Part II — Linear Analysis

Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

*Part IB Linear Algebra, Analysis II and Metric and Topological Spaces are essential*

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces.

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications.


Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel’s inequality, the Parseval equation, and the Riesz-Fischer theorem. Duality; the self duality of Hilbert space.

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators.
Contents

0 Introduction 3

1 Normed vector spaces 4
   1.1 Bounded linear maps .............................. 4
   1.2 Dual spaces .................................. 4
   1.3 Adjoint ........................................ 5
   1.4 The double dual ................................ 5
   1.5 Isomorphism .................................... 5
   1.6 Finite-dimensional normed vector spaces .......... 5
   1.7 Hahn–Banach Theorem ............................ 5

2 Baire category theorem 6
   2.1 The Baire category theorem ....................... 6
   2.2 Some applications ............................... 6

3 The topology of $\mathcal{C}(K)$ 7
   3.1 Normality of compact Hausdorff spaces .......... 7
   3.2 Tietze-Urysohn extension theorem ............... 7
   3.3 Arzelà-Ascoli theorem ............................ 7
   3.4 Stone–Weierstrass theorem ...................... 7

4 Hilbert spaces 9
   4.1 Inner product spaces ............................. 9
   4.2 Riesz representation theorem .................... 9
   4.3 Orthonormal systems and basis .................. 9
   4.4 The isomorphism with $\ell_2$ ..................... 9
   4.5 Operators ....................................... 9
   4.6 Self-adjoint operators .......................... 10
0 Introduction
1 Normed vector spaces

Definition (Normed vector space). A normed vector space is a pair \((V, \| \cdot \|)\), where \(V\) is a vector space over a field \(F\) and \(\| \cdot \| : V \mapsto \mathbb{R}\), known as the norm, satisfying

(i) \(\|v\| \geq 0\) for all \(v \in V\), with equality iff \(v = 0\).

(ii) \(\|\lambda v\| = |\lambda|\|v\|\) for all \(\lambda \in F, v \in V\).

(iii) \(\|v + w\| \leq \|v\| + \|w\|\) for all \(v, w \in V\).

Definition (Topological vector space). A topological vector space \((V, U)\) is a vector space \(V\) together with a topology \(U\) such that addition and scalar multiplication are continuous maps, and moreover singleton points \(\{v\}\) are closed sets.

Definition (Absolute convexity). Let \(V\) be a vector space. Then \(C \subseteq V\) is absolutely convex (or balanced convex) if for any \(\lambda, \mu \in F\) such that \(|\lambda| + |\mu| \leq 1\), we have \(\lambda C + \mu C \subseteq C\). In other words, if \(c_1, c_2 \in C\), we have \(\lambda c_1 + \mu c_2 \in C\).

Definition (Bounded subset). Let \(V\) be a topological vector space. Then \(B \subseteq V\) is bounded if for every open neighbourhood \(U \subseteq V\) of \(0\), there is some \(s > 0\) such that \(B \subseteq tU\) for all \(t > s\).

Definition (Banach spaces). A normed vector space is a Banach space if it is complete as a metric space, i.e. every Cauchy sequence converges.

1.1 Bounded linear maps

Definition (Bounded linear map). \(T : X \rightarrow Y\) is a bounded linear map if there is a constant \(C > 0\) such that \(\|Tx\|_Y \leq C\|x\|_X\) for all \(x \in X\). We write \(\mathcal{B}(X, Y)\) for the set of bounded linear maps from \(X\) to \(Y\).

Definition (Norm on \(\mathcal{B}(X, Y)\)). Let \(T : X \rightarrow Y\) be a bounded linear map. Define \(\|T\|_{\mathcal{B}(X, Y)}\) by

\[\|T\|_{\mathcal{B}(X, Y)} = \sup_{\|x\| \leq 1} \|Tx\|_Y.\]

1.2 Dual spaces

Definition (Dual space). Let \(V\) be a normed vector space. The dual space is

\[V^* = \mathcal{B}(V, F)\].

We call the elements of \(V^*\) functionals. The algebraic dual of \(V\) is

\[V' = \mathcal{L}(V, F)\],

where we do not require boundedness.
1.3 Adjoint

**Definition** (Adjoint). Let $X, Y$ be normal vector spaces. Given $T \in \mathcal{B}(X, Y)$, we define the adjoint of $T$, denoted $T^*$, as a map $T^* \in \mathcal{B}(Y^*, X^*)$ given by

$$T^*(g)(x) = g(T(x))$$

for $x \in X$, $y \in Y^*$. Alternatively, we can write

$$T^*(g) = g \circ T.$$

1.4 The double dual

**Definition** (Double dual). Let $V$ be a normed vector space. Define $V^{**} = (V^*)^*$.

1.5 Isomorphism

**Definition** (Isomorphism). Let $X, Y$ be normed vector spaces. Then $T : X \to Y$ is an isomorphism if it is a bounded linear map with a bounded linear inverse (i.e. it is a homeomorphism).

We say $X$ and $Y$ are isomorphic if there is an isomorphism $T : X \to Y$.

We say that $T : X \to Y$ is an isometric isomorphism if $T$ is an isomorphism and $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

$X$ and $Y$ are isometrically isomorphic if there is an isometric isomorphism between them.

1.6 Finite-dimensional normed vector spaces

**Definition** (Equivalent norms). Let $V$ be a vector space, and $\| \cdot \|_1, \| \cdot \|_2$ be norms on $V$. We say that these are equivalent if there exists a constant $C > 0$ such that for any $v \in V$, we have

$$C^{-1}\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2.$$

1.7 Hahn–Banach Theorem

**Definition** (Partial order). A relation $\leq$ on a set $X$ is a partial order if it satisfies

(i) $x \leq x$ (reflexivity)

(ii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry)

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity)

**Definition** (Total order). Let $(S, \leq)$ be a partial order. $T \subseteq S$ is totally ordered if for all $x, y \in T$, either $x \leq y$ or $y \leq x$, i.e. every two things are related.

**Definition** (Upper bound). Let $(S, \leq)$ be a partial order. $S' \subseteq S$ subset. We say $b \in S$ is an upper bound of this subset if $x \leq b$ for all $x \in S'$.

**Definition** (Maximal element). Let $(S, \leq)$ be a partial order. Then $m \in S$ is a maximal element if $x \geq m$ implies $x = m$.

**Definition** (Reflexive). We say $V$ is reflexive if $\phi(V) = V^{**}$. 

2 Baire category theorem

2.1 The Baire category theorem

Definition (Nowhere dense set). Let $X$ be a topological space. A subset $E \subseteq X$ is nowhere dense if $\bar{E}$ has empty interior.

Definition (First/second category, meagre and residual). Let $X$ be a topological space. We say that $Z \subseteq X$ is of first category or meagre if it is a countable union of nowhere dense sets.

A subset is of second category or non-meagre if it is not of first category.

A subset is residual if $X \setminus Z$ is meagre.

2.2 Some applications
3 The topology of \( C(K) \)

**Definition** (Hausdorff space). A topological space \( X \) is **Hausdorff** if for all distinct \( p, q \in X \), there are \( U_p, U_q \subseteq X \) that are open subsets of \( X \) such that \( p \in U_p, q \in U_q \) and \( U_p \cap U_q = \emptyset \).

**Notation.** \( C(K) \) is the set of continuous functions \( f : K \to \mathbb{R} \) with the norm \( \|f\|_{C(K)} = \sup_{x \in K} |f(x)| \).

3.1 Normality of compact Hausdorff spaces

**Definition** (Normal space). A topological space \( X \) is **normal** if for every disjoint pair of closed subsets \( C_1, C_2 \) of \( X \), there exists \( U_1, U_2 \subseteq X \) disjoint open such that \( C_1 \subseteq U_1, C_2 \subseteq U_2 \).

**Definition** (\( T_i \) space). A topological space \( X \) has the **\( T_1 \) property** if for all \( x, y \in X \), where \( x \neq y \), there exists \( U \subseteq X \) open such that \( x \in U \) and \( y \notin U \).

A topological space \( X \) has the **\( T_2 \) property** if \( X \) is Hausdorff.

A topological space \( X \) has the **\( T_3 \) property** if for any \( x \in X \), \( C \subseteq X \) closed with \( x \notin C \), then there are \( U_x, U_C \) disjoint open such that \( x \in U_x, C \subseteq U_C \). These spaces are called **regular**.

A topological space \( X \) has the **\( T_4 \) property** if \( X \) is normal.

3.2 Tietze-Urysohn extension theorem

3.3 Arzelà-Ascoli theorem

**Definition** (Equicontinuous). Let \( K \) be a topological space, and \( F \subseteq C(K) \).

We say \( F \) is **equicontinuous at** \( x \in K \) if for every \( \varepsilon \), there is some \( U \) which is an open neighbourhood of \( x \) such that \( (\forall f \in F)(\forall y \in U) |f(y) - f(x)| < \varepsilon \).

We say \( F \) is **equicontinuous** if it is equicontinuous at \( x \) for all \( x \in K \).

**Definition** (\( \varepsilon \)-net). Let \( X \) be a metric space, and let \( E \subseteq X \). For \( \varepsilon > 0 \), we say that \( N \subseteq X \) is an \( \varepsilon \)-net for \( E \) if and only if \( \bigcup_{x \in E} B(x, \varepsilon) \supseteq E \).

**Definition** (Totally bounded subset). Let \( X \) be a metric space, and \( E \subseteq X \). We say that \( E \) is **totally bounded** for every \( \varepsilon \), there is a finite \( \varepsilon \)-net \( N_{\varepsilon} \) for \( E \).

3.4 Stone–Weierstrass theorem

**Definition** (Algebra). A vector space \( (V, +) \) is called an **algebra** if there is an operation (called multiplication) \( : V \to V \) such that \( (V, +, \) is a **rng** (i.e. ring not necessarily with multiplicative identity). Also, \( \lambda(v \cdot w) = (\lambda v) \cdot w = v \cdot (\lambda w) \) for all \( \lambda \in F, v, w \in V \).

If \( V \) is in addition a normed vector space and

\[
\|v \cdot w\|_V \leq \|v\|_V \cdot \|w\|_V
\]

for all \( v, w \in V \), then we say \( V \) is a **normed algebra**.
If $V$ complete normed algebra, we say $V$ is a Banach algebra.

If $V$ is an algebra that is commutative as a rng, then we say $V$ is a commutative algebra.

If $V$ is an algebra with multiplicative identity, then $V$ is a unital algebra.
4 Hilbert spaces

4.1 Inner product spaces

**Definition** (Inner product). Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. We say $p : V \times V \to \mathbb{R}$ or $\mathbb{C}$ is an *inner product* on $V$ if it satisfies

(i) \[ p(v, w) = \overline{p(w, v)} \] for all $v, w \in V$. (antisymmetry)

(ii) \[ p(\lambda_1 v_1 + \lambda_2 v_2, u) = \lambda_1 p(v_1, w) + \lambda_2 p(v_2, w). \] (linearity in first argument)

(iii) $p(v, v) \geq 0$ for all $v \in V$ and equality holds iff $v = 0$. (non-negativity)

We will often denote an inner product by $p(v, w) = \langle v, w \rangle$. We call $(V, \langle \cdot, \cdot \rangle)$ an *inner product space*.

**Definition** (Orthogonality). In an inner product space, $v$ and $w$ are *orthogonal* if $\langle v, w \rangle = 0$.

**Definition** (Euclidean space). A normed vector space $(V, \| \cdot \|)$ is a *Euclidean space* if there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\|v\| = \sqrt{\langle v, v \rangle}$.

**Definition** (Hilbert space). A Euclidean space $(E, \| \cdot \|)$ is a *Hilbert space* if it is complete.

**Definition** (Orthogonal space). Let $E$ be a Euclidean space and $S \subseteq E$ an arbitrary subset. Then the *orthogonal space* of $S$, denoted by $S^\perp$, is given by $S^\perp = \{ v \in E : \forall w \in S, \langle v, w \rangle = 0 \}$.

4.2 Riesz representation theorem

4.3 Orthonormal systems and basis

**Definition** (Orthonormal system). Let $E$ be a Euclidean space. A set of unit vectors $\{e_\alpha\}_{\alpha \in A}$ is called an *orthonormal system* if $\langle e_\alpha, e_\beta \rangle = 0$ if $\alpha \neq \beta$.

**Definition** (Maximal orthonormal system). Let $E$ be a Euclidean space. An orthonormal system is called *maximal* if it cannot be extended to a strictly larger orthonormal system.

**Definition** (Hilbert space basis). Let $H$ be a Hilbert space. A maximal orthonormal system is called a *Hilbert space basis*.

4.4 The isomorphism with $\ell_2$

4.5 Operators

**Definition** (Spectrum and resolvent set). Let $X$ be a Banach space and $T \in B(X)$, we define the *spectrum* of $T$, denoted by $\sigma(T)$ by $\sigma(t) = \{ \lambda \in \mathbb{C} : T - \lambda I$ is not invertible $\}$. The *resolvent set*, denoted by $\rho(T)$, is $\rho(t) = \mathbb{C} \setminus \sigma(T)$. 

Definition (Resolvent). Let $X$ be a Banach space. The resolvent is the map $R : \rho(T) \to B(X)$ given by $\lambda \mapsto (T - \lambda I)^{-1}$.

Definition (Eigenvalue). We say $\lambda$ is an eigenvalue of $T$ if $\ker(T - \lambda I) \neq \{0\}$.

Definition (Point spectrum). Let $X$ be a Banach space. The point spectrum is $\sigma_p(T) = \{ \lambda \in \mathbb{C} : \lambda$ is an eigenvalue of $T \}$.

Definition (Approximate point spectrum). Let $X$ be a Banach space. The approximate point spectrum is defined as $\sigma_{ap}(X) = \{ \lambda \in \mathbb{C} : \exists \{x_n\} \subseteq X : \|x_n\|_X = 1$ and $(T - \lambda I)x_n \to 0 \}$.

Definition (Compact operator). Let $X, Y$ be Banach spaces. We say $T \in B(X,Y)$ is compact if for every bounded subset $E$ of $X$, $T(E)$ is totally bounded. We write $B_0(X)$ for the set of all compact operators $T \in B(X)$.

4.6 Self-adjoint operators

Definition (Self-adjoint operator). Let $H$ be a Hilbert space, $T \in B(H)$. Then $T$ is self-adjoint or Hermitian if for all $x, y \in H$, we have $\langle Tx, y \rangle = \langle x, Ty \rangle$. 