Part IV — Bounded Cohomology

Definitions

Based on lectures by M. Burger
Notes taken by Dexter Chua
Easter 2017

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The cohomology of a group or a topological space in degree \( k \) is a real vector space which describes the “holes” bounded by \( k \) dimensional cycles and encodes their relations. Bounded cohomology is a refinement which provides these vector spaces with a (semi) norm and hence topological objects acquire mysterious numerical invariants. This theory, introduced in the beginning of the 80’s by M. Gromov, has deep connections with the geometry of hyperbolic groups and negatively curved manifolds. For instance, hyperbolic groups can be completely characterized by the “size” of their bounded cohomology.

The aim of this course is to give an introduction to the bounded cohomology of groups, and treat more in detail one of its important applications to the study of groups acting by homeomorphisms on the circle. More precisely we will treat the following topics:

(i) Ordinary and bounded cohomology of groups: meaning of these objects in low degrees, that is, zero, one and two; relations with quasimorphisms. Proof that the bounded cohomology in degree two of a non abelian free group contains an isometric copy of the Banach space of bounded sequences of reals. Examples and meaning of bounded cohomology classes of geometric origin with non trivial coefficients.

(ii) Actions on the circle, the bounded Euler class: for a group acting by orientation preserving homeomorphisms of the circle, Ghys has introduced an invariant, the bounded Euler class of the action, and shown that it characterizes (minimal) actions up to conjugation. We will treat in some detail this work as it leads to important applications of bounded cohomology to the question of which groups can act non trivially on the circle: for instance \( \text{SL}(2, \mathbb{Z}) \) can, while lattices in “higher rank Lie groups”, like \( \text{SL}(n, \mathbb{Z}) \) for \( n \) at least 3, can’t.

(iii) Amenability and resolutions: we will set up the abstract machinery of resolutions and the notions of injective modules in ordinary as well as bounded cohomology; this will provide a powerful way to compute these objects in important cases. A fundamental role in this theory is played by various notions of amenability; the classical notion of amenability for a group, and amenability of a group action on a measure space, due to R. Zimmer. The goal is then to describe applications of this machinery to various rigidity questions, and in particular to the theorem due, independently to Ghys, and Burger–Monod, that lattices in higher rank groups don’t act on the circle.
Pre-requisites

Prerequisites for this course are minimal: no prior knowledge of group cohomology of any form is needed; we'll develop everything we need from scratch. It is however an advantage to have a “zoo” of examples of infinite groups at one’s disposal: for example free groups and surface groups. In the third part, we’ll need basic measure theory; amenability and ergodic actions will play a role, but there again everything will be built up on elementary measure theory.

Contents

1 Quasi-homomorphisms 4
   1.1 Quasi-homomorphisms .......................... 4
   1.2 Relation to commutators ......................... 4
   1.3 Poincare translation quasimorphism .......... 4

2 Group cohomology and bounded cohomology 6
   2.1 Group cohomology ................................ 6
   2.2 Bounded cohomology of groups ................. 6

3 Actions on $S^1$ 7
   3.1 The bounded Euler class ....................... 7
   3.2 The real bounded Euler class ................. 7

4 The relative homological approach 8
   4.1 Injective modules .............................. 8
   4.2 Amenable actions .............................. 8
1 Quasi-homomorphisms

1.1 Quasi-homomorphisms

Definition (Quasi-homomorphism). Let $G$ be a group. A function $f: G \to A$ is a quasi-homomorphism if the function

$$df: G \times G \to A \quad (x, y) \mapsto f(x) + f(y) - f(xy)$$

is bounded. We define the defect of $f$ to be

$$D(f) = \sup_{x, y \in G} |df(x, y)|.$$ 

We write $QH(G, A)$ for the $A$-module of quasi-homomorphisms.

Notation. We write $\ell^\infty(G, A) = \{f: G \to A : f \text{ is bounded}\}$.

Definition (Homogeneous function). A function $f: G \to \mathbb{R}$ is homogeneous if for all $n \in \mathbb{Z}$ and $g \in G$, we have $f(g^n) = nf(g)$.

Notation. We write $QH^h(G, \mathbb{R})$ for the vector space of homogeneous quasi-homomorphisms $G \to \mathbb{R}$.

1.2 Relation to commutators

Definition (Commutator length). Let $a \in [G, G]$. Then commutator length $\text{cl}(a)$ is the word length with respect to the generators

$$\{[x, y] : x, y \in G\}.$$ 

In other words, it is the smallest $n$ such that

$$a = [x_1, y_1][x_2, y_2] \cdots [x_n, y_n]$$

for some $x_i, y_i \in G$.

Definition (Stable commutator length). The stable commutator length is defined by

$$\text{ scl}(a) = \lim_{n \to \infty} \frac{\text{cl}(a^n)}{n}.$$ 

1.3 Poincare translation quasimorphism

Definition (Positively-oriented triple). We say a triple of points $x_1, x_2, x_3 \in S^1$ is positively-oriented if they are distinct and ordered as follows:

$$x_1, x_2, x_3$$
More formally, recall that there is a natural covering map \( \pi: \mathbb{R} \to S^1 \) given by quotienting by \( \mathbb{Z} \). Formally, we let \( \tilde{x}_1 \in \mathbb{R} \) be any lift of \( x_1 \). Then let \( \tilde{x}_2, \tilde{x}_3 \) be the unique lifts of \( x_2 \) and \( x_3 \) respectively to \([\tilde{x}_1, \tilde{x}_1 + 1)\). Then we say \( x_1, x_2, x_3 \) are positively-oriented if \( \tilde{x}_2 < \tilde{x}_3 \).

**Definition** (Orientation-preserving map). A map \( S^1 \to S^1 \) is orientation-preserving if it sends positively-oriented triples to positively-oriented triples. We write \( \text{Homeo}^+(S^1) \) for the group of orientation-preserving homeomorphisms of \( S^1 \).

**Notation.** We write \( \text{Rot} \) for the group of rotations in \( \text{Homeo}^+(S^1) \). This corresponds to the subgroup \( T_{\mathbb{R}} \subseteq \text{Homeo}_Z^+(\mathbb{R}) \).

**Definition** (Poincare translation quasimorphism). The Poincare translation quasimorphism \( T: \text{Homeo}_Z^+(\mathbb{R}) \to \mathbb{R} \) is the homogenization of \( F \).

**Definition** (Rotation number). The rotation number of \( \varphi \in \text{Homeo}^+(S^1) \) is \( T(\tilde{\varphi}) \mod \mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z} \), where \( \tilde{\varphi} \) is a lift of \( \varphi \) to \( \text{Homeo}_Z^+(\mathbb{R}) \).
2 Group cohomology and bounded cohomology

2.1 Group cohomology

Definition (Homogeneous k-cochain). A homogeneous k-cochain with values in A is a function \( f: \Gamma^{k+1} \to A \). The set \( C(\Gamma^{k+1}, A) \) is an abelian group and \( \Gamma \) acts on it by automorphisms in the following way:

\[
(\gamma \ast f)(\gamma_0, \cdots, \gamma_m) = f(\gamma^{-1} \gamma_0, \cdots, \gamma^{-1} \gamma_k).
\]

By convention, we set \( C(\Gamma^0, A) \cong A \).

Definition (Differential \( d^{(k)} \)). We define the differential \( d^{(k)}: C(\Gamma^k, A) \to C(\Gamma^{k+1}, A) \) by

\[
(d^{(k)}f)(\gamma_0, \cdots, \gamma_k) = \sum_{j=0}^{k} (-1)^j f(\gamma_0, \cdots, \hat{\gamma}_j, \cdots, \gamma_k).
\]

In particular, we set \( d^{(0)}(a) \) to be the function that is constantly \( a \).

Definition (k-cocycle and k-coboundaries).

– The k-cocycles are \( \ker d^{(k+1)} \).
– The k-coboundaries are \( \text{im} d^{(k)} \).

Definition (Group cohomology \( H^k(\Gamma, A) \)). We define the \( k \)-th cohomology group to be

\[
H^k = \frac{(\ker d^{(k+1)})^\Gamma}{d^{(k)}(C(\Gamma^k, A)^\Gamma)} = \frac{(d^{(k)}(C(\Gamma^k, A)))^\Gamma}{d^{(k)}(C(\Gamma^k, A)^\Gamma)}.
\]

Definition (Central extension). Let \( A \) be an abelian group, and \( \Gamma \) a group. Then a central extension of \( \Gamma \) by \( A \) is an exact sequence

\[
0 \longrightarrow A \longrightarrow \hat{\Gamma} \longrightarrow \Gamma \longrightarrow 0
\]

such that the image of \( A \) is contained in the center of \( \hat{\Gamma} \).

Definition (Euler class). The Euler class of the \( \Gamma \)-action by orientation-preserving homeomorphisms of \( S^1 \) is

\[
h^*(e) \in H^2(\Gamma, \mathbb{Z}),
\]

where \( h: \Gamma \to \text{Homeo}^+(S^1) \) is the map defining the action.

2.2 Bounded cohomology of groups

Definition (Bounded cohomology). The \( k \)-th bounded cohomology group of \( \Gamma \) with coefficients in \( A \) is

\[
H^k_b(\Gamma, A) = \frac{\ker(d^{(k+1)}: C_0(\Gamma^{k+1}, A)^\Gamma \to C_0(\Gamma^{k+2}, A)^\Gamma)}{d^{(k)}(C_0(\Gamma^k, A)^\Gamma)}.
\]

Definition (Amenable group). A discrete group \( \Gamma \) is amenable if there is a linear form \( m: \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{R} \) such that

– \( m(f) \geq 0 \) if \( f \geq 0 \);
– \( m(1) = 1 \); and
– \( m \) is left-invariant, i.e. \( m(\gamma \ast f) = m(f) \), where \( (\gamma \ast f)(x) = f(\gamma^{-1}x) \).
3 Actions on $S^1$

3.1 The bounded Euler class

**Definition** (Bounded Euler class). The bounded Euler class

$$e^b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{Z})$$

is the bounded cohomology class represented by the cocycle $c$.

**Definition** (Bounded Euler class of action). The bounded Euler class of an action $h: \Gamma \to \text{Homeo}^+(S^1)$ is $h^*(e^b) \in H^2_b(\Gamma, \mathbb{Z})$.

**Definition** (Increasing map of degree 1). A map $\varphi: S^1 \to S^1$ is increasing of degree 1 if there is some $\tilde{\varphi}: \mathbb{R} \to \mathbb{R}$ lifting $\varphi$ such that $\tilde{\varphi}$ is is monotonically increasing and

$$\tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + 1$$

for all $x \in \mathbb{R}$.

**Definition** (Semiconjugate action). Two actions $h_1, h_2: \Gamma \to \text{Homeo}^+(S^1)$ are semi-conjugate if there are increasing maps of degree 1 $\varphi_1, \varphi_2: S^1 \to S^1$ such that

(i) $h_1(\gamma) \varphi_1 = \varphi_1 h_2(\gamma)$ for all $\gamma \in \Gamma$;

(ii) $h_2(\gamma) \varphi_2 = \varphi_2 h_1(\gamma)$ for all $\gamma \in \Gamma$.

**Definition** (Minimal action). An action on $S^1$ is minimal if every orbit is dense.

3.2 The real bounded Euler class

**Definition** (Real bounded Euler class). The real bounded Euler class is the class $e^b \in H^2_b(\text{Homeo}^+(S^1), \mathbb{R})$ obtained by change of coefficients from $\mathbb{Z} \to \mathbb{R}$.

The real bounded Euler class of an action $h: \Gamma \to \text{Homeo}^+(S^1)$ is the pullback

$$h^*(e^b) \in H^2_b(\Gamma, \mathbb{R})$$

**Definition** (Strongly proximal action). A $\Gamma$-action by homeomorphisms on a compact metrizable space $X$ is strongly proximal if for all probability measures $\mu$ on $X$, the weak-$*$ closure $\Gamma_\mu$ contains a Dirac mass.

**Definition** (Lattice). A lattice in a locally compact group $G$ is a discrete subgroup $\Gamma$ such that on $\Gamma \setminus G$, there is a $G$-invariant probability measure.
4 The relative homological approach

4.1 Injective modules

Definition (Banach Γ module). A Banach Γ-module is a Banach space $V$ together with an action $\Gamma \times V \to V$ by linear isometries.

Definition (Admissible morphism). An injective morphism $i: A \to B$ of Banach spaces is admissible if there exists $\sigma: B \to A$ with

- $\sigma i = \text{id}_A$; and
- $\|\sigma\| \leq 1$.

Definition (Injective Banach Γ-module). A Banach Γ-module is injective if for any diagram

\begin{equation}
\begin{array}{ccc}
A & \rightarrow^i & B \\
\downarrow^\alpha & & \\
E & \leftarrow^\beta & \\
\end{array}
\end{equation}

where $i$ and $\alpha$ are morphisms of Γ-modules, and $i$ is injective admissible, then there exists $\beta: B \to E$ a morphism of Γ-modules such that

\begin{equation}
\begin{array}{ccc}
A & \rightarrow^i & B \\
\downarrow^\alpha & & \\
E & \leftarrow^\beta & \\
\end{array}
\end{equation}

commutes and $\|\beta\| \leq \|\alpha\|$.

Definition (Injective resolution). Let $V$ be a Banach Γ-module. An injective resolution of $V$ is an exact sequence

\begin{equation}
\begin{array}{ccc}
V & \rightarrow & E_0 \\
& & \rightarrow \\
& & \rightarrow \\
& & \rightarrow \\
& & \rightarrow \\
& & \rightarrow \cdots
\end{array}
\end{equation}

where each $E_k$ is injective.

4.2 Amenable actions

Definition (Conditional expectation). A conditional expectation on $G \times S$ is a linear map $M: L^\infty(G \times S) \to L^\infty(S)$ such that

(i) $M(1) = 1$;

(ii) If $M \geq 0$, then $M(f) \geq 0$; and

(iii) $M$ is $L^\infty(S)$-linear.

Definition (Amenable action). A $G$-action on $S$ is amenable if there exists a $G$-equivariant conditional expectation.