Part IV — Bounded Cohomology

Theorems

Based on lectures by M. Burger

Notes taken by Dexter Chua

Easter 2017

The cohomology of a group or a topological space in degree k is a real vector space which describes the "holes" bounded by k dimensional cycles and encodes their relations. Bounded cohomology is a refinement which provides these vector spaces with a (semi) norm and hence topological objects acquire mysterious numerical invariants. This theory, introduced in the beginning of the 80's by M. Gromov, has deep connections with the geometry of hyperbolic groups and negatively curved manifolds. For instance, hyperbolic groups can be completely characterized by the "size" of their bounded cohomology.

The aim of this course is to give an introduction to the bounded cohomology of groups, and treat more in detail one of its important applications to the study of groups acting by homeomorphisms on the circle. More precisely we will treat the following topics:

(i) Ordinary and bounded cohomology of groups: meaning of these objects in low degrees, that is, zero, one and two; relations with quasimorphisms. Proof that the bounded cohomology in degree two of a non abelian free group contains an isometric copy of the Banach space of bounded sequences of reals. Examples and meaning of bounded cohomology classes of geometric origin with non trivial coefficients.

(ii) Actions on the circle, the bounded Euler class: for a group acting by orientation preserving homeomorphisms of the circle, Ghys has introduced an invariant, the bounded Euler class of the action, and shown that it characterizes (minimal) actions up to conjugation. We will treat in some detail this work as it leads to important applications of bounded cohomology to the question of which groups can act non trivially on the circle: for instance $\mathrm{SL}(2, \mathbb{Z})$ can, while lattices in "higher rank Lie groups", like $\mathrm{SL}(n, \mathbb{Z})$ for n at least 3, can't.

(iii) Amenability and resolutions: we will set up the abstract machinery of resolutions and the notions of injective modules in ordinary as well as bounded cohomology; this will provide a powerful way to compute these objects in important cases. A fundamental role in this theory is played by various notions of amenability; the classical notion of amenability for a group, and amenability of a group action on a measure space, due to R. Zimmer. The goal is then to describe applications of this machinery to various rigidity questions, and in particular to the theorem due, independently to Ghys, and Burger–Monod, that lattices in higher rank groups don't act on the circle.

Pre-requisites

Prerequisites for this course are minimal: no prior knowledge of group cohomology of any form is needed; we'll develop everything we need from scratch. It is however an advantage to have a "zoo" of examples of infinite groups at one's disposal: for example free groups and surface groups. In the third part, we'll need basic measure theory; amenability and ergodic actions will play a role, but there again everything will be built up on elementary measure theory.

The basic reference for this course is R. Frigerio, "Bounded cohomology of discrete groups", arXiv:1611.08339, and for part 3, M. Burger & A. Iozzi, "A useful formula from bounded cohomology", available at: `https://people.math.ethz.ch/~iozzi/publications.html`.

Contents

1 Quasi-homomorphisms **4**
1.1 Quasi-homomorphisms . . . 4
1.2 Relation to commutators . . . 4
1.3 Poincare translation quasimorphism . . . 5

2 Group cohomology and bounded cohomology **6**
2.1 Group cohomology . . . 6
2.2 Bounded cohomology of groups . . . 6

3 Actions on S^1 **7**
3.1 The bounded Euler class . . . 7
3.2 The real bounded Euler class . . . 7

4 The relative homological approach **8**
4.1 Injective modules . . . 8
4.2 Amenable actions . . . 8

1 Quasi-homomorphisms

1.1 Quasi-homomorphisms

Lemma. Let $f \in \mathcal{QH}(G, A)$. Then for every $g \in G$, the limit

$$Hf(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}$$

exists in $\mathbb{R}$. Moreover,

(i) $Hf\colon G \to \mathbb{R}$ is a homogeneous quasi-homomorphism.

(ii) $f - Hf \in \ell^\infty(G, \mathbb{R})$.

Corollary. We have

$$\mathcal{QH}(G, \mathbb{R}) = \mathcal{QH}_h(G, \mathbb{R}) \oplus \ell^\infty(G, \mathbb{R})$$

Lemma. Let $f\colon G \to \mathbb{R}$ be a homogeneous quasi-homomorphism.

(i) We have $f(xyx^{-1}) = f(y)$ for all $x, y \in G$.

(ii) If G is abelian, then f is in fact a homomorphism. Thus

$$\mathcal{QH}_h(G, \mathbb{R}) = \mathrm{Hom}(G, \mathbb{R}).$$

Theorem (P. Rolli, 2009)**.** The function $f_{\alpha,\beta}$ is a quasi-homomorphism, and the map

$$\ell^\infty_{\mathrm{odd}}(\mathbb{Z}) \oplus \ell^\infty_{\mathrm{odd}}(\mathbb{Z}) \to \frac{\mathcal{QH}(F_2, \mathbb{R})}{\ell^\infty(F_2, \mathbb{R}) + \mathrm{Hom}(F_2, \mathbb{R})}$$

is injective.

Theorem (Hull–Osin 2013)**.** The space

$$\frac{\mathcal{QH}(G, \mathbb{R})}{\ell^\infty(G, \mathbb{R}) + \mathrm{Hom}(G, \mathbb{R})}$$

is infinite-dimensional if G is acylindrically hyperbolic.

1.2 Relation to commutators

Lemma. If f is a homogeneous quasi-homomorphism and $x, y \in G$, then

$$|f([x, y])| \leq D(f).$$

Lemma (Bavard, 1992)**.** If f is a homogeneous quasi-homomorphism, then

$$\sup_{x,y} |f([x, y])| = D(f).$$

Lemma. For $a \in [G, G]$, we have

$$|f(a)| \leq 2D(f)\,\mathrm{cl}(a).$$

Proposition.

$$|f(a)| \leq 2D(f)\mathrm{scl}(a).$$

Theorem (Bavard, 1992)**.** For all $a \in [G, G]$, we have

$$\mathrm{scl}(a) = \frac{1}{2} \sup_{\phi \in \mathcal{QH}_h(G,\mathbb{R})} \frac{|\phi(a)|}{|D(\phi)|},$$

where, of course, we skip over those $\phi \in \mathrm{Hom}(G, \mathbb{R})$ in the supremum to avoid division by zero.

Corollary. The stable commutator length vanishes identically iff every homogeneous quasi-homomorphism is a homomorphism.

Theorem (Carder–Keller 1983)**.** For $n \geq 3$, we have

$$\mathrm{SL}(n, \mathbb{Z}) = [\mathrm{SL}(n, \mathbb{Z}), \mathrm{SL}(n, \mathbb{Z})],$$

and the commutator length is bounded.

Theorem (D. Witte Morris, 2007)**.** Let $\mathcal{O}$ be the ring of integers of some number field. Then $\mathrm{cl}\colon [\mathrm{SL}(n, \mathcal{O}), \mathrm{SL}(n, \mathcal{O})] \to \mathbb{R}$ is bounded iff $n \geq 3$ or $n = 2$ and $\mathcal{O}^\times$ is infinite.

Theorem (Burger–Monod, 2002)**.** Let $\Gamma < G$ be an irreducible lattice in a connected semisimple group G with finite center and rank $G \geq 2$. Then every homogeneous quasimorphism $\Gamma \to \mathbb{R}$ is $\equiv 0$.

Theorem (Burger–Monod, 2009)**.** Let Γ be a finitely-generated group and let μ be a symmetric probability measure on Γ whose support generates Γ. Then every class in $\mathcal{QH}(\Gamma, \mathbb{R})/\ell^\infty(\Gamma, \mathbb{R})$ has a unique μ-harmonic representative. In addition, this harmonic representative f satisfies the following:

$$\|\mathrm{d}f\|_\infty \leq \|\mathrm{d}g\|_\infty$$

for any $g \in f + \ell^\infty(\Gamma, R)$.

1.3 Poincare translation quasimorphism

Proposition. Every lift $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ of an orientation preserving homeomorphism $\varphi\colon S^1 \to S^1$ is a monotone increasing homeomorphism of $\mathbb{R}$, commuting with translation by $\mathbb{Z}$, i.e.

$$\tilde{\varphi} \circ T_m = T_m \circ \tilde{\varphi}$$

for all $m \in \mathbb{Z}$.

Conversely, any such map is a lift of an orientation-preserving homeomorphism.

Lemma. The function $F\colon \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ given by $\varphi \mapsto \varphi(0)$ is a quasi-homomorphism.

2 Group cohomology and bounded cohomology

2.1 Group cohomology

Lemma.

(i) $d^{(k)}$ is a Γ-equivariant group homomorphism.

(ii) $d^{(k+1)} \circ d^{(k)} = 0$. So $\operatorname{im} d^{(k)} \subseteq \ker d^{(k+1)}$.

(iii) In fact, we have $\operatorname{im} d^{(k)} = \ker d^{(k+1)}$.

Lemma. A homomorphism $f : \Gamma \to \Gamma'$ of groups induces a natural map $f^* : H^k(\Gamma', \mathbb{Z}) \to H^k(\Gamma, \mathbb{Z})$ for all k. Moreover, if $g : \Gamma' \to \Gamma''$ is another group homomorphism, then $f^* \circ g^* = (gf)^*$.

Proposition. $H^0(\Gamma, A) \cong A$.

Proposition. $H^1(\Gamma, A) = \operatorname{Hom}(\Gamma, A)$.

Proposition. $H^2(\Gamma, A)$ parametrizes the set of isomorphism classes of central extensions of Γ by A.

Lemma. We have $c(f_1, f_2) \in \{0, 1\}$.

Theorem (Milnor–Wood)**.** If $h\colon \Gamma_g \to \operatorname{Homeo}^+(S^1)$, then $|h^*(e)| \leq 2g - 2$.

Theorem (Gauss–Bonnet)**.** If $h\colon \Gamma_g \to \operatorname{PSL}(2, \mathbb{R}) \subseteq \operatorname{Homeo}^+(S^1)$ is the holonomy representation of a hyperbolic structure, then

$$h^*(e) = \pm(2g - 2).$$

Theorem (Matsumoko, 1986)**.** If h defines a minimal action of Γ_g on S^1 and $|h^*(e)| = 2g - 2$, then h is conjugate to a hyperbolization.

2.2 Bounded cohomology of groups

Proposition. The map $\bar{d}^2$ induces an isomorphism

$$\frac{\mathcal{QH}(\Gamma, A)}{\ell^\infty(\Gamma, A) + \operatorname{Hom}(\Gamma, A)} \cong \ker c_2.$$

Theorem. Assume Γ is finitely-generated. Let G_α be the central extension of Γ by $\mathbb{Z}$, defined by a class in $H^2(\Gamma, \mathbb{Z})$ which admits a bounded representative. Then with any word metric, Γ_α is quasi-isometric to $\Gamma \times \mathbb{Z}$ via the "identity map".

Proposition. Let Γ be an amenable group. Then $H_b^k(\Gamma, \mathbb{R}) = 0$ for $k \geq 1$.

3 Actions on S^1

3.1 The bounded Euler class

Lemma. If h_1 and h_2 are minimal actions that are semiconjugate via φ_1 and φ_2, then φ_1 and φ_2 are homeomorphisms and are inverses of each other.

Theorem (F. Ghys, 1984)**.** Two actions h_1 and h_2 are semiconjugate iff $h_1^*(e^b) = h_2^*(e^b)$.

3.2 The real bounded Euler class

Corollary. An action h is semi-conjugate to an action by rotations iff $h^*(e^b_{\mathbb{R}}) = 0$.

Theorem. Let $h\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ be an action. Then one of the following holds:

(i) There is a finite orbit, and all finite orbits have the same cardinality.

(ii) The action is minimal.

(iii) There is a closed, minimal, invariant, infinite, proper subset $K \subsetneq S^1$ such that any $x \in S^1$, the closure of the orbit $\overline{h(\Gamma)x}$ contains K.

Corollary. Let $h\colon \Gamma \to S^1$ be an action. Then one of the following is true:

(i) $h^*(e^b_{\mathbb{R}}) = 0$ and h is semi-conjugate to an action by rotations.

(ii) $h^*(e^b_{\mathbb{R}}) \neq 0$, and then h is semi-conjugate to a minimal *unbounded* action, i.e. $\{h(\gamma) : \gamma \in \Gamma\}$ is not equicontinuous.

Lemma. A minimal compact subgroup $U \subseteq \mathrm{Homeo}^+(S^1)$ is conjugate to a subgroup of Rot.

Theorem (Ghys, Margulis)**.** If $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ is an action which is minimal and unbounded. Then the centralizer $C_{\mathrm{Homeo}^+(S^1)}(\rho(\Gamma))$ is finite cyclic, say $\langle\varphi\rangle$, and the factor action ρ_0 on $S^1/\langle\varphi\rangle \cong S^1$ is minimal and strongly proximal. We call this action the *strongly proximal quotient* of ρ.

Theorem (Burger, 2007)**.** Let G be a second-countable locally compact group, and $\Gamma < G$ be a lattice, and $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ a minimal unbounded action. Then the following are equivalent:

- $\rho^*(e^b_{\mathbb{R}})$ is in the image of the restriction map $H^2_{bc}(G, \mathbb{R}) \to H^2_b(\Gamma, \mathbb{R})$
- The strongly proximal quotient $\rho_{ss}\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ extends continuously to G.

Theorem (Burger–Monod, 2002)**.** The restriction map $H^2_{bc}(G) \to H^2_b(\Gamma, \mathbb{R})$ is an isomorphism in the following cases:

(i) $G = G_1 \times \cdots \times G_n$ is a cartesian product of locally compact groups and Γ has dense projections on each individual factor.

(ii) G is a connected semisimple Lie group with finite center and rank $G \geq 2$, and Γ is irreducible.

4 The relative homological approach

4.1 Injective modules

Theorem. Let $E^\bullet$ be an injective resolution of $\mathbb{R}$ Then

$$H^\bullet(E^{\bullet\Gamma}) \cong H_b^\bullet(\Gamma, \mathbb{R})$$

as topological vector spaces.

In case $E^\bullet$ admits contracting homotopies, this isomorphism is semi-norm decreasing.

Lemma.

- $\ell^\infty(\Gamma^n)$ for $n \geq 1$ are all injective Banach Γ-modules.
- $\ell^\infty_{\text{alt}}(\Gamma^n)$ for $n \geq 1$ are injective Banach Γ-modules as well.

Proposition. The trivial Γ-module $\mathbb{R}$ is injective iff Γ is amenable.

4.2 Amenable actions

Theorem (Burger–Monod, 2002)**.** Let $G \times S \to S$ be a non-singular action. Then the following are equivalent:

(i) The G action is amenable.

(ii) $L^\infty(S)$ is an injective G-module.

(iii) $L^\infty(S^n)$ for all $n \geq 1$ is injective.

Corollary. If (S, μ) is an amenable G-space, then we have an isometric isomorphism $H^\bullet(L^\infty(S^n, \mu)^G, d_n) \cong H^\bullet(L^\infty_{\text{alt}}(S^n, \mu)^G, d_n) \cong H_b(G, \mathbb{R})$.