Probability and Measure 1

1.1. Let $E$ be a set and let $S$ be a set of $\sigma$-algebras on $E$. Define
\[ \mathcal{E}^* = \{ A \subseteq E : A \in \mathcal{E} \text{ for all } \mathcal{E} \in S \}. \]
Show that $\mathcal{E}^*$ is a $\sigma$-algebra on $E$. Show, on the other hand, by example, that the union of two $\sigma$-algebras on the same set need not be a $\sigma$-algebra.

1.2. Show that the following sets of subsets of $\mathbb{R}$ all generate the same $\sigma$-algebra:
\begin{itemize}
  \item[(a)] $\{(a, b) : a < b\}$,
  \item[(b)] $\{(a, b] : a < b\}$,
  \item[(c)] $\{(-\infty, b] : b \in \mathbb{R}\}$.
\end{itemize}

1.3. Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4. Show that a $\pi$-system which is also a $d$-system is a $\sigma$-algebra.

1.5. Let $\mu$ be a finite-valued additive set function on a ring $A$. Show that $\mu$ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $A$, with $\cap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6. Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$,
\[ \mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n). \]
Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events $A_n$ are independent if and only if the $\sigma$-algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8. Let $B$ be a Borel subset of the interval $[0, 1]$. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \Delta B$ ($= (A^c \cap B) \cup (A \cap B^c)$) is less than $\varepsilon$. Show further that this remains true for every Borel set in $\mathbb{R}$ of finite Lebesgue measure.

1.9. Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ null if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{ A \cup N : A \in \mathcal{E}, N \in \mathcal{N} \}$ is a $\sigma$-algebra and show that $\mu$ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the completion of $\mathcal{E}$ with respect to $\mu$. Suppose now that $E$ is $\sigma$-finite and write $\mu^*$ for the outer measure associated to $\mu$, as in the proof of Carathéodory’s Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of $\mu^*$-measurable sets.
2.1. Let \((f_n : n \in \mathbb{N})\) be a sequence of measurable functions on a measurable space \((E, \mathcal{E})\). Show that the following functions are also measurable: \(f_1 + f_2, f_1 f_2, \inf_n f_n, \sup_n f_n, \lim \inf_n f_n, \lim \sup_n f_n\). Show also that \(\{x \in E : f_n(x) \text{ converges as } n \to \infty\} \in \mathcal{E}\).

2.2. Let \((E, \mathcal{E})\) and \((G, \mathcal{G})\) be measurable spaces, let \(\mu\) be a measure on \(\mathcal{E}\), and let \(f : E \to G\) be a measurable function. Show that we can define a measure \(\nu\) on \(\mathcal{G}\) by setting \(\nu(A) = \mu(f^{-1}(A))\) for each \(A \in \mathcal{G}\).

2.3. Show that the following condition implies that random variables \(X\) and \(Y\) are independent: 
\[
\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) \mathbb{P}(Y \leq y)
\]
for all \(x, y \in \mathbb{R}\).

2.4. Let \((A_n : n \in \mathbb{N})\) be a sequence of events, with \(\mathbb{P}(A_n) = 1/n^2\) for all \(n\). Set \(X_n = n^2 1_{A_n} - 1\) and set \(X_n = (X_1 + \cdots + X_n)/n\). Show that \(\mathbb{E}(X_n) = 0\) for all \(n\), but that \(X_n \to -1\) almost surely as \(n \to \infty\).

2.5. The zeta function is defined for \(s > 1\) by \(\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\). Let \(X\) and \(Y\) be independent random variables with 
\[
\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).
\]
Write \(A_n\) for the event that \(n\) divides \(X\). Show that the events \((A_p : p \text{ prime})\) are independent and deduce Euler’s formula
\[
\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).
\]
Show also that \(\mathbb{P}(X\text{ is square-free}) = 1/\zeta(2s)\). Write \(H\) for the highest common factor of \(X\) and \(Y\). Show finally that \(\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)\).

2.6. Let \((X_n : n \in \mathbb{N})\) be independent \(N(0, 1)\) random variables. Prove that 
\[
\lim_{n \to \infty} \sup_n \left(\frac{X_n}{\sqrt{2 \log n}}\right) = 1 \quad \text{a.s.}
\]

2.7. Let \(C_n\) denote the \(n\)th approximation to the Cantor set \(C\): thus \(C_0 = [0, 1]\), \(C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]\), \(C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]\), etc. and \(C_n \downarrow C\) as \(n \to \infty\). Denote by \(F_n\) the distribution function of a random variable uniformly distributed on \(C_n\). Show that

(a) \(C\) is uncountable and has Lebesgue measure 0,
(b) for all \(x \in [0, 1]\), the limit \(F(x) = \lim_{n \to \infty} F_n(x)\) exists,
(c) the function \(F\) is continuous on \([0, 1]\), with \(F(0) = 0\) and \(F(1) = 1\),
(d) for almost all \(x \in [0, 1]\), \(F\) is differentiable at \(x\) with \(F'(x) = 0\).

*Hint: express \(F_{n+1}\) recursively in terms of \(F_n\) and use this relation to obtain a uniform estimate on \(F_{n+1} - F_n\).*
Probability and Measure 2

3.1. Suppose that a simple function $f$ has two representations

$$ f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{j=1}^{n} b_k 1_{B_k}. $$

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \ldots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly $B_\delta$ for $\delta \in \{0, 1\}^n$. Then set

$$ f_{\varepsilon, \delta} = \sum_{k=1}^{m} \varepsilon_k a_k $$

if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon, \delta} = 0$ otherwise. Show that, for any measure $\mu$,

$$ \sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon, \delta} f_{\varepsilon, \delta} \mu(A_\varepsilon \cap B_\delta) $$

and deduce that

$$ \sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j). $$

3.2. Let $\mu$ and $\nu$ be finite Borel measures on $\mathbb{R}$. Let $f$ be a continuous bounded function on $\mathbb{R}$. Show that $f$ is integrable with respect to $\mu$ and $\nu$. Show further that, if $\mu(f) = \nu(f)$ for all such $f$, then $\mu = \nu$.

3.3. Let $f$ be an integrable function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that, for some $\pi$-system $\mathcal{A}$ containing $E$ and generating $\mathcal{E}$, we have $\mu(f 1_A) = 0$ for all $A \in \mathcal{A}$. Show that $f = 0$ a.e.

3.4. Let $X$ be a non-negative integer-valued random variable. Show that

$$ \mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n). $$

Deduce that, if $\mathbb{E}(X) = \infty$ and $X_1, X_2, \ldots$ is a sequence of independent random variables with the same distribution as $X$, then, almost surely, $\limsup_n (X_n/n) \geq 1$, and moreover $\limsup_n (X_n/n) = \infty$.

Now suppose that $Y_1, Y_2, \ldots$ is any sequence of independent identically distributed random variables with $\mathbb{E}|Y_1| = \infty$. Show that, almost surely, $\limsup_n (|Y_n|/n) = \infty$, and moreover $\limsup_n (|Y_1 + \cdots + Y_n|/n) = \infty$.

3.5. For $\alpha \in (0, \infty)$ and $x \in (0, \infty)$, define $f_\alpha(x) = x^{-\alpha}$. Show that $f_\alpha$ is integrable with respect to Lebesgue measure on $(0, 1]$ if and only if $\alpha < 1$. Show also that $f_\alpha$ is integrable with respect to Lebesgue measure on $[1, \infty)$ if and only if $\alpha > 1$. 

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3.6. Show that the function \(\sin x/x\) is not Lebesgue integrable over \([1, \infty)\) but that integral \(\int_1^\infty (\sin x/x)\,dx\) converges as \(N \to \infty\).

3.7. Show that, as \(n \to \infty\),
\[
\int_0^\infty \frac{\sin(e^x)}{1 + nx^2}\,dx \to 0 \quad \text{and} \quad \int_0^1 \frac{(n \cos x)/(1 + n^2 x^2)}{\cos \left(\frac{x}{n}\right)}\,dx \to 0.
\]

3.8. Let \(u\) and \(v\) be differentiable functions on \(\mathbb{R}\) with continuous derivatives \(u'\) and \(v'\). Suppose that \(uv'\) and \(u'v\) are integrable on \(\mathbb{R}\) and \(u(x)v(x) \to 0\) as \(|x| \to \infty\). Show that
\[
\int_{\mathbb{R}} u(x)v'(x)\,dx = - \int_{\mathbb{R}} u'(x)v(x)\,dx.
\]

3.9. Let \((E, \mathcal{E})\) and \((G, \mathcal{G})\) be measurable spaces and let \(f : E \to G\) be a measurable function. Given a measure \(\mu\) on \((E, \mathcal{E})\), consider the image measure \(\nu = \mu \circ f^{-1}\) on \((G, \mathcal{G})\). Show that \(\nu(g) = \mu(g \circ f)\) for all non-negative measurable functions \(g\) on \(G\).

3.10. The moment generating function \(\phi\) of a real-valued random variable \(X\) is defined by \(\phi(\theta) = \mathbb{E}(e^{\theta X})\), \(\theta \in \mathbb{R}\).
Suppose that \(\phi\) is finite on an open interval containing 0. Show that \(\phi\) has derivatives of all orders at 0 and that \(X\) has finite moments of all orders given by
\[
\mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n_{\theta=0} \phi(\theta).
\]

3.11. Let \(X_1, \ldots, X_n\) be random variables with density functions \(f_1, \ldots, f_n\) respectively. Suppose that the \(\mathbb{R}^n\)-valued random variable \(X = (X_1, \ldots, X_n)\) also has a density function \(f\). Show that \(X_1, \ldots, X_n\) are independent if and only if
\[
f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n) \quad \text{a.e.}
\]

3.12. Show that, for all non-negative measurable functions \(f\) on \([0, \infty)\), the function \((x, y) \mapsto f(||(x, y)||)\) is measurable on \(\mathbb{R}^2\) and (without using the Jacobian formula)
\[
\int_{\mathbb{R}^2} f(||(x, y)||)\,dxdy = 2\pi \int_0^\infty rf(r)\,dr.
\]
Hence show that \((2\pi)^{-1/2}e^{-x^2/2}\) is a probability density function.

3.13. Let \(\mu\) and \(\nu\) be probability measures on \((E, \mathcal{E})\) and let \(f : E \to [0, R]\) be a measurable function.
Suppose that \(\nu(A) = \mu(f1_A)\) for all \(A \in \mathcal{E}\). Let \((X_n : n \in \mathbb{N})\) be a sequence of independent random variables in \(E\) with law \(\mu\) and let \((U_n : n \in \mathbb{N})\) be a sequence of independent \(U[0, 1]\) random variables. Set
\[
T = \min\{n \in \mathbb{N} : RU_n \leq f(X_n)\}, \quad Y = X_T.
\]
Show that \(Y\) has law \(\nu\). (This justifies simulation by rejection sampling.)
Probability and Measure 3

4.1. Let \((f_n : n \in \mathbb{N})\) be a sequence of integrable functions and suppose that \(f_n \to f\) a.e. for some integrable function \(f\). Show that, if \(\|f_n\|_1 \to \|f\|_1\), then \(\|f_n - f\|_1 \to 0\).

4.2. Let \(X\) be a random variable and let \(1 \leq p < \infty\). Show that, if \(X \in L^p(\mathbb{P})\), then \(P(|X| \geq \lambda) = O(\lambda^{-q})\) as \(\lambda \to \infty\), where \(q = \frac{p}{p-1}\). Prove the identity

\[
E(|X|^p) = \int_0^\infty p \lambda^{p-1} P(|X| \geq \lambda) d\lambda
\]

and deduce that, for all \(q > p\), if \(P(|X| \geq \lambda) = O(\lambda^{-q})\) as \(\lambda \to \infty\), then \(X \in L^p(\mathbb{P})\).

4.3. Give a simple proof of Schwarz’ inequality \(\|fg\|_1 \leq \|f\|_2 \|g\|_2\) for measurable functions \(f\) and \(g\).

4.4. Show that \(\|XY\|_1 = \|X\|_1 \|Y\|_1\) for independent random variables \(X\) and \(Y\). Show further that, if \(X\) and \(Y\) are also integrable, then \(E(XY) = E(X)E(Y)\).

4.5. A stepfunction \(f : \mathbb{R} \to \mathbb{R}\) is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions \(\mathcal{I}\) is dense in \(L^p(\mathbb{R})\) for all \(p \in [1, \infty)\): that is, for all \(f \in L^p(\mathbb{R})\) and all \(\varepsilon > 0\) there exists \(g \in \mathcal{I}\) such that \(\|f - g\|_p < \varepsilon\). Deduce that the set of continuous functions of compact support is also dense in \(L^p(\mathbb{R})\) for all \(p \in [1, \infty)\).

4.6. Let \((X_n : n \in \mathbb{N})\) be an identically distributed sequence in \(L^2(\mathbb{P})\). Show that \(n P(|X_1| > \varepsilon \sqrt{n}) \to 0\) as \(n \to \infty\), for all \(\varepsilon > 0\). Deduce that \(n^{-1/2} \max_{k \leq n} |X_k| \to 0\) in probability.

5.1. Let \((E, \mathcal{E}, \mu)\) be a measure space and let \(V_1 \leq V_2 \leq \ldots\) be an increasing sequence of closed subspaces of \(L^2 = L^2(E, \mathcal{E}, \mu)\) for \(f \in L^2\), denote by \(f_n\) the orthogonal projection of \(f\) on \(V_n\). Show that \(f_n\) converges in \(L^2\).

5.2. Let \(X = (X_1, \ldots, X_n)\) be a random variable, with all components in \(L^2(\mathbb{P})\). The covariance matrix \(\text{var}(X) = (c_{ij} : 1 \leq i, j \leq n)\) of \(X\) is defined by \(c_{ij} = \text{cov}(X_i, X_j)\). Show that \(\text{var}(X)\) is a non-negative definite matrix.

6.1. Find a uniformly integrable sequence of random variables \((X_n : n \in \mathbb{N})\) such that both \(X_n \to 0\) a.s. and \(E(\sup_n |X_n|) = \infty\).

6.2. Let \((X_n : n \in \mathbb{N})\) be an identically distributed sequence in \(L^2(\mathbb{P})\). Show that

\[
E(\max_{k \leq n} |X_k|) / \sqrt{n} \to 0 \quad \text{as} \quad n \to \infty.
\]
7.1. Let $u, v \in L^1(\mathbb{R}^d)$ and define $f : \mathbb{R}^d \to \mathbb{C}$ by $f(x) = u(x) + iv(x)$. Set
\[
\int_{\mathbb{R}^d} f(x) dx = \int_{\mathbb{R}^d} u(x) dx + i \int_{\mathbb{R}^d} v(x) dx.
\]
Show that, for all $y \in \mathbb{R}^d$, we have
\[
\int_{\mathbb{R}^d} f(x - y) dx = \int_{\mathbb{R}^d} f(x) dx = \int_{\mathbb{R}^d} f(-x) dx
\]
and show that
\[
\left| \int_{\mathbb{R}^d} f(x) dx \right| \leq \int_{\mathbb{R}^d} |f(x)| dx.
\]

7.2. Show that the Fourier transform of a finite Borel measure on $\mathbb{R}^d$ is a bounded continuous function.

7.3. Determine which of the following distributions on $\mathbb{R}$ have an integrable characteristic function: $N(\mu, \sigma^2)$, Bin($N, p$), Poisson($\lambda$), $U[0, 1]$.

7.4. For a finite Borel measure $\mu$ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of $\mu$ has a $k$th continuous derivative, which at 0 is given by
\[
\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).
\]

7.5. Define a function $\psi$ on $\mathbb{R}$ by setting $\psi(x) = C \exp\{-(1 - x^2)^{-1}\}$ for $|x| < 1$ and $\psi(x) = 0$ otherwise, where $C$ is a constant chosen so that $\int_{\mathbb{R}} \psi(x) dx = 1$. For $f \in L^1(\mathbb{R})$ of compact support, show that $f * \psi$ is $C^\infty$ and of compact support.

7.6. (i) Show that for any real numbers $a, b$ one has $\int_a^b e^{ix} dx \to 0$ as $|t| \to \infty$.
(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform
\[
\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx
\]
tends to 0 as $|t| \to \infty$. This is the Riemann–Lebesgue Lemma.

7.7. Say that $f \in L^2(\mathbb{R})$ is $L^2$-differentiable with $L^2$-derivative $Df$ if
\[
\|\tau_h f - f - hDf\|_2 / h \to 0 \quad \text{as} \quad h \to 0,
\]
where $\tau_h f(x) = f(x + h)$. Show that the function $f(x) = \max(1 - |x|, 0)$ is $L^2$-differentiable and find its $L^2$-derivative.

Suppose that $f \in L^1 \cap L^2$ is $L^2$-differentiable. Show that $u\hat{f}(u) \in L^2$. Deduce that $f$ has a continuous version and that $\|f\|_\infty \leq C \| (1 + |u|) \hat{f}(u) \|_2$ for some absolute constant $C < \infty$, to be determined. This is a simple example of a Sobolev inequality.
Probability and Measure 4

7.8. Let \((X_n : n \in \mathbb{N})\) be a sequence of random variables in \(\mathbb{R}\) and let \(X\) be another such random variable. Show that \(X_n \to X\) weakly if and only if \(X_n \to X\) in distribution.

7.9. Let \(\mu\) be a Borel probability measure on \(\mathbb{R}^d\) and let \((\mu_n : n \in \mathbb{N})\) be a sequence of such measures. Suppose that \(\mu_n(f) \to \mu(f)\) for all \(C^\infty\) functions on \(\mathbb{R}^d\) of compact support. Show that \(\mu_n\) converges weakly to \(\mu\) on \(\mathbb{R}^d\).

8.1. Let \(X = (X_1, \ldots, X_n)\) be a Gaussian random variable in \(\mathbb{R}^n\) with mean \(\mu\) and covariance matrix \(V\). Assume that \(V\) is invertible write \(V^{-1/2}\) for the positive-definite square root of \(V^{-1}\). Set \(Y = (Y_1, \ldots, Y_n) = V^{-1/2}(X - \mu)\). Show that \(Y_1, \ldots, Y_n\) are independent \(N(0, 1)\) random variables. Show further that we can write \(X_2\) in the form \(X_2 = aX_1 + Z\) where \(Z\) is independent of \(X_1\) and determine the distribution of \(Z\).

8.2. Let \(X_1, \ldots, X_n\) be independent \(N(0, 1)\) random variables. Show that
\[
\left(\frac{X}{\sqrt{n}}, \sum_{m=1}^n (X_m - \overline{X})^2\right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2\right)
\]
have the same distribution, where \(\overline{X} = (X_1 + \cdots + X_n)/n\).

9.1. Let \((E, \mathcal{E}, \mu)\) be a measure space and \(\tau : E \to E\) a measure-preserving transformation. Show that \(\mathcal{E}_\tau := \{A \in \mathcal{E} : \tau^{-1}(A) = A\}\) is a \(\sigma\)-algebra, and that a measurable function \(f\) is \(\mathcal{E}_\tau\)-measurable if and only if it is invariant, that is \(f \circ \tau = f\).

9.2. Show that, if \(\theta\) is an ergodic measure-preserving transformation and \(f\) is a \(\theta\)-invariant function, then there exists a constant \(c \in \mathbb{R}\) such that \(f = c\) a.e..

9.3. For \(x \in [0, 1)\), set \(\tau(x) = 2x \text{ mod } 1\). Show that \(\tau\) is a measure-preserving transformation of \(([0, 1), \mathcal{B}([0, 1]), dx)\), and that \(\tau\) is ergodic. Identify the invariant function \(\overline{f}\) corresponding to each integrable function \(f\).

9.4. Fix \(a \in [0, 1)\) and define, for \(x \in [0, 1)\), \(\tau(x) = x + a \text{ mod } 1\). Show that \(\tau\) is also a measure-preserving transformation of \(([0, 1), \mathcal{B}([0, 1]), dx)\). Determine for which values of \(a\) the transformation \(\tau\) is ergodic. Hint: you may use the fact that any integrable function \(f\) on \([0, 1)\) whose Fourier coefficients all vanish must itself vanish a.e.. Identify, for all values of \(a\), the invariant function \(\overline{f}\) corresponding to an integrable function \(f\).
9.5. Call a sequence of random variables \((X_n : n \in \mathbb{N})\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) stationary if for each \(n, k \in \mathbb{N}\) the random vectors \((X_1, \ldots, X_n)\) and \((X_{k+1}, \ldots, X_{k+n})\) have the same distribution: for \(A_1, \ldots, A_n \in \mathcal{B}\),
\[
\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \ldots, X_{k+n} \in A_n).
\]
Show that, if \((X_n : n \in \mathbb{N})\) is a stationary sequence and \(X_1 \in L^p\), for some \(p \in [1, \infty)\), then
\[
\frac{1}{n} \sum_{i=1}^{n} X_i \to X \quad \text{a.s. and in } L^p,
\]
for some random variable \(X \in L^p\) and find \(\mathbb{E}(X)\).

10.1. Let \((X_n : n \in \mathbb{N})\) be a sequence of independent random variables, such that \(\mathbb{E}(X_n) = \mu\) and \(\mathbb{E}(X_n^2) \leq M\) for all \(n\), for some constants \(\mu \in \mathbb{R}\) and \(M < \infty\). Set \(P_n = X_1X_2 + X_2X_3 + \cdots + X_{n-1}X_n\). Show that \(P_n/n\) converges a.s. as \(n \to \infty\) and identify the limit.

10.2. The Cauchy distribution has density function \(f(x) = \pi^{-1}(1+x^2)^{-1}\) for \(x \in \mathbb{R}\). Show that the corresponding characteristic function is given by \(\varphi(u) = e^{-|u|}\). Show also that, if \(X_1, \ldots, X_n\) are independent Cauchy random variables, then the random variable \((X_1 + \cdots + X_n)/n\) is also Cauchy.

10.3. Let \(f\) be a bounded continuous function on \((0, \infty)\), having Laplace transform
\[
\hat{f}(\lambda) = \int_{0}^{\infty} e^{-\lambda x} f(x) dx, \quad \lambda \in (0, \infty).
\]
Let \((X_n : n \in \mathbb{N})\) be a sequence of independent exponential random variables, of parameter \(\lambda\). Show that \(\hat{f}\) has derivatives of all orders on \((0, \infty)\) and that, for all \(n \in \mathbb{N}\), for some \(C(\lambda, n) \neq 0\) independent of \(f\), we have
\[
(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n) \mathbb{E}(f(S_n))
\]
where \(S_n = X_1 + \cdots + X_n\). Deduce that if \(\hat{f} \equiv 0\) then also \(f \equiv 0\).

10.4. For each \(n \in \mathbb{N}\), there is a unique probability measure \(\mu_n\) on the unit sphere \(S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}\) such that \(\mu_n(A) = \mu_n(UA)\) for all Borel sets \(A\) and all orthogonal \(n \times n\) matrices \(U\). Fix \(k \in \mathbb{N}\) and, for \(n \geq k\), let \(\gamma_n\) denote the probability measure on \(\mathbb{R}^k\) which is the law of \(\sqrt{n}(x^1, \ldots, x^k)\) under \(\mu_n\). Show
\[
\text{(a) if } X \sim N(0, I_n) \text{ then } X/|X| \sim \mu_n,
\]
\[
\text{(b) if } (X_n : n \in \mathbb{N}) \text{ is a sequence of independent } N(0,1) \text{ random variables and if } R_n = \sqrt{X_1^2 + \cdots + X_n^2} \text{ then } R_n/\sqrt{n} \to 1 \text{ a.s.,}
\]
\[
\text{(c) } \gamma_n \text{ converges weakly to the standard Gaussian distribution on } \mathbb{R}^k \text{ as } n \to \infty.
\]