlands correspondence

Local class field theory says we have an isomorphism

$$W_F^{ab} \cong F^\times.$$

If we want to state this in terms of the full Weil group, we can talk about the one-dimensional representations of W_F, and write local class field theory as a correspondence

$$\left\{ \text{characters of } \mathrm{GL}_1(F) \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{1-dimensional representations} \\ \text{of } W_F \end{array} \right\}$$

The Langlands correspondence aims to understand the representations of $\mathrm{GL}_n(F)$, and it turns out this corresponds to n-dimensional representations of $\mathcal{L}_F$. That is, if we put enough adjectives in front of these words.

4.1 Representations of groups

The adjectives we need are fairly general. The group $\mathrm{GL}_n(F)$ contains a profinite open subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. The general theory applies to any topological group with a profinite open subgroup K, with G/K countable.

Definition (Smooth representation)**.** A *smooth representation* of G is a continuous representation of G over $\mathbb{C}$, where $\mathbb{C}$ is given the discrete topology. That is, it is a pair (π, V) where V is a complex vector space and $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ a homomorphism such that for every $v \in V$, the stabilizer of v in G is open.

Note that we can replace $\mathbb{C}$ with any field, but we like $\mathbb{C}$. Typically, V is an *infinite-dimensional* vector space. To retain some sanity, we often desire the following property:

Definition (Admissible representation)**.** We say (π, V) is *admissible* if for every open compact subgroup $K \subseteq G$ the fixed set $V^K = \{v \in V : \pi(g)v = v \ \forall g \in K\}$ is finite-dimensional.

Example. Take $G = \mathrm{GL}_2(F)$. Then $\mathbb{P}^1(F)$ has a right action of G by linear transformations. In fact, we can write $\mathbb{P}^1(F)$ as

$$\mathbb{P}^1(F) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \setminus G.$$

Let V be the space of all locally constant functions $f : \mathbb{P}^1(F) \to \mathbb{C}$. There are lots of such functions, because $\mathbb{P}^1(F)$ is totally disconnected. However, since $\mathbb{P}^1(F)$ is compact, each such function can only take finitely many values.

We let

$$\pi(g)f = (x \mapsto f(xg)).$$

It is not difficult to see that this is an infinite-dimensional admissible representation.

Of course, any finite-dimensional representation is an example, but $\mathrm{GL}_n(F)$ does not have very interesting finite-dimensional representations.

Proposition. Let $G = \mathrm{GL}_n(F)$. If (π, V) is a smooth representation with $\dim V < \infty$, then

$$\pi = \sigma \circ \det$$

for some $\sigma : F^\times \to \mathrm{GL}_\mathbb{C}(V)$.

So these are pretty boring.

Proof. If $V = \bigoplus_{i=1}^d Ce_i$, then

$$\ker \pi = \bigcap_{i=1}^d (\text{stabilizers of } e_i)$$

is open. It is also a normal subgroup, so

$$\ker \pi \supseteq K_m = \{g \in \mathrm{GL}_n(\mathcal{O}) : g \equiv I \bmod \varpi^m\}$$

for some m, where ϖ is a uniformizer of F. In particular, $\ker \pi$ contains $E_{ij}(x)$ for some $i \neq j$ and x, which is the matrix that is the identity except at entry (i, j), where it is x.

But since $\ker \pi$ is normal, conjugation by diagonal matrices shows that it contains all $E_{ij}(x)$ for all $x \in F$ and $i \neq j$. For any field, these matrices generate $\mathrm{SL}_n(F)$. So we are done. □

So the interesting representations are all infinite-dimensional. Fortunately, a lot of things true for finite-dimensional representations also hold for these. For example,

Lemma (Schur's lemma)**.** Let (π, V) be an irreducible representation. Then every endomorphism of V commuting with π is a scalar.

In particular, there exists $\omega_\pi : Z(G) \to \mathbb{C}^\times$ such that

$$\pi(zg) = \omega_\pi(z)\pi(g)$$

for all $z \in Z(G)$ and $g \in G$. This is called the *central character*. □

At this point, we are already well-equipped to state a high-level description of the local Langlands correspondence.

Theorem (Harris–Taylor, Henniart)**.** There is a bijection

$$\left\{\begin{array}{c}\text{irreducible, admissible}\\ \text{representations of } \mathrm{GL}_n(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{semi-simple } n\text{-dimensional}\\ \text{representations of } \mathcal{L}_F\end{array}\right\}.$$

In the next section, we will introduce the Hecke algebra, which allows us to capture these scary infinite dimensional representations of $\mathrm{GL}_n(F)$ in terms of something finite-dimensional.

Afterwards, we are going to state the Langlands classification of irreducible admissible representations of $\mathrm{GL}_n(F)$. We can then state a detailed version of the local Langlands correspondence in terms of this classification.

4.2 Hecke algebras

Let G, K be as before.

Notation. We write $C_c^\infty(G)$ for the vector space of locally constant functions $f : G \to \mathbb{C}$ of compact support.

Definition (Hecke algebra)**.** The *Hecke algebra* is defined to be

$$\mathcal{H}(G, K) = \{\varphi \in C_c^\infty(G) : \varphi(kgk') = \varphi(g) \text{ for all } k, k' \in K\}.$$

This is spanned by the characteristic functions of double cosets KgK.

This algebra comes with a product called the convolution product. To define this, we need the Haar measure on G. This is a functional $C_c^\infty(G) \to \mathbb{C}$, written

$$f \mapsto \int_G f(g)\;\mathrm{d}\mu(g),$$

that is invariant under left translation, i.e. for all $h \in G$,we have

$$\int f(hg)\;\mathrm{d}\mu(g) = \int f(g)\;\mathrm{d}\mu(g).$$

To construct the Haar measure, we take $\mu(1_K) = 1$. Then if $K' \subseteq K$ is an open subgroup, then it is of finite index, and since we want $\mu(1_{xK'}) = \mu(1_{K'})$, we must have

$$\mu(1_{K'}) = \frac{1}{(K : K')}.$$

We then set $\mu(1_{xK'}) = \mu(1_{K'})$ for any $x \in G$, and since these form a basis of the topology, this defines μ.

Definition (Convolution product)**.** The *convolution product* on $\mathcal{H}(G, K)$ is

$$(\varphi * \varphi')(g) = \int_G \varphi(x)\varphi'(x^{-1}g)\;\mathrm{d}\mu(x).$$

Observe that this integral is actually a finite sum.

It is an exercise to check that this is a $\mathbb{C}$-algebra with unit

$$e_K = \frac{1}{\mu(K)} 1_K,$$

Now if (π, V) is a smooth representation, then for all $v \in V$ and $\varphi \in \mathcal{H}(G, K)$, consider the expression

$$\pi(\varphi)v = \int_G \varphi(g)\pi(g)v\;\mathrm{d}\mu(g).$$

Note that since the stabilizer of v is open, the integral is actually a finite sum, so we can make sense of it. One then sees that

$$\pi(\varphi)\pi(\varphi') = \pi(\varphi * \varphi').$$

This would imply V is a $\mathcal{H}(G,K)$-module, if the unit acted appropriately. It doesn't, however, since in fact $\pi(\varphi)$ always maps into V^K. Indeed, if $k \in K$, then

$$\pi(k)\pi(\varphi)v = \int_G \varphi(g)\pi(kg)v \, \mathrm{d}\mu(g) = \int \varphi(g)\pi(g) \, \mathrm{d}\mu(g) = \pi(\varphi)v,$$

using that $\varphi(g) = \varphi(k^{-1}g)$ and $\mathrm{d}\mu(g) = \mathrm{d}\mu(k^{-1}g)$.

So our best hope is that V^K is an $\mathcal{H}(G,K)$-module, and one easily checks that $\pi(e_K)$ indeed acts as the identity. We also have a canonical projection $\pi(e_K) : V \twoheadrightarrow V^K$.

In good situations, this Hecke module determines V.

Proposition. There is a bijection between isomorphism classes of irreducible admissible (π, V) with $V^K \neq 0$ and isomorphism classes of simple finite-dimensional $\mathcal{H}(G,K)$-modules, which sends (π, V) to V^K with the action we described.

If we replace K by a smaller subgroup $K' \subseteq K$, then we have an inclusion

$$\mathcal{H}(G,K) \hookrightarrow \mathcal{H}(G,K'),$$

which does *not* take e_K to $e_{K'}$. We can then form the union of all of these, and let

$$\mathcal{H}(G) = \varinjlim_K \mathcal{H}(G,K)$$

which is an algebra without unit. Heuristically, the unit should be the delta function concentrated at the identity, but that is not a function.

This $\mathcal{H}(G)$ acts on any smooth representation, and we get an equivalence of categories

$$\left\{\begin{array}{c}\text{smooth}\\ G\text{-representations}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{non-degenerate}\\ \mathcal{H}(G)\text{-modules}\end{array}\right\}.$$

The non-degeneracy condition is $V = \mathcal{H}(G)V$.

Note that if $\varphi \in \mathcal{H}(G)$ and (π, V) is admissible, then the $\operatorname{rank} \pi(\varphi) < \infty$, using that V^K is finite-dimensional. So the trace is well-defined. The *character* of (π, V) is then the map

$$\varphi \mapsto \operatorname{tr} \pi(\varphi)$$

In this sense, admissible representations have traces.

4.3 The Langlands classification

Recall that the group algebra $\mathbb{C}[G]$ is an important tool in the representation theory of finite groups. This decomposes as a direct sum over all irreducible representations

$$\mathbb{C}[G] = \bigoplus_\pi \pi^{\dim(\pi)}.$$

The same result is true for compact groups, if we replace $\mathbb{C}[G]$ by $L^2(G)$. We get a decomposition

$$L^2(G) = \hat{\bigoplus_\pi} \pi^{\dim(\pi)},$$

where L^2 is defined with respect to the Haar measure, and the sum is over all (finite dimensional) irreducible representations of G. The hat on the direct sum says it is a Hilbert space direct sum, which is the completion of the vector space direct sum. This result is known as the *Peter–Weyl theorem*. For example,

$$L^2(\mathbb{R}/\mathbb{Z}) = \hat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C} \cdot e^{2\pi i n y}.$$

However, if G is non-compact, then this is no longer true.

Sometimes, we can salvage this a bit by replacing the discrete direct sum with a continuous version. For example, the characters of $\mathbb{R}$ are those of the form

$$x \mapsto e^{2\pi i x y},$$

which are not L^2 functions. But we can write any function in $L^2(\mathbb{R})$ as

$$x \mapsto \int_y \varphi(y) e^{2\pi i x y}\,\mathrm{d}y.$$

So in a sense, $L^2(\mathbb{R})$ is the "continuous direct sum" of irreducible representations.

In general, $L^2(G)$ decomposes as a sum of irreducible representations, and contains both a discrete sum and a continuous part. However, there are irreducible representations that *don't* appear in $L^2(G)$, discretely or continuously. These are known as the *complementary series* representations. This happens, for example, for $G = \mathrm{SL}_2(\mathbb{R})$ (Bargmann 1947).

We now focus on the case $G = \mathrm{GL}_n(F)$, or any reductive F-group (it doesn't hurt to go for generality if we are not proving anything anyway). It turns out in this case, we can describe the representations that appear in $L^2(G)$ pretty explicitly. These are characterized by the matrix coefficients.

If $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ is a finite-dimensional representation, then the matrix coefficients $\pi(g)_{ij}$ can be written as

$$\pi(g)_{ij} = \delta^i(\pi(g)e_j),$$

where $e_j \in \mathbb{C}^n$ is the j^{th} basis vector and $\delta^i \in (\mathbb{C}^n)^*$ is the i^{th} dual basis vector.

More generally, if $\pi : G \to \mathrm{GL}(V)$ is a finite-dimensional representation, and $v \in V$, $\ell \in V^*$, then we can think of

$$\pi_{v,\ell}(g) = \ell(\pi(g)v)$$

as a matrix element of $\pi(g)$, and this defines a function $\pi_{v,\ell} : G \to \mathbb{C}$.

In the case of the G we care about, our representations are fancy infinite-dimensional representations, and we need a fancy version of the dual known as the *contragredient*.

Definition (Contragredient)**.** Let (π, V) be a smooth representation. We define $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and the representation (π^*, V^*) of G is defined by

$$\pi_*(g)\ell = (v \mapsto \ell(\pi(g^{-1})v)).$$

We then define the *contragredient* $(\tilde{\pi}, \tilde{v})$ to be the subrepresentation

$$\tilde{V} = \{\ell \in V^* \text{ with open stabilizer}\}.$$

This contragredient is quite pleasant. Recall that

$$V = \bigcup_K V^K.$$

We then have

$$\tilde{V} = \bigcup (V^*)^K.$$

Using the projection $\pi(e_K) : V \to V^K$, we can identify

$$(V^*)^K = (V^K)^*.$$

So in particular, if π is admissible, then so is $\tilde{\pi}$, and we have a canonical isomorphism

$$V \to \tilde{\tilde{V}}.$$

Definition (Matrix coefficient)**.** Let (π, V) be a smooth representation, and $v \in V$, $\ell \in \tilde{V}$. The *matrix coefficient* $\pi_{v,\ell}$ is defined by

$$\pi_{v,\ell}(g) = \ell(\pi(g)v).$$

This is a locally constant function $G \to \mathbb{C}$.

We can now make the following definition:

Definition (Square integrable representation)**.** Let (π, V) be an irreducible smooth representation of G. We say it is *square integrable* if ω_π is unitary and

$$|\pi_{v,\ell}| \in L^2(G/Z)$$

for all (v, ℓ).

Note that the fact that ω_π is unitary implies $|\pi_{v,\ell}|$ is indeed a function on $L^2(G/Z)$. In general, it is unlikely that $\pi_{v,\ell}$ is in $L^2(G)$.

If Z is finite, then ω_π is automatically unitary and we don't have to worry about quotienting about the center. Moreover, π is square integrable iff $\pi_{v,\ell} \in L^2(G)$. In this case, if we pick $\ell \in \tilde{V}$ non-zero, then $v \mapsto \pi_{v,\ell}$ gives an embedding of V into $L^2(G)$. In general, we have

$$V \subseteq L^2(G, \omega_\pi) = \{f : G \to \mathbb{C} : f(zg) = \omega_\pi(z) f(z), \quad |f| \in L^2(G/Z)\},$$

A slight weakening of square integrability is the following weird definition:

Definition (Tempered representation)**.** Let (π, V) be irreducible, ω_π unitary. We say it is *tempered* if for all (v, ℓ) and $\varepsilon > 0$, we have

$$|\pi_{v,\ell}| \in L^{2+\varepsilon}(G/Z).$$

The reason for this definition is that π is tempered iff it occurs in $L^2(G)$, not necessarily discretely.

Weakening in another direction gives the following definition:

Definition (Essentially square integrable)**.** Let (π, V) be irreducible. Then (π, V) is *essentially square integrable* (or *essentially tempered*) if

$$\pi \otimes (\chi \circ \det)$$

is square integrable (or tempered) for some character $\chi : F^\times \to \mathbb{C}$.

Note that while these definitions seem very analytic, there are in fact purely algebraic interpretations of these definitions, using Jacquet modules.

A final category of representations is the following:

Definition (Supercuspidal representation)**.** We say π is *supercuspidal* if for all (v, ℓ), the support of $\pi_{v,\ell}$ is compact mod Z.

These are important because they are building blocks of *all* irreducible representations of $\mathrm{GL}_n(F)$, in a sense we will make precise.

The key notion is that of parabolic induction, which takes a list of representations σ_i of $\mathrm{GL}_{n_i}(F)$ to a representation of $\mathrm{GL}_N(F)$, where $N = \sum n_i$.

We first consider a simpler case, where we have an n-tuple $\chi = (\chi_1, \ldots, \chi_n) : (F^\times)^n \to \mathbb{C}^\times$ of characters. The group $G = \mathrm{GL}_n(F)$ containing the Borel subgroup B of upper-triangular matrices. Then B is the semi-direct product TN, where $T \cong (F^\times)^n$ consists of the diagonal matrices and N the unipotent ones. We can then view χ as a character $\chi : B \to B/N = T \to \mathbb{C}^\times$.

We then induce this up to a representation of G. Here the definition of an induced representation is not the usual one, but has a twist.

Definition (Induced representation)**.** Let $\chi : B \to \mathbb{C}$ be a character. We define the induced representation $\mathrm{Ind}_B^G(\chi)$ to be the space of locally constant functions $f : g \to \mathbb{C}$ such that

$$f(bg) = \chi(b)\delta_B(b)^{1/2} f(g)$$

for all $b \in B$ and $g \in G$, where G acts by

$$\pi(g)f : x \mapsto f(xg).$$

The function $\delta_B(b)^{1/2}$ is defined by

$$\delta_B(b) = |\det \mathrm{ad}_B(b)|.$$

More explicitly, if the diagonal entries of $b \in B$ are $x_1, \ldots, x_n$, then

$$\delta_B(b) = \prod_{i=1}^n |x_i|^{n+1-2i} = |x_1|^{n-1}|x_2|^{n-3} \cdots |x_n|^{-n+1}$$

This is a smooth representation since $B \backslash G$ is compact. In fact, it is admissible and of finite length.

When this is irreducible, it is said to be a *principle series representation* of G.

Example. Recall that $\mathbb{P}^1(F) = B\backslash \mathrm{GL}_2(F)$. In this terminology,

$$C^\infty(\mathbb{P}^1(F)) = \mathrm{Ind}_B^G(\delta_B^{-1/2}).$$

This is not irreducible, since it contains the constant functions, but quotienting by these does give an irreducible representation. This is called the *Steinberg representation*.

In general, we start with a *parabolic subgroup* $P \subseteq \mathrm{GL}_n(F) = G$, i.e. one conjugate to block upper diagonal matrices with a partition $n = n_1 + \cdots + n_r$. This then decomposes into MN, where

$$M \cong \prod_i \mathrm{GL}_{n_i}(F), \quad N = \left\{ \begin{pmatrix} I_{n_1} & \cdots & * \\ & \ddots & \vdots \\ & & I_{n_r} \end{pmatrix} \right\}.$$

This is an example of a *Levi decomposition*, and M is a *Levi subgroup*.

To perform *parabolic induction*, we let (σ, U) be a smooth representation of M, e.g. $\sigma_1 \otimes \cdots \otimes \sigma_r$, where each σ_i is a representation of $\mathrm{GL}_{n_i}(F)$. This then defines a representation of P via $P \to P/N = M$, and we define $\mathrm{Ind}_P^G(\sigma)$ to be the space of all locally constant functions $f : G \to U$ such that

$$f(pg) = \delta_P(p)\sigma(p)f(g)$$

for all $p \in P, g \in G$ and δ_P is again defined by

$$\delta_P(p) = |\det \mathrm{ad}_P(p)|.$$

This is again a smooth representation.

Proposition.

(i) σ is admissible implies $\pi = \mathrm{Ind}_P^G \sigma$ is admissible.

(ii) σ is unitary implies π is unitary.

(iii) $\mathrm{Ind}_P^G(\tilde{\sigma}) = \tilde{\pi}$. □

(ii) and (iii) are the reasons for the factor $\delta_P^{1/2}$.

Example. Observe

$$\widetilde{C^\infty(\mathbb{P}(F))} = \mathrm{Ind}(\delta_B^{1/2}) = \{f : G \to \mathbb{C} : f(bg) = \delta_B(b)f(g)\}.$$

There is a linear form to $\mathbb{C}$ given by integrating f over $\mathrm{GL}_2(\mathcal{O})$ (we can't use F since $\mathrm{GL}_2(F)$ is not compact). In fact, this map is G-invariant, and not just $\mathrm{GL}_2(\mathcal{O})$-invariant. This is dual to the constant subspace of $C^\infty(\mathbb{P}^1(F))$.

The rough statement of the classification theorem is that every irreducible admissible representation of $G = \mathrm{GL}_n(F)$, is a subquotient of an induced representation $\mathrm{Ind}_P^G \sigma$ for some supercuspidal representation of a Levi subgroup $P = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}(F)$. This holds for *any* reductive G/F.

For $\mathrm{GL}_n(F)$, we can be more precise. This is called the *Langlands classification.* We first classify all the essentially square integrable representations:

Theorem. Let $n = mr$ with $m, r \geq 1$. Let σ be any supercuspidal representation of $\mathrm{GL}_m(F)$. Let

$$\sigma(x) = \sigma \otimes |\det{}_m|^x.$$

Write $\Delta = (\sigma, \sigma(1), \ldots, \sigma(r-1))$, a representation of $\mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$. Then $\mathrm{Ind}_P^G(\Delta)$ has a unique irreducible subquotient $Q(\Delta)$, which is essentially square integrable.

Moreover, $Q(\Delta)$ is square integrable iff the central character is unitary, iff $\sigma(\frac{r-1}{2})$ is square-integrable, and every essentially square integrable π is a $Q(\Delta)$ for a unique Δ.

Example. Take $n = 2 = r$, $\sigma = |\cdot|^{-1/2}$. Take

$$P = B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Then

$$\operatorname{Ind}_B^G(|\cdot|^{-1/2}, |\cdot|^{-1/2}) = C^\infty(B\backslash G) \supseteq \mathbb{C},$$

where $\mathbb{C}$ is the constants. Then $C^\infty(B\backslash G)$ is not supercuspidal, but the quotient is the Steinberg representation, which is square integrable. Thus, every two-dimensional essentially square integrable representation which is not supercuspidal is a twist of the Steinberg representation by $\chi \circ \det$.

We can next classify tempered representations.

Theorem. The tempered irreducible admissible representations of $\mathrm{GL}_n(F)$ are precisely the representations $\operatorname{Ind}_P^G \sigma$, where σ is irreducible square integrable. In particular, $\operatorname{Ind}_P^G \sigma$ are always irreducible when σ is square integrable.

Example. For GL_2, we seek a π which is tempered but not square integrable. This must be of the form

$$\pi = \operatorname{Ind}_B^G(\chi_1, \chi_2),$$

where $|\chi_1| = |\chi_2| = 1$. If we want it to be essentially tempered, then we only need $|\chi_1| = |\chi_2|$.

Finally, we classify all irreducible (admissible) representations.

Theorem. Let $n = n_1 + \cdots + n_r$ be a partition, and σ_i tempered representation of $\mathrm{GL}_{n_i}(F)$. Let $t_i \in \mathbb{R}$ with $t_1 > \cdots > t_r$. Then $\operatorname{Ind}_P^G(\sigma_1(t_1), \ldots, \sigma_r(t_r))$ has a unique irreducible quotient *Langlands quotient*, and every π is (uniquely) of this form.

Example. For GL_2, the remaining (i.e. not essentially tempered) representations are the irreducible subquotients of

$$\operatorname{Ind}_B^G(\chi_1, \chi_2),$$

where

$$|\chi_i| = |\cdot|_F^{t_i}, \quad t_1 > t_2.$$

Note that the one-dimensional representations must occur in this set, because we haven't encountered any yet.

For example, if we take $\chi_1 = |\cdot|^{1/2}$ and $\chi_2 = |\cdot|^{-1/2}$, then

$$\operatorname{Ind}_B^G(\chi_1, \chi_2) = \widetilde{C^\infty(B}\backslash G),$$

which has the trivial representation as its irreducible quotient.

4.4 Local Langlands correspondence

Theorem (Harris–Taylor, Henniart)**.** There is a bijection

$$\left\{\begin{matrix}\text{irreducible, admissible} \\ \text{representations of } \mathrm{GL}_n(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{semi-simple } n\text{-dimensional} \\ \text{representations of } \mathcal{L}_F\end{matrix}\right\}.$$

Moreover,

- For $n = 1$, this is the same as local class field theory.
- Under local class field theory, this corresponds between ω_π and $\det \sigma$.

– The supercuspidals correspond to the irreducible representations of W_F itself.

– If a supercuspidal π_0 corresponds to the representation σ_0 of W_F, then the essentially square integrable representation $\pi = Q(\pi_0(-\frac{r-1}{2}), \ldots, \pi_0(\frac{r-1}{2}))$ corresponds to $\sigma = \sigma_0 \otimes \mathrm{Sym}^{r-1}\mathbb{C}^2$.

– If π_i correspond to σ_i, where σ_i are irreducible and unitary, then the tempered representation $\mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$ corresponds to $\sigma_1 \oplus \cdots \oplus \sigma_r$.

– For general representations, if π is the Langlands quotient of

$$\mathrm{Ind}(\pi_1(t_1), \ldots, \pi_r(t_r))$$

with each π_i tempered, and π_i corresponds to unitary representations σ_i of $\mathcal{L}_F$, then π corresponds to $\bigoplus \sigma_i \otimes |\mathrm{Art}_F^{-1}|_F^{t_i}$.

The hard part of the theorem is the correspondence between the supercuspidal representations and irreducible representations of W_F. This correspondence is characterized by ε-factors of pairs.

Recall that for an irreducible representation of W_F, we had an ε factor $\varepsilon(\sigma, \mu_F, \psi)$. If we have two representations, then we can just take the tensor product $\varepsilon(\sigma_1 \otimes \sigma_2, \mu_F, \psi)$. It turns out for supercuspidals, we can also introduce ε-factors $\varepsilon(\pi, \mu_F, \psi)$. There are also ε factors for pairs, $\varepsilon(\pi_1, \pi_2, \mu_F, \psi)$. Then the correspondence is such that if π_i correspond to σ_i, then

$$\varepsilon(\sigma_1 \otimes \sigma_2, \mu_F, \psi) = \varepsilon(\pi_1, \pi_2, \mu_F, \psi).$$

When $n = 1$, we get local class field theory. Recall that we actually have a *homomorphic* correspondence between characters of $F^\times$ and characters of W_F, and the correspondence is uniquely determined by

(i) The behaviour on unramified characters, which is saying that the Artin map sends uniformizers to geometric Frobenii

(ii) The base change property: the restriction map $W_{F'}^{\mathrm{ab}} \to W_F^{\mathrm{ab}}$ correspond to the norm map of fields

If we want to extend this to the local Langlands correspondence, the corresponding picture will include

(i) Multiplication: taking multiplications of GL_n and GL_m to representations of GL_{mn}

(ii) Base change: sending representations of $\mathrm{GL}_n(F)$ to representations of $\mathrm{GL}_n(F')$ for a finite extension F'/F

These thing exist by virtue of local Langlands correspondence (much earlier, base change for *cyclic extensions* was constructed by Arthur–Clozel).

Proposition. Let $\sigma : W_F \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Then the following are equivalent:

(i) For some $g \in W_F \setminus I_F$, $\sigma(g)$ has an eigenvalue of absolute value 1.

(ii) $\operatorname{im}\sigma$ is relatively compact, i.e. has compact closure, i.e. is bounded.

(iii) σ is unitary.

Proof. The only non-trivial part is (i) $\Rightarrow$ (ii). We know

$$\operatorname{im}\sigma = \langle \sigma(\Phi), \sigma(I_F) = H\rangle,$$

where Φ is some lift of the Frobenius and H is a finite group. Moreover, I_F is normal in W_F. So for some $n \geq 1$, $\sigma(\Phi^n)$ commutes with H. Thus, replacing g and Φ^n with suitable non-zero powers, we can assume $\sigma(g) = \sigma(\Phi^n)h$ for some $h \in \mathbb{H}$. Since H is finite, and $\sigma(\Phi^n)$ commutes with h, we may in fact assume $\sigma(g) = \sigma(\Phi)^n$. So we know $\sigma(\Phi)$ has eigenvalue with absolute value 1.

Let $V_1 \subseteq V = \mathbb{C}^n$ be a sum of eigenspaces for $\sigma(\Phi)^n$ with all eigenvalues having absolute value 1. Since $\sigma(\Phi^n)$ is central, we know V_1 is invariant, and hence V is irreducible. So $V_1 = V$. So all eigenvalues of $\sigma(\Phi)$ have eigenvalue 1. Since V is irreducible, we know it is F-semisimple. So $\sigma(\Phi)$ is semisimple. So $\langle\sigma(\Phi)\rangle$ is bounded. So $\operatorname{im}\sigma$ is bounded. $\square$

5 Modular forms and representation theory

Recall that a modular form is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(z) = j(\gamma, z)^{-k} f(\gamma(z))$$

for all γ in some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma(z) = \frac{az+b}{cz+d}, \quad j(\gamma, z) = cz + d.$$

Let M_k be the set of all such f.

Consider the group

$$\mathrm{GL}_2(\mathbb{Q})^+ = \{g \in \mathrm{GL}_2(\mathbb{Q}) : \det g > 0\}.$$

This acts on M_k on the left by

$$g : f \mapsto j_1(g^{-1}, z)^{-k} f(g^{-1}(z)), \quad j_1(g, z) = |\det g|^{-1/2} j(g, z).$$

The factor of $|\det g|^{-1/2}$ makes the diagonal $\mathrm{diag}(\mathbb{Q}^\times_{>0})$ act trivially. Note that later we will consider some g with negative determinant, so we put the absolute value sign in.

For any $f \in M_k$, the stabilizer contains some $\Gamma(N)$, which we can think of as some continuity condition. To make this precise, we can form the completion of $\mathrm{GL}_2(\mathbb{Q})^+$ with respect to $\{\Gamma(N) : N \geq 1\}$, and the action extends to a representation π' of this completion. In fact, this completion is

$$G' = \{g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty) : \det(g) \in \mathbb{Q}^\times_{>0}\}.$$

This is a closed subgroup of $G = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$, and in fact

$$G = G' \cdot \begin{pmatrix} \hat{\mathbb{Z}} & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact G is a semidirect product of the groups.

The group G' seems quite nasty, since the determinant condition is rather unnatural. It would be nice to get a representation of G itself, and the easy way to do so is by induction. What is this? By definition, it is

$$\mathrm{Ind}_{G'}^G(M_k) = \{\varphi : G \to M_k : \forall h \in G', \varphi(hg) = \pi'(h)\varphi(g)\}.$$

Equivalently, this consists of functions $F : \mathbb{H} \times G \to \mathbb{C}$ such that for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$, we have

$$j_1(\gamma, z)^{-k} F(\gamma(z), \gamma g) = F(z, g),$$

and for every $g \in G$, there is some open compact $K \subseteq G$ such that

$$F(z, g) = F(z, gh) \text{ for all } h \in K,$$

and that F is holomorphic in z (and at the cusps).

To get rid of the plus, we can just replace $\mathrm{GL}_2(\mathbb{Q})^+, \mathbb{H}$ with $\mathrm{GL}_2(\mathbb{Q}), \mathbb{C} \setminus \mathbb{R} = \mathbb{H}^\pm$. These objects are called *adelic modular forms*.

If F is an adelic modular form, and we fix g, then the function $f(z) = F(z, g)$ is a usual modular form. Conversely, if F invariant under $\ker(\mathbb{G}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$, then F corresponds to a tuple of $\Gamma(N)$-modular forms indexed by $(\mathbb{Z}/N\mathbb{Z})^\times$. This has an action of $G = \prod_p' \mathrm{GL}_2(\mathbb{Q}_p)$ (which is the restricted product with respect to $\mathrm{GL}_2(\mathbb{Z}_p)$).

The adelic modular forms contain the cusp forms, and any $f \in M_k$ generates a subrepresentation of the space of adelic forms.

Theorem.

(i) The space V_f of adelic cusp forms generated by $f \in S_k(\Gamma_1(N))$ is irreducible iff f is a T_p eigenvector for all $p \nmid n$.

(ii) This gives a bijection between irreducible G-invariant spaces of adelic cusp forms and Atkin–Lehner newforms.

Note that it is important to stick to cusp forms, where there is an inner product, because if we look at the space of adelic modular forms, it is not completely decomposable.

Now suppose (π, V) is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty) = \prod' G_p = \mathrm{GL}_2(\mathbb{Q}_p)$, and take a maximal compact subgroups $K_p^0 = \mathrm{GL}_2(\mathbb{Z}_p) \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$. Then general facts about irreducible representations of products imply irreducibility (and admissibility) is equivalent to the existence of irreducible admissible representations (π_p, V_p) of G_p for all p such that

(i) For almost all p, $\dim V_p^{K_p^0} \geq 1$ (for $G_p = \mathrm{GL}_n(\mathbb{Q}_p)$, this implies the dimension is 1). Fix some non-zero $f_p^0 \in V_p^{K_p^0}$.

(ii) We have

$$\pi = \otimes_p' \pi_p,$$

the restricted tensor product, which is generated by $\bigotimes_p v_p$ with $v_p = f_p^0$ for almost all p. To be precise,

$$\otimes_p' \pi_p = \varinjlim_{\text{finite } S} \bigotimes_{p \in S} \pi_p.$$

The use of v_p is to identify smaller tensor products with larger tensor products.

Note that (i) is equivalent to the assertion that (π_p, V_p) is an irreducible principal series representation $\mathrm{Ind}_{B_p}^{G_p}(\chi_1, \chi_2)$ where χ_i are *unramified* characters. These unramified characters are determined by $\chi_p(p) = \alpha_{p,i}$.

If $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ is a normalized eigenform with character $\omega : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, and if f corresponds to $\pi = \bigotimes' \pi_p$, then for every $p \nmid N$, we have $\pi_p = \mathrm{Ind}_{B_p}^{G_p}(\chi_1, \chi_2)$ is an unramified principal series, and

$$\begin{aligned} a_p &= p^{(k-1)/2}(\alpha_{p,1} + \alpha_{p,2}) \\ \omega(p)^{-1} &= \alpha_{p,1}\alpha_{p,2}. \end{aligned}$$

We can now translate difficult theorems about modular forms into representation theory.

Example. The Ramanujan conjecture (proved by Deligne) says

$$|a_p| \leq 2p^{(k-1)/2}.$$

If we look at the formulae above, since $\omega(p)^{-1}$ is a root of unity, this is equivalent to the statement that $|\alpha_{p,i}| = 1$. This is true iff π_p is tempered.

We said above that if $\dim V_p^{K_p^0} \geq 1$, then in fact it is equal to 1. There is a generalization of that.

Theorem (Local Atkin–Lehner theorem)**.** If (π, V) is an irreducible representation of $\mathrm{GL}_2(F)$, where $F/\mathbb{Q}_p$ and $\dim V = \infty$, then there exists a unique $n_\pi > 0$ such that

$$V^{K_n} = \begin{cases} 0 & n < n_\pi \\ \text{one-dimensional} & n = n_\pi \end{cases}, \quad K_n = \left\{ g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \varpi^n \right\}.$$

Taking the product of these invariant vectors for $n = n_\pi$ over all p gives Atkin–Lehner newform.

What about the other primes? i.e. the primes at infinity?

We have a map $f : \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty) \to \mathbb{C}$. Writing $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$, which has an action of Γ, we can convert this to an action of $\mathrm{SO}(2)$ on $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$. Consider the function

$$\Phi_f : \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty) = \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

given by

$$\Phi_f(h_\infty, h^\infty) = j_1(h_\infty, i)^{-k} f(h_\infty(i), h^\infty).$$

Then this is invariant under $\mathrm{GL}_2(\mathbb{Q})^+$, i.e.

$$\Phi_f(\gamma h_\infty, \gamma h^\infty) = \Phi(h_\infty, h^\infty).$$

Now if we take

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2),$$

then

$$\Phi_f(h_\infty k_\theta, h^\infty) = e^{ik\theta} \Phi_f(h_\infty, h^\infty).$$

So we get invariance under γ, but we need to change what $\mathrm{SO}(2)$ does. In other words, Φ_f is now a function

$$\Phi_f : \mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{G}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

satisfying various properties:

- It generates a finite-dimensional representation of $\mathrm{SO}(2)$
- It is invariant under an open subset of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$
- It satisfies growth condition and cuspidality, etc.

By the Cauchy–Riemann equations, the holomorphicity condition of f says Φ_f satisfies some differential equation. In particular, that says Φ_f is an eigenfunction for the Casimir in the universal enveloping algebra of $\mathfrak{sl}_2$. These conditions together define the space of *automorphic forms*.

Example. Take the non-holomorphic Eisenstein series

$$E(z, s) = \sum_{(c,d)\neq(0,0)} \frac{1}{|cz + d|^{2s}}.$$

This is a real analytic function on $\Gamma\backslash\mathbb{H} \to \mathbb{C}$. Using the above process, we get an automorphic form on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $k = 0$. So it actually invariant under $\mathrm{SO}(2)$. It satisfies

$$\Delta E = s(1 - s)E.$$

There exist automorphic cusp forms which are invariant under $\mathrm{SO}(2)$, known as Maass forms. They also satisfy a differential equation with a Laplacian eigenvalue λ. A famous conjecture of Selberg says $\lambda \geq \frac{1}{4}$. What this is equivalent to is that the representation of $\mathrm{GL}_2(\mathbb{R})$ they generate is tempered.

In terms of representation theory, this looks quite similar to Ramanujan's conjecture!

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