Part II — Integrable Systems

Theorems with proof

Based on lectures by A. Ashton
Notes taken by Dexter Chua

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Part IB Methods, and Complex Methods or Complex Analysis are essential; Part II Classical Dynamics is desirable.

Integrability of ordinary differential equations: Hamiltonian systems and the Arnol'd–Liouville Theorem (sketch of proof). Examples. [3]

Integrability of partial differential equations: The rich mathematical structure and the universality of the integrable nonlinear partial differential equations (Korteweg-de Vries, sine–Gordon). Backlund transformations and soliton solutions. [2]

The inverse scattering method: Lax pairs. The inverse scattering method for the KdV equation, and other integrable PDEs. Multi soliton solutions. Zero curvature representation. [6]

Hamiltonian formulation of soliton equations. [2]

Painleve equations and Lie symmetries: Symmetries of differential equations, the ODE reductions of certain integrable nonlinear PDEs, Painleve equations. [3]

Contents

0 Introduction

1 Integrability of ODE's

1.1 Vector fields and flow maps

Proposition.

(i) $g^0 = \mathrm{id}$

(ii) $g^{t+s} = g^t g^s$

(iii) $(g^t)^{-1} = g^{-t}$

Proof. The equality $g^0 = \mathrm{id}$ is by definition of g, and the last equality follows from the first two since $t + (-t) = 0$. To see the second, we need to show that

$$g^{t+s}\mathbf{x}_0 = g^t(g^s\mathbf{x}_0)$$

for any $\mathbf{x}_0$. To do so, we see that both of them, as a function of t, are solutions to

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}), \quad \mathbf{x}(0) = g^s\mathbf{x}_0.$$

So the result follows since solutions are unique. □

Proposition. Let $\mathbf{V}_1, \mathbf{V}_2$ be vector fields with flows g_1^t and g_2^s. Then we have

$$[\mathbf{V}_1, \mathbf{V}_2] = 0 \quad \Longleftrightarrow \quad g_1^t g_2^s = g_2^s g_1^t.$$

Proof. See example sheet 1. □

1.2 Hamiltonian dynamics

Proposition.

(i) This is linear in each argument.

(ii) This is antisymmetric, i.e. $\{f, g\} = -\{g, f\}$.

(iii) This satisfies the Leibniz property:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

(iv) This satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

(v) We have

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.$$

Proof. Just write out the definitions. In particular, you will be made to write out the 24 terms of the Jacobi identity in the first example sheet. □

Proposition. Let $f : M \to \mathbb{R}$ be a smooth function. If $\mathbf{x}(t)$ evolves according to Hamilton's equation, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}.$$

Proposition. We have

$$[\mathbf{V}_f, \mathbf{V}_g] = -\mathbf{V}_{\{f,g\}}.$$

Proof. See first example sheet. □

1.3 Canonical transformations

Proposition. A map $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ is canonical iff $D\mathbf{y}$ is *symplectic*, i.e.

$$D\mathbf{y}J(D\mathbf{y})^T = J.$$

1.4 The Arnold-Liouville theorem

Theorem (Arnold-Liouville theorem)**.** We let (M, H) be an integrable $2n$-dimensional Hamiltonian system with independent, involutive first integrals $f_1, \cdots, f_n$, where $f_1 = H$. For any fixed $\mathbf{c} \in \mathbb{R}^n$, we set

$$M_{\mathbf{c}} = \{(\mathbf{q}, \mathbf{p}) \in M : f_i(\mathbf{q}, \mathbf{p}) = c_i, i = 1, \cdots, n\}.$$

Then

(i) $M_{\mathbf{c}}$ is a smooth n-dimensional surface in M. If $M_{\mathbf{c}}$ is compact and connected, then it is diffeomorphic to

$$T^n = S^1 \times \cdots \times S^1.$$

(ii) If $M_{\mathbf{c}}$ is compact and connected, then locally, there exists canonical coordinate transformations $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ called the *action-angle coordinates* such that the angles $\{\phi_k\}_{k=1}^n$ are coordinates on $M_{\mathbf{c}}$; the actions $\{I_k\}_{k=1}^n$ are first integrals, and $H(\mathbf{q}, \mathbf{p})$ does not depend on $\boldsymbol{\phi}$. In particular, Hamilton's equations

$$\dot{\mathbf{I}} = 0, \quad \dot{\boldsymbol{\phi}} = \frac{\partial \tilde{H}}{\partial \mathbf{I}} = \text{constant}.$$

Proof sketch. The first part is pure differential geometry. To show that $M_{\mathbf{c}}$ is smooth and n-dimensional, we apply the preimage theorem you may or may not have learnt from IID Differential Geometry (which is in turn an easy consequence of the inverse function theorem from IB Analysis II). The key that makes this work is that the constraints are independent, which is the condition that allows the preimage theorem to apply.

We next show that $M_{\mathbf{c}}$ is diffeomorphic to the torus if it is compact and connected. Consider the Hamiltonian vector fields defined by

$$\mathbf{V}_{f_i} = J\frac{\partial f_i}{\partial \mathbf{x}}.$$

We claim that these are *tangent* to the surface $M_{\mathbf{c}}$. By differential geometry, it suffices to show that the derivative of the $\{f_j\}$ in the direction of $\mathbf{V}_{f_i}$ vanishes. We can compute

$$\left(\mathbf{V}_{f_i} \cdot \frac{\partial}{\partial \mathbf{x}}\right) f_j = \frac{\partial f_j}{\partial \mathbf{x}} J \frac{\partial f_i}{\partial \mathbf{x}} = \{f_j, f_i\} = 0.$$

Since this vanishes, we know that $\mathbf{V}_{f_i}$ is a tangent to the surface. Again by differential geometry, the flow maps $\{g_i\}$ must map $M_{\mathbf{c}}$ to itself. Also, we know that the flow maps commute. Indeed, this follows from the fact that

$$[\mathbf{V}_{f_i}, \mathbf{V}_{f_j}] = -\mathbf{V}_{\{f_i, f_j\}} = -\mathbf{V}_0 = 0.$$

So we have a whole bunch of commuting flow maps from $M_{\mathbf{c}}$ to itself. We set

$$g^{\mathbf{t}} = g_1^{t_1} g_2^{t_2} \cdots g_n^{t_n},$$

where $\mathbf{t} \in \mathbb{R}^n$. Then because of commutativity, we have

$$g^{\mathbf{t}_1 + \mathbf{t}_2} = g^{\mathbf{t}_1} g^{\mathbf{t}_2}.$$

So this is gives a group action of $\mathbb{R}^n$ on the surface $M_{\mathbf{c}}$. We fix $\mathbf{x} \in M_{\mathbf{c}}$. We define

$$\operatorname{stab}(\mathbf{x}) = \{\mathbf{t} \in \mathbb{R}^n : g^{\mathbf{t}}\mathbf{x} = \mathbf{x}\}.$$

We introduce the map

$$\phi : \frac{\mathbb{R}^n}{\operatorname{stab}(\mathbf{x})} \to M_{\mathbf{c}}$$

given by $\phi(\mathbf{t}) = g^{\mathbf{t}}\mathbf{x}$. By the orbit-stabilizer theorem, this gives a bijection between $\mathbb{R}^n / \operatorname{stab}(\mathbf{x})$ and the orbit of $\mathbf{x}$. It can be shown that the orbit of $\mathbf{x}$ is exactly the connected component of $\mathbf{x}$. Now if $M_{\mathbf{c}}$ is connected, then this must be the whole of $\mathbf{x}$! By general differential geometry theory, we get that this map is indeed a diffeomorphism.

We know that $\operatorname{stab}(\mathbf{x})$ is a subgroup of $\mathbb{R}^n$, and if the g_i are non-trivial, it can be seen (at least intuitively) that this is discrete. Thus, it must be isomorphic to something of the form $\mathbb{Z}^k$ with $1 \leq k \leq n$.

So we have

$$M_{\mathbf{c}} \cong \mathbb{R}^n / \operatorname{stab}(\mathbf{x}) \cong \mathbb{R}^n / \mathbb{Z}^k \cong \mathbb{R}^k / \mathbb{Z}^k \times \mathbb{R}^{n-k} \cong T^k \times \mathbb{R}^{n-k}.$$

Now if $M_{\mathbf{c}}$ is compact, we must have $n - k = 0$, i.e. $n = k$, so that we have no factors of $\mathbb{R}$. So $M_{\mathbf{c}} \cong T^n$.

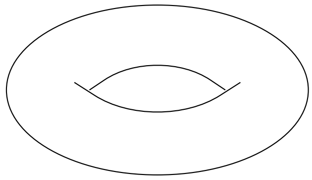
With all the differential geometry out of the way, we can now construct the action-angle coordinates.

For simplicity of presentation, we only do it in the case when $n = 2$. The proof for higher dimensions is entirely analogous, except that we need to use a higher-dimensional analogue of Green's theorem, which we do not currently have.

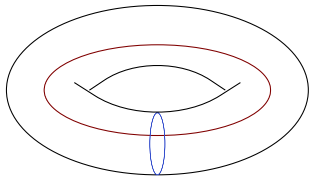
We note that it is currently trivial to re-parameterize the phase space with coordinates $(\mathbf{Q}, \mathbf{P})$ such that $\mathbf{P}$ is constant within the Hamiltonian flow, and each coordinate of $\mathbf{Q}$ takes values in S^1. Indeed, we just put $\mathbf{P} = \mathbf{c}$ and use the diffeomorphism $T^n \cong M_{\mathbf{c}}$ to parameterize each $M_{\mathbf{c}}$ as a product of n copies of S^n. However, this is not good enough, because such an arbitrary transformation will almost certainly not be canonical. So we shall try to find a more natural and in fact canonical way of parametrizing our phase space.

We first work on the generalized momentum part. We want to replace $\mathbf{c}$ with something nicer. We will do something analogous to the simple harmonic oscillator we've got.

So we fix a $\mathbf{c}$, and try to come up with some numbers $\mathbf{I}$ that labels this $M_{\mathbf{c}}$. Recall that our surface $M_{\mathbf{c}}$ looks like a torus:



Up to continuous deformation of loops, we see that there are two non-trivial "single" loops in the torus, given by the red and blue loops:



More generally, for an n torus, we have n such distinct loops $\Gamma_1, \cdots, \Gamma_n$. More concretely, after identifying $M_{\mathbf{c}}$ with S^n, these are the loops given by

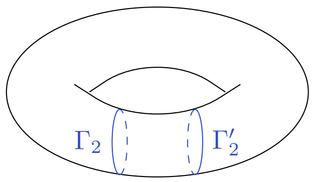
$$\{0\} \times \cdots \times \{0\} \times S^1 \times \{0\} \times \cdots \times \{0\} \subseteq S^1.$$

We now attempt to define:

$$I_j = \frac{1}{2\pi} \oint_{\Gamma_j} \mathbf{p} \cdot \mathrm{d}\mathbf{q},$$

This is just like the formula we had for the simple harmonic oscillator.

We want to make sure this is well-defined — recall that Γ_i actually represents a *class* of loops identified under continuous deformation. What if we picked a different loop?



On $M_{\mathbf{c}}$, we have the equation

$$f_i(\mathbf{q}, \mathbf{p}) = \mathbf{c}_i.$$

We will have to assume that we can invert this equation for $\mathbf{p}$ locally, i.e. we can write

$$\mathbf{p} = \mathbf{p}(\mathbf{q}, \mathbf{c}).$$

The condition for being able to do so is just

$$\det\left(\frac{\partial f_i}{\partial p_j}\right) \neq 0,$$

which is not hard.

Then by definition, the following holds identically:

$$f_i(\mathbf{q}, \mathbf{p}(\mathbf{q}, \mathbf{c})) = c_i.$$

We an then differentiate this with respect to q_k to obtain

$$\frac{\partial f_i}{\partial q_k} + \frac{\partial f_i}{\partial p_\ell}\frac{\partial p_\ell}{\partial q_k} = 0$$

on $M_{\mathbf{c}}$. Now recall that the $\{f_i\}$'s are in involution. So on $M_{\mathbf{c}}$, we have

$$\begin{aligned}
0 &= \{f_i, f_j\}\\
&= \frac{\partial f_i}{\partial q_k}\frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\frac{\partial f_j}{\partial q_k}\\
&= \left(-\frac{\partial f_i}{\partial p_\ell}\frac{\partial p_\ell}{\partial q_k}\right)\frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\left(-\frac{\partial f_j}{\partial p_\ell}\frac{\partial p_\ell}{\partial q_k}\right)\\
&= \left(-\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_\ell}\right)\frac{\partial f_j}{\partial p_\ell} - \frac{\partial f_i}{\partial p_k}\left(-\frac{\partial f_j}{\partial p_\ell}\frac{\partial p_\ell}{\partial q_k}\right)\\
&= \frac{\partial f_i}{\partial p_k}\left(\frac{\partial p_\ell}{\partial q_k} - \frac{\partial p_k}{\partial q_\ell}\right)\frac{\partial f_j}{\partial p_\ell}.
\end{aligned}$$

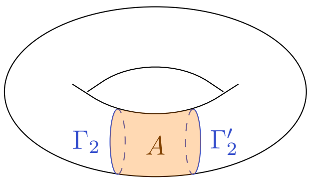
Recall that the determinants of the matrices $\frac{\partial f_i}{\partial p_k}$ and $\frac{\partial f_j}{\partial p_\ell}$ are non-zero, i.e. the matrices are invertible. So for this to hold, the middle matrix must vanish! So we have

$$\frac{\partial p_\ell}{\partial q_k} - \frac{\partial p_k}{\partial q_\ell} = 0.$$

In our particular case of $n = 2$, since ℓ, k can only be $1, 2$, the only non-trivial thing this says is

$$\frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1} = 0.$$

Now suppose we have two "simple" loops Γ_2 and Γ_2'. Then they bound an area A:



Then we have

$$\begin{aligned}
\left(\oint_{\Gamma_2} - \oint_{\Gamma_2'}\right)\mathbf{p}\cdot \mathrm{d}\mathbf{q} &= \oint_{\partial A}\mathbf{p}\cdot \mathrm{d}\mathbf{q}\\
&= \iint_A \left(\frac{\partial p_2}{\partial q_1} - \frac{\partial p_1}{\partial q_2}\right)\,\mathrm{d}q_1\,\mathrm{d}q_2\\
&= 0
\end{aligned}$$

by Green's theorem.

So I_j is well-defined, and

$$\mathbf{I} = \mathbf{I}(\mathbf{c})$$

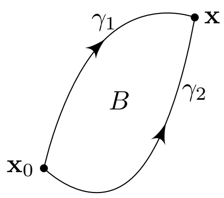
is just a function of c. This will be our new "momentum" coordinates. To figure out what the angles $\boldsymbol{\phi}$ should be, we use generating functions. For now, we assume that we can invert $\mathbf{I}(\mathbf{c})$, so that we can write

$$\mathbf{c} = \mathbf{c}(\mathbf{I}).$$

We arbitrarily pick a point $\mathbf{x}_0$, and define the generating function

$$S(\mathbf{q}, \mathbf{I}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{p}(\mathbf{q}', \mathbf{c}(\mathbf{I})) \cdot \mathrm{d}\mathbf{q}',$$

where $\mathbf{x} = (\mathbf{q}, \mathbf{p}) = (\mathbf{q}, \mathbf{p}(\mathbf{q}, \mathbf{c}(\mathbf{I})))$. However, this is not *a priori* well-defined, because we haven't said how we are going to integrate from $\mathbf{x}_0$ to $\mathbf{x}$. We are going to pick paths arbitrarily, but we want to make sure it is well-defined. Suppose we change from a path γ_1 to γ_2 by a little bit, and they enclose a surface B.

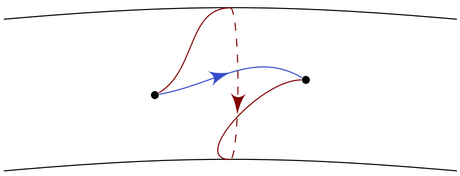


Then we have

$$S(\mathbf{q}, \mathbf{I}) \mapsto S(\mathbf{q}, \mathbf{I}) + \oint_{\partial B} \mathbf{p} \cdot \mathrm{d}\mathbf{q}.$$

Again, we are integrating $\mathbf{p} \cdot \mathrm{d}\mathbf{q}$ around a boundary, so there is no change.

However, we don't live in flat space. We live in a torus, and we can have a crazy loop that does something like this:



Then what we have effectively got is that we added a loop (say) Γ_2 to our path, and this contributes a factor of $2\pi I_2$. In general, these transformations give changes of the form

$$S(\mathbf{q}, \mathbf{I}) \mapsto S(\mathbf{q}, \mathbf{I}) + 2\pi I_j.$$

This is the only thing that can happen. So differentiating with respect to I, we know that

$$\boldsymbol{\phi} = \frac{\partial S}{\partial \mathbf{I}}$$

is well-defined modulo 2π. These are the *angles coordinates*. Note that just like angles, we can pick $\boldsymbol{\phi}$ consistently locally without this ambiguity, as long as we stay near some fixed point, but when we want to talk about the whole surface, this ambiguity necessarily arises. Now also note that

$$\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}.$$

Indeed, we can write

$$S = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{x}',$$

where

$$\mathbf{F} = (\mathbf{p}, 0).$$

So by the fundamental theorem of calculus, we have

$$\frac{\partial S}{\partial \mathbf{x}} = \mathbf{F}.$$

So we get that

$$\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}.$$

In summary, we have constructed on $M_{\mathbf{c}}$ the following: $\mathbf{I} = \mathbf{I}(\mathbf{c})$, $S(\mathbf{q}, I)$, and

$$\boldsymbol{\phi} = \frac{\partial S}{\partial \mathbf{I}}, \quad \mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}.$$

So S is a generator for the canonical transformation, and $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ is a canonical transformation.

Note that at any point $\mathbf{x}$, we know $\mathbf{c} = \mathbf{f}(\mathbf{x})$. So $I(\mathbf{c}) = I(\mathbf{f})$ depends on the first integrals only. So we have

$$\dot{\mathbf{I}} = 0.$$

So Hamilton's equations become

$$\dot{\boldsymbol{\phi}} = \frac{\partial \tilde{H}}{\partial \mathbf{I}}, \quad \dot{\mathbf{I}} = 0 = \frac{\partial \tilde{H}}{\partial \boldsymbol{\phi}}.$$

So the new Hamiltonian depends only on $\mathbf{I}$. So we can integrate up and get

$$\boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 + \Omega t, \quad \mathbf{I}(t) = \mathbf{I}_0,$$

where

$$\Omega = \frac{\partial \tilde{H}}{\partial \mathbf{I}}(\mathbf{I}_0).$$

□

2 Partial Differential Equations

2.1 KdV equation

2.2 Sine–Gordon equation

2.3 Bäcklund transformations

3 Inverse scattering transform

3.1 Forward scattering problem

3.1.1 Continuous spectrum

3.1.2 Discrete spacetime and bound states

3.1.3 Summary of forward scattering problem

3.2 Inverse scattering problem

Theorem (GLM inverse scattering theorem)**.** A potential $u = u(x)$ that decays rapidly to 0 as $|x| \to \infty$ is completely determined by its scattering data

$$S = \left\{\{\chi_n, c_n\}_{n=1}^N, R(k)\right\}.$$

Given such a scattering data, if we set

$$F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x} + \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx} R(k)\;\mathrm{d}k,$$

and define $k(x, y)$ to be the *unique* solution to

$$k(x, y) + F(x + y) + \int_x^\infty k(x, z) f(z + y)\;\mathrm{d}z = 0,$$

then

$$u(x) = -2\frac{\mathrm{d}}{\mathrm{d}x} k(x, x).$$

Proof. Too hard. □

3.3 Lax pairs

Theorem (Isospectral flow theorem)**.** Let (L, A) be a Lax pair. Then the discrete eigenvalues of L are time-independent. Also, if $L\psi = \lambda\psi$, where λ is a discrete eigenvalue, then

$$L\tilde{\psi} = \lambda\tilde{\psi},$$

where

$$\tilde{\psi} = \psi_t + A\psi.$$

Proof. We will assume that the eigenvalues at least vary smoothly with t, so that for each eigenvalue λ_0 at $t = 0$ with eigenfunction $\psi_0(x)$, we can find some $\lambda(t)$ and $\psi(x, t)$ with $\lambda(0) = \lambda_0$, $\psi(x, 0) = \psi_0(x)$ such that

$$L(t)\psi(x, t) = \lambda(t)\psi(x, t).$$

We will show that in fact $\lambda(t)$ is constant in time. Differentiating with respect to t and rearranging, we get

$$\begin{aligned}
\lambda_t\psi &= L_t\psi + L\psi_t - \lambda\psi_t\\
&= LA\psi - AL\psi + L\psi_t - \lambda\psi_t\\
&= LA\psi - \lambda A\psi + L\psi_t - \lambda\psi_t\\
&= (L - \lambda)(\psi_t + A\psi)
\end{aligned}$$

We now take the inner product ψ, and use that $\|\psi\| = 1$. We then have

$$\begin{aligned}\lambda_t &= \langle \psi, \lambda_t \psi \rangle \\ &= \langle \psi, (L - \lambda)(\psi_t + A_\psi) \rangle \\ &= \langle (L - \lambda)\psi, \psi_t + A_\psi \rangle \\ &= 0,\end{aligned}$$

using the fact that L, hence $L - \lambda$ is self-adjoint.

So we know that $\lambda_t = 0$, i.e. that λ is time-independent. Then our above equation gives

$$L\tilde{\psi} = \lambda\tilde{\psi},$$

where

$$\tilde{\psi} = \psi_t + A\psi.$$

□

3.4 Evolution of scattering data

3.4.1 Continuous spectrum ($\lambda = k^2 > 0$)

3.4.2 Discrete spectrum ($\lambda = -\kappa^2 < 0$)

3.4.3 Summary of inverse scattering transform

3.5 Reflectionless potentials

3.6 Infinitely many first integrals

4 Structure of integrable PDEs

4.1 Infinite dimensional Hamiltonian system

Proposition. If $u_t = \mathcal{J}\delta H$ and $I = I[u]$, then

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \{I, H\}.$$

In particular $I[u]$ is a first integral of $u_t = \mathcal{J}\delta H$ iff $\{I, H\} = 0$.

Proof.

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\varepsilon \to 0} \frac{I[u + \varepsilon u_t] - I[u]}{\varepsilon} = \langle \delta I, u_t \rangle = \langle \delta I, \mathcal{J}\delta H \rangle = \{I, H\}.$$ □

4.2 Bihamiltonian systems

Theorem. Suppose a system is bi-Hamiltonian via $(\mathcal{J}_0, H_0)$ and $(\mathcal{J}_1, H_1)$. It is a fact that we can find a sequence $\{H_n\}_{n \geq 0}$ such that

$$\mathcal{J}_1 \delta H_{n+1} = \mathcal{J}_0 \delta H_n.$$

Under these definitions, $\{H_n\}$ are all first integrals of the system and are in involution, i.e.

$$\{H_n, H_m\} = 0$$

for all $n, m \geq 0$, where the Poisson bracket is taken with respect to $\mathcal{J}_1$.

Proof. We notice the following interesting fact: for $m \geq 1$, we have

$$\begin{aligned}
\{H_n, H_m\} &= \langle \delta H_n, \mathcal{J}_1 \delta H_m \rangle \\
&= \langle \delta H_n, \mathcal{J}_0 \delta H_{m-1} \rangle \\
&= -\langle \mathcal{J}_0 \delta H_n, \delta H_{m-1} \rangle \\
&= -\langle \mathcal{J}_1 \delta H_{n+1}, \delta H_{m-1} \rangle \\
&= \langle \delta H_{n+1}, \mathcal{J}_1 \delta H_{m-1} \rangle \\
&= \{H_{n+1}, H_{m-1}\}.
\end{aligned}$$

Iterating this many times, we find that for any n, m, we have

$$\{H_n, H_m\} = \{H_m, H_n\}.$$

Then by antisymmetry, they must both vanish. So done. □

4.3 Zero curvature representation

4.4 From Lax pairs to zero curvature

5 Symmetry methods in PDEs

5.1 Lie groups and Lie algebras

5.2 Vector fields and one-parameter groups of transformations

5.3 Symmetries of differential equations

5.4 Jets and prolongations

Proposition (Prolongation formula)**.** Let

$$V(x,u) = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}.$$

Then we have

$$\mathrm{pr}^{(n)}V = V + \sum_{k=1}^{n} \eta_k \frac{\partial}{\partial u^{(k)}},$$

where

$$\begin{aligned} \eta_0 &= \eta(x,u) \\ \eta_{k+1} &= \mathrm{D}_x \eta_k - u^{(k+1)} D_x \xi. \end{aligned}$$

5.5 Painlevé test and integrability