# Part II — Galois Theory

## Theorems with proof


Based on lectures by C. Birkar

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*Groups, Rings and Modules is essential*

Field extensions, tower law, algebraic extensions; irreducible polynomials and relation with simple algebraic extensions. Finite multiplicative subgroups of a field are cyclic. Existence and uniqueness of splitting fields. [6]

Existence and uniqueness of algebraic closure. [1]

Separability. Theorem of primitive element. Trace and norm. [3]

Normal and Galois extensions, automorphic groups. Fundamental theorem of Galois theory. [3]

Galois theory of finite fields. Reduction mod $p$. [2]

Cyclotomic polynomials, Kummer theory, cyclic extensions. Symmetric functions. Galois theory of cubics and quartics. [4]

Solubility by radicals. Insolubility of general quintic equations and other classical problems. [3]

Artin's theorem on the subfield fixed by a finite group of automorphisms. Polynomial invariants of a finite group; examples. [2]

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# 0 Introduction

# 1 Solving equations

# 2 Field extensions

## 2.1 Field extensions

**Theorem** (Tower Law)**.** Let $F/L/K$ be field extensions. Then

$$[F : K] = [F : L][L : K]$$

*Proof.* Assume $[F : L]$ and $[L : K]$ are finite. Let $\{\alpha_1, \cdots, \alpha_m\}$ be a basis for $L$ over $K$, and $\{\beta_1, \cdots, \beta_n\}$ be a basis for $F$ over $L$. Pick $\gamma \in F$. Then we can write

$$\gamma = \sum_i b_i \beta_i, \quad b_i \in L.$$

For each $b_i$, we can write as

$$b_i = \sum_j a_{ij}\alpha_j, \quad a_{ij} \in K.$$

So we can write

$$\gamma = \sum_i \left(\sum_j a_{ij}\alpha_j\right)\beta_i = \sum_{i,j} a_{ij}\alpha_j\beta_i.$$

So $T = \{\alpha_j\beta_i\}_{i,j}$ spans $F$ over $K$. To show that this is a basis, we have to show that they are linearly independent. Consider the case where $\gamma = 0$. Then we must have $b_i = 0$ since $\{\beta_i\}$ is a basis of $F$ over $L$. Hence each $a_{ij} = 0$ since $\{\alpha_j\}$ is a basis of $L$ over $K$.

This implies that $T$ is a basis of $F$ over $K$. So

$$[F : K] = |T| = nm = [F : L][L : K].$$

Finally, if $[F : L] = \infty$ or $[L : K] = \infty$, then clearly $[F : K] = \infty$ as well. So equality holds as well. □

**Lemma.** Let $L/K$ be a finite extension. Then $L$ is algebraic over $K$.

*Proof.* Let $n = [L : K]$, and let $\alpha \in L$. Then $1, \alpha, \alpha^2, \cdots, \alpha^n$ are linearly dependent over $K$ (since there are $n + 1$ elements). So there exists some $a_i \in K$ (not all zero) such that

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0.$$

So we have a non-trivial polynomial that vanishes at $\alpha$. So $\alpha$ is algebraic over $K$.

Since $\alpha$ was arbitrary, $L$ itself is algebraic. □

**Proposition.** Let $L/K$ be a field extension, $\alpha \in L$ algebraic over $K$, and $P_\alpha$ the minimal polynomial. Then $P_\alpha$ is irreducible in $K[t]$.

*Proof.* Assume that $P_\alpha = QR$ in $K[t]$. So $0 = P_\alpha(\alpha) = Q(\alpha)R(\alpha)$. So $Q(\alpha) = 0$ or $R(\alpha) = 0$. Say $Q(\alpha) = 0$. So $Q \in I_\alpha$. So $Q$ is a multiple of $P_\alpha$. However, we also know that $P_\alpha$ is a multiple of $Q_\alpha$. This is possible only if $R$ is a unit in $K[t]$, i.e. $R \in K$. So $P_\alpha$ is irreducible. □

**Theorem.** Let $L/K$ a field extension, $\alpha \in L$ algebraic. Then

(i) $K(\alpha)$ is the image of the (ring) homomorphism $\phi : K[t] \to L$ defined by $f \mapsto f(\alpha)$.

(ii) $[K(\alpha) : K] = \deg P_\alpha$, where $P_\alpha$ is the minimal polynomial of $\alpha$ over $K$.

*Proof.*

(i) Let $F$ be the image of $\phi$. The first step is to show that $F$ is indeed a field. Since $F$ is the image of a ring homomorphism, we know $F$ is a subring of $L$. Given $\beta \in F$ non-zero, we have to find an inverse.

By definition, $\beta = f(\alpha)$ for some $f \in K[t]$. The idea is to use Bézout's identity. Since $\beta \neq 0$, $f(\alpha) \neq 0$. So $f \notin I_\alpha = \langle P_\alpha \rangle$. So $P_\alpha \nmid f$ in $K[t]$. Since $P_\alpha$ is irreducible, $P_\alpha$ and $f$ are coprime. Then there exists some $g, h \in K[t]$ such that $fg + hP_\alpha = 1$. So $f(\alpha)g(\alpha) = f(\alpha)g(\alpha) + h(\alpha)P_\alpha(\alpha) = 1$. So $\beta g(\alpha) = 1$. So $\beta$ has an inverse. So $F$ is a field.

From the definition of $F$, we have $K \subseteq F$ and $\alpha \in F$, using the constant polynomials $f = c \in K$ and the identity $f = t$.

Now, if $K \subseteq G \subseteq L$ and $\alpha \in G$, then $G$ contains all the polynomial expressions of $\alpha$. Hence $F \subseteq G$. So $K(\alpha) = F$.

(ii) Let $n = \deg P_\alpha$. We show that $\{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$ is a basis for $K(\alpha)$ over $K$.

First note that since $\deg P_\alpha = n$, we can write

$$\alpha^n = \sum_{i=0}^{n-1} a_i \alpha^i.$$

So any other higher powers are also linear combinations of the $\alpha^i$s (by induction). This means that $K(\alpha)$ is spanned by $1, \cdots, \alpha^{n-1}$ as a $K$ vector space.

It remains to show that $\{1, \cdots, \alpha^{n-1}\}$ is linearly independent. Assume not. Then for some $b_i$, we have

$$\sum_{i=0}^{n-1} b_i \alpha^i = 0.$$

Let $f = \sum b_i t^i$. Then $f(\alpha) = 0$. So $f \in I_\alpha = \langle P_\alpha \rangle$. However, $\deg f < \deg P_\alpha$. So we must have $f = 0$. So all $b_i = 0$. So $\{1, \cdots, \alpha^{n-1}\}$ is a basis for $K(\alpha)$ over $K$. So $[K(\alpha) : K] = n$. □

**Corollary.** Let $L/K$ be a field extension, $\alpha \in L$. Then $\alpha$ is algebraic over $K$ if and only if $K(\alpha)/K$ is a finite extension.

*Proof.* If $\alpha$ is algebraic, then $[K(\alpha) : K] = \deg P_\alpha < \infty$ by above. So the extension is finite.

If $K \subseteq K(\alpha)$ is a finite extension, then by previous lemma, the entire $K(\alpha)$ is algebraic over $K$. So $\alpha$ is algebraic over $K$. □

**Theorem.** Suppose that $L/K$ is a field extension.

(i) If $\alpha_1, \cdots, \alpha_n \in L$ are algebraic over $K$, then $K(\alpha_1, \cdots, \alpha_n)/K$ is a finite extension.

(ii) If we have field extensions $L/F/K$ and $F/K$ is a finite extension, then $F = K(\alpha_1, \cdots, \alpha_n)$ for some $\alpha_1, \cdots, \alpha_n \in L$.

*Proof.*

(i) We prove this by induction. Since $\alpha_1$ is algebraic over $K$, $K \subseteq K(\alpha_1)$ is a finite extension.

For $1 \leq i < n$, $\alpha_{i+1}$ is algebraic over $K$. So $\alpha_{i+1}$ is also algebraic over $K(\alpha_1, \cdots, \alpha_i)$. So $K(\alpha_1, \cdots, \alpha_i) \subseteq K(\alpha_1, \cdots, \alpha_i)(\alpha_{i+1})$ is a finite extension. But $K(\alpha_1, \cdots, \alpha_i)(\alpha_{i+1}) = K(\alpha_1, \cdots, \alpha_{i+1})$. By the tower law, $K \subseteq K(\alpha_i, \cdots, \alpha_{i+1})$ is a finite extension.

(ii) Since $F$ is a finite dimensional vector space over $K$, we can take a basis $\{\alpha_1, \cdots, \alpha_n\}$ of $F$ over $K$. Then it should be clear that $F = K(\alpha_1, \cdots, \alpha_n)$. □

**Proposition** (Eisenstein's criterion)**.** Let $f = a_n t^n + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$. Assume that there is some prime number $p$ such that

(i) $p \mid a_i$ for all $i < n$.

(ii) $p \nmid a_n$

(iii) $p^2 \nmid a_0$.

Then $f$ is irreducible in $\mathbb{Q}[t]$.

## 2.2 Ruler and compass constructions

**Theorem.** Let $S \subseteq \mathbb{R}^2$ be finite. Then

(i) If $R$ is 1-step constructible from $S$, then $[\mathbb{Q}(S \cup \{R\}) : \mathbb{Q}(S)] = 1$ or 2.

(ii) If $T \subseteq \mathbb{R}^2$ is finite, $S \subseteq T$, and the points in $T$ are constructible from $S$, Then $[\mathbb{Q}(S \cup T) : \mathbb{Q}(S)] = 2^k$ for some $k$ (where $k$ can be 0).

*Proof.* By assumption, there are distinct lines or circles $C, C'$ constructed from $S$ using ruler and compass, such that $R \in C \cap C'$. By elementary geometry, $C$ and $C'$ can be given by the equations

$$\begin{aligned} C &: a(x^2 + y^2) + bx + cy + d = 0, \\ C' &: a'(x^2 + y^2) + b'x + c'y + d' = 0. \end{aligned}$$

where $a, b, c, d, a', b', c', d' \in \mathbb{Q}(S)$. In particular, if we have a line, then we can take $a = 0$.

Let $R = (r_1, r_2)$. If $a = a' = 0$ (i.e. $C$ and $C'$ are lines), then solving the two linear equations gives $r_1, r_2 \in \mathbb{Q}(S)$. So $[\mathbb{Q}(S \cup \{R\}) : \mathbb{Q}(S)] = 1$.

So we can now assume wlog that $a \neq 0$. We let

$$p = a'b - ab', \quad q = a'c - ac', \quad \ell = a'd - ad',$$

which are the coefficients when we perform $a' \times C - a \times C'$. Then by assumption, $p \neq 0$ or $q \neq 0$. Otherwise, $c$ and $c'$ would be the same curve. wlog $p \neq 0$. Then since $(r_1, r_2)$ satisfy both equations of $C$ and $C'$, they satisfy

$$px + qy + \ell = 0.$$

In other words, $pr_1 + qr_2 + \ell = 0$. This tells us that

$$r_1 = -\frac{qr_2 + \ell}{p}. \tag{$*$}$$

If we put $r_1, r_2$ into the equations of $C$ and $C'$ and use $(*)$, we get an equation of the form

$$\alpha r_2^2 + \beta r_2 + \gamma = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{Q}(S)$. So we can find $r_2$ (and hence $r_1$ using linear relations) using only a single radical of degree 2. So

$$[\mathbb{Q}(S \cup \{R\}) : \mathbb{Q}(S)] = [\mathbb{Q}(S)(r_2) : \mathbb{Q}(S)] = 1 \text{ or } 2,$$

since the minimal polynomial of $r_2$ over $\mathbb{Q}(S)$ has degree 1 or 2.

Then (ii) follows directly from induction, using the tower law. □

**Corollary.** It is impossible to "double the cube".

*Proof.* Consider the cube with unit side length, i.e. we are given the set $S = \{(0, 0), (1, 0)\}$. Then doubling the cube would correspond to constructing a side of length $\ell$ such that $\ell^3 = 2$, i.e. $\ell = \sqrt[3]{2}$. Thus we need to construct a point $R = (\sqrt[3]{2}, 0)$ from $S$.

If we can indeed construct this $R$, then we need

$$[\mathbb{Q}(S \cup \{R\}) : \mathbb{Q}(S)] = 2^k$$

for some $k$. But we know that $\mathbb{Q}(S) = \mathbb{Q}$ and $\mathbb{Q}(S \cup \{R\}) = \mathbb{Q}(\sqrt[3]{2})$, and that

$$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3.$$

This is a contradiction since 3 is not a power of 2. □

## 2.3 $K$-homomorphisms and the Galois Group

## 2.4 Splitting fields

**Lemma.** Let $L/K$ be a field extension, $f \in K[t]$ irreducible, $\deg f > 0$. Then there is a 1-to-1 correspondence

$$\mathrm{Root}_f(L) \longleftrightarrow \mathrm{Hom}_K(K[t]/\langle f\rangle, L).$$

*Proof.* Since $f$ is irreducible, $\langle f\rangle$ is a maximal ideal. So $K[t]/\langle f\rangle$ is a field. Also, there is a natural inclusion $K \hookrightarrow K[t]/\langle f\rangle$. So it makes sense to talk about $\mathrm{Hom}_K(K[t]/\langle f\rangle, L)$.

To any $\beta \in \mathrm{Root}_f(L)$, we assign $\phi : K[t]/\langle f\rangle \to L$ where we map $\bar{t} \mapsto \beta$ ($\bar{t}$ is the equivalence class of $t$). This is well defined since if $\bar{t} = \bar{g}$, then $g = t + hf$ for some $h \in K[t]$. So $\phi(\bar{g}) = \phi(\overline{t + hf}) = \beta + h(\beta)f(\beta) = \beta$.

Conversely, given any $K$-homomorphism $\phi : K[t]/\langle f\rangle \to L$, we assign $\beta = \phi(\bar{t})$. This is a root since $f(\beta) = f(\phi(\bar{t})) = \phi(f(\bar{t})) = \phi(0) = 0$.

This assignments are inverses to each other. So we get a one-to-one correspondence. □

**Corollary.** Let $L/K$ be a field extension, $f \in K[t]$ irreducible, $\deg f > 0$. Then

$$|\mathrm{Hom}_K(K[t]/\langle f\rangle, L)| \leq \deg f.$$

In particular, if $E = K[t]/\langle f\rangle$, then

$$|\operatorname{Aut}_K(E)| = |\operatorname{Root}_f(E)| \leq \deg f = [E : K].$$

So $E/K$ is a Galois extension iff $|\operatorname{Root}_f(E)| = \deg f$.

*Proof.* This follows directly from the following three facts:

- $|\operatorname{Root}_f(L)| \leq \deg f$
- $\operatorname{Aut}_K(E) = \operatorname{Hom}_K(E, E)$
- $\deg f = [K(\alpha) : K] = [E : K]$. □

**Theorem.** Let $K$ be a field, $f \in K[t]$. Then

(i) There is a splitting field of $f$.

(ii) The splitting field is unique (up to $K$-isomorphism).

*Proof.*

(i) If $\deg f = 0$, then $K$ is a splitting field of $f$. Otherwise, we add the roots of $f$ one by one.

Pick $g \mid f$ in $K[t]$, where $g$ is irreducible and $\deg g > 0$. We have the field extension $K \subseteq K[t]/\langle g\rangle$. Let $\alpha_1 = \bar{t}$. Then $g(\alpha_1) = 0$ which implies that $f(\alpha_1) = 0$. Hence we can write $f = (t - \alpha_1)h$ in $K(\alpha_1)[t]$. Note that $\deg h < \deg f$. So we can repeat the process on $h$ iteratively to get a field extensions $K \subseteq K(\alpha_1, \cdots, \alpha_n)$. This $K(\alpha_1, \cdots, \alpha_n)$ is a splitting field of $f$.

(ii) Assume $L$ and $L'$ are both splitting fields of $f$ over $K$. We want to find a $K$-isomorphism from $L$ to $L'$.

Pick largest $F, F'$ such that $K \subseteq F \subseteq L$ and $K \subseteq F' \subseteq L'$ are field extensions and there is a $K$-isomorphism from $\psi : F \to F'$. By "largest", we mean we want to maximize $[F : K]$.

We want to show that we must have $F = L$. Then we are done because this means that $F'$ is a splitting field, and hence $F' = L'$.

So suppose $F \neq L$. We will try to produce a larger $\tilde{F}$ with $K$-isomorphism $\tilde{F} \to \tilde{F}' \subseteq L'$.

Since $F \neq L$, we know that there is some $\alpha \in \operatorname{Root}_f(L)$ such that $\alpha \notin F$. Then there is some irreducible $g \in K[t]$ with $\deg g > 0$ such that $g(\alpha) = 0$ and $g \mid f$. Say $f = gh$.

Now we know there is an isomorphism $F[t]/\langle g\rangle \to F(\alpha)$ by $\bar{t} \mapsto \alpha$. The isomorphism $\psi : F \to F'$ extends to a isomorphism $\mu : F[t] \to F'[t]$. Then since the coefficients of $f$ are in $K$, we have $f = \mu(f) = \mu(g)\mu(h)$. So $\mu(g) \mid f$ in $F'[t]$. Since $g$ is irreducible in $F[t]$, $\mu(g)$ is irreducible in $F'[t]$. So there is some $\alpha' \in \operatorname{Root}_{\mu(g)}(L') \subseteq \operatorname{Root}_f(L')$ and isomorphism $F'[t]/\langle \mu(g)\rangle \to F'(\alpha')$.

Now $\mu$ induces a $K$-isomorphism $F[t]/\langle g\rangle \to F'[t]/\langle \mu(g)\rangle$, which in turn induces a $K$-isomorphism $F(\alpha) \to F'(\alpha')$. This contradicts the maximality of $F$. So we must have had $F = L$. □

## 2.5 Algebraic closures

**Lemma.** If $R$ is a commutative ring, then it has a maximal ideal. In particular, if $I$ is an ideal of $R$, then there is a maximal ideal that contains $I$.

*Proof.* Let
$$\mathcal{P} = \{I : I \text{ is an ideal of } R, I \neq R\}.$$
If $I_1 \subseteq I_2 \subseteq \cdots$ is any chain of $I_i \in P$, then $I = \bigcup I_i \in \mathcal{P}$. By Zorn's lemma, there is a maximal element of $\mathcal{P}$ (containing $I$). So $R$ has at least one maximal ideal (containing $I$). □

**Theorem** (Existence of algebraic closure)**.** Any field $K$ has an algebraic closure.

*Proof.* Let
$$\mathcal{A} = \{\lambda = (f, j) : f \in K[t] \text{ irreducible monic}, 1 \leq j \leq \deg f\}.$$
We can think of $j$ as labelling which root of $f$ we want. For each $\lambda \in \mathcal{A}$, we assign a variable $t_\lambda$. We take
$$R = K[t_\lambda : \lambda \in \mathcal{A}]$$
to be the polynomial ring over $K$ with variables $t_\lambda$. This $R$ contains all the "roots" of the polynomials in $K$. However, we've got a bit too much. For example, (if $K = \mathbb{Q}$), in $R$, $\sqrt{3}$ and $\sqrt{3}+1$ would be put down as separate, unrelated variables. So we want to quotient this $R$ by something.

For every monic and irreducible $f \in K[t]$, we define
$$\tilde{f} = f - \prod_{j=1}^{\deg f} (t - t_{(f,j)}) \in R[t].$$
If we want the $t_{(f,j)}$ to be roots of $f$, then $\tilde{f}$ should vanish for all $f$. Denote the coefficient of $t^\ell$ in $\tilde{f}$ by $b_{(f,\ell)}$. Then we want $b_{(f,\ell)} = 0$ for all $f, \ell$.

To do so, let $I \subseteq R$ be the ideal generated by all such coefficients. We now want to quotient $R$ by $I$. We first have to check that $I \neq R$.

Suppose not. So there are $b_{(f_1,\ell_1)}, \cdots, b_{(f_r,\ell_r)}$ with $g_1, \cdots, g_r \in R$ such that
$$g_1 b_{(f_1,\ell_1)} + \cdots + g_r b_{(f_r,\ell_r)} = 1. \tag{$*$}$$
We will attempt to reach a contradiction by constructing a homomorphism $\phi$ that sends each $b_{(f_i,\ell_i)}$ to 0.

Let $E$ be a splitting field of $f_1 f_2 \cdots f_r$. So in $E[t]$, for each $i$, we can write
$$f_i = \prod_{j=1}^{\deg f_i} (t - \alpha_{i,j}).$$
Then we define a homomorphism $\phi : R \to E$ by
$$\begin{cases} \phi(t_{(f_i,j)}) = \alpha_{i,j} & \\ \phi(t_\lambda) = 0 & \text{otherwise} \end{cases}$$
This induces a homomorphism $\tilde{\phi} : R[t] \to E[t]$.

Now apply

$$\begin{aligned}\tilde{\phi}(\tilde{f}_i) &= \tilde{\phi}(f_i) - \prod_{j=1}^{\deg f_i} \tilde{\phi}(t - t_{(f_i,j)})\\ &= f_i - \prod_{j=1}^{\deg f_i} (t - \alpha_{i,j})\\ &= 0\end{aligned}$$

So $\phi(b_{(f_i,\ell_i)}) = 0$ as $b_{(f_i,\ell_i)}$ is a coefficient of $f_i$.

Now we apply $\phi$ to $(*)$ to obtain

$$\phi(g_1 b_{(f_1,\ell_1)} + \cdots + g_r b_{(f_r,\ell_r)}) = \phi(1).$$

But this is a contradiction since the left had side is 0 while the right is 1. Hence we must have $I \neq R$.

We would like to quotient by $I$, but we have to be a bit more careful, since the quotient need not be a field. Instead, pick a maximal ideal $M$ containing $I$, and consider $L = R/M$. Then $L$ is a field. Moreover, since we couldn't have quotiented out anything in $K$ (any ideal containing anything in $K$ would automatically contain all of $R$), this is a field extension $L/K$. We want to show that $L$ is an algebraic closure.

Now we show that $L$ is algebraic over $K$. This should all work out smoothly, since that's how we constructed $L$. First we pick $\alpha \in L$. Since $L = R/M$ and $R$ is generated by the terms $t_\lambda$, there is some $(f_1, j_1), \cdots, (f_r, j_r)$ such that

$$\alpha \in K(\bar{t}_{(f_i,j_i)}, \cdots, \bar{t}_{(f_r,j_r)}).$$

So $\alpha$ is algebraic over $K$ if each $\bar{t}_{(f_i,j_i)}$ is algebraic over $K$. To show this, note that $\tilde{f}_i = 0$, since we've quotiented out each of its coefficients. So by definition,

$$0 = f_i(t) - \prod_{j=1}^{\deg f_i} (t - \bar{t}_{(f_i,j)}).$$

So $f_i(\bar{t}_{(f_i,j_i)}) = 0$. So done.

Finally, we have to show that $L$ is algebraically closed. Suppose $L \subseteq E$ is a finite (and hence algebraic) extension. We want to show that $L = E$.

Consider arbitrary $\beta \in E$. Then $\beta$ is algebraic over $L$, say a root of $f \in L[t]$. Since every coefficient of $f$ can be found in some finite extension $K(\bar{t}_{(f_i,j_i)}, \cdots, \bar{t}_{(f_r,j_r)})$, there is a finite extension $F$ of $K$ that contains all coefficients of $f$. Since $F(\beta)$ is a finite extension of $F$, we know $F(\beta)$ is a finite and hence algebraic extension of $K$. In particular, $\beta$ is algebraic in $K$.

Let $P_\beta$ be the minimal polynomial of $\beta$ over $K$. Since all polynomials in $K$ split over $L$ by construction ($f(t) = \prod(t - \bar{t}_{(f,j)})$), its roots must in $L$. In particular, $\beta \in L$. So $L = E$. □

**Theorem** (Uniqueness of algebraic closure)**.** Any field $K$ has a unique algebraic closure up to $K$-isomorphism.

*Proof.* (sketch) Suppose $L, L'$ are both algebraic closures of $K$. Let

$$\mathcal{H} = \{(F, \psi) : K \subseteq F \subseteq L, \psi \in \mathrm{Hom}_K(F, L')\}.$$

We define a partial order on $\mathcal{H}$ by $(F_1, \psi_1) \leq (F_2, \psi_2)$ if $F_1 \leq F_2$ and $\psi_1 = \psi_2|_{F_1}$.

We have to show that chains have upper bounds. Given a chain $\{(F_\alpha, \psi_\alpha)\}$, we define

$$F = \bigcup F_\alpha, \quad \psi(x) = \psi_\alpha(x) \text{ for } x \in F_\alpha.$$

Then $(F, \psi) \in \mathcal{H}$. Then applying Zorn's lemma, there is a maximal element of $\mathcal{H}$, say $(F, \psi)$.

Finally, we have to prove that $F = L$, and that $\psi(L) = L'$. Suppose $F \neq L$. Then we attempt to produce a larger $\tilde{F}$ and a $K$-isomorphism $\tilde{F} \to \tilde{F}' \subseteq L'$. Since $F \neq L$, there is some $\alpha \in L \setminus F$. Since $L$ is an algebraic extension of $K$, there is some irreducible $g \in K[t]$ such that $\deg g > 0$ and $g(\alpha) = 0$.

Now there is an isomorphism $F[t]/\langle g\rangle \to F(\alpha)$ defined by $\bar{t} \mapsto \alpha$. The isomorphism $\psi : F \to F'$ then extends to an isomorphism $\mu : F[t] \to F'[t]$ and thus to $\mathbb{F}[t]/\langle g\rangle \to F'[t]/\langle \mu(g)\rangle$. Then if $\alpha'$ is a root of $\mu(g)$, then we have $F'[t]//\langle \mu(g)\rangle \cong F'(\alpha')$. So this gives an isomorphism $F(\alpha) \to F(\alpha')$. This contradicts the maximality of $\phi$.

By doing the argument the other way round, we must have $\psi(L) = L'$. So done. □

## 2.6 Separable extensions

**Lemma.** Let $K$ be a field, $f, g \in K[t]$. Then

(i) $(f + g)' = f' + g'$, $(fg)' = fg' + f'g$.

(ii) Assume $f \neq 0$ and $L$ is a splitting field of $f$. Then $f$ has a repeated root in $L$ if and only if $f$ and $f'$ have a common (non-constant) irreducible factor in $K[t]$ (if and only if $f$ and $f'$ have a common root in $L$).

*Proof.*

(i) $(f + g)' = f' + g'$ is true by linearity.

To show that $(fg)' = fg' + f'g$, we use linearity to reduce to the case where $f = t^n, g = t^m$. Then both sides are $(n + m)t^{n+m-1}$. So this holds.

(ii) First assume that $f$ has a repeated root. So let $f = (t - \alpha)^2 h \in L[t]$ where $\alpha \in L$. Then $f' = 2(t - \alpha)h + (t - \alpha)^2 h' = (t - \alpha)(2h + (t - \alpha)h')$. So $f(\alpha) = f'(\alpha) = 0$. So $f$ and $f'$ have common roots. However, we want a common irreducible factor in $K[t]$, not $L[t]$. So we let $P_\alpha$ be the minimal polynomial of $\alpha$ over $K$. Then $P_\alpha \mid f$ and $P_\alpha \mid f'$. So done.

Conversely, suppose $e$ is a common irreducible factor of $f$ and $f'$ in $K[t]$, with $\deg e > 0$. Pick $\alpha \in \mathrm{Root}_e(L)$. Then $\alpha \in \mathrm{Root}_f(L) \cap \mathrm{Root}_{f'}(L)$.

Since $\alpha$ is a root of $f$, we can write $f = (t - \alpha)q \in L[t]$ for some $q$. Then

$$f' = (t - \alpha)q' + q.$$

Since $(t - \alpha) \mid f'$, we must have $(t - \alpha) \mid q$. So $(t - \alpha)^2 \mid f$. □

**Corollary.** Let $K$ be a field, $f \in K[t]$ non-zero irreducible. Then

(i) If $\operatorname{char} K = 0$, then $f$ is separable.

(ii) If $\operatorname{char} K = p > 0$, then $f$ is not separable iff $\deg f > 0$ and $f \in K[t^p]$. For example, $t^{2p} + 3t^p + 1$ is not separable.

*Proof.* By definition, for irreducible $f$, $f$ is not separable iff $f$ has a repeated root. So by our previous lemma, $f$ is not separable if and only if $f$ and $f'$ have a common irreducible factor of positive degree in $K[t]$. However, since $f$ is irreducible, its only factors are 1 and itself. So this can happen if and only if $f' = 0$.

To make it more explicit, we can write

$$f = a_n t^n + \cdots + a_1 t + a_0.$$

Then we can write

$$f' = n a_n t^{n-1} + \cdots + a_1.$$

Now $f' = 0$ if and only if all coefficients $i a_i = 0$ for all $i$.

(i) Suppose $\operatorname{char} K = 0$, then if $\deg f = 0$, then $f$ is trivially separable. If $\deg f > 0$, then $f$ is not separable iff $f' = 0$ iff $i a_i = 0$ for all $i$ iff $a_i = 0$ for $i \geq 1$. But we cannot have a polynomial of positive degree with all its coefficients zero (apart from the constant term). So $f$ must be separable.

(ii) If $\deg f = 0$, then $f$ is trivially separable. So assume $\deg f > 0$.

Then $f$ is not separable $\Leftrightarrow f' = 0 \Leftrightarrow i a_i = 0$ for $i \geq 0 \Leftrightarrow a_i = 0$ for all $i \geq 1$ not multiples of $p \Leftrightarrow f \in K[t^p]$. □

**Lemma.** Let $L/F/K$ be finite extensions, and $E/K$ be a field extension. Then for all $\alpha \in L$, we have

$$|\operatorname{Hom}_K(F(\alpha), E)| \leq [F(\alpha) : F] |\operatorname{Hom}_K(F, E)|.$$

*Proof.* We show that for each $\psi \in \operatorname{Hom}_K(F, E)$, there are at most $[F(\alpha) : F]$ $K$-isomorphisms in $\operatorname{Hom}_K(F(\alpha), E)$ that restrict to $\psi$ in $F$. Since each $K$-isomorphism in $\operatorname{Hom}_K(F(\alpha), E)$ has to restrict to something, it follows that there are at most $[F(\alpha) : F] |\operatorname{Hom}_K(F, E)|$ $K$-homomorphisms from $F(\alpha)$ to $E$.

Now let $P_\alpha$ be the minimal polynomial for $\alpha$ in $F$, and let $\psi \in \operatorname{Hom}_K(F, E)$. To extend $\psi$ to a morphism $F(\alpha) \to E$, we need to decide where to send $\alpha$. So there should be some sort of correspondence

$$\operatorname{Root}_{P_\alpha}(E) \longleftrightarrow \{\phi \in \operatorname{Hom}_K(F(\alpha), E) : \phi|_F = \psi\}.$$

Except that the previous sentence makes no sense, since $P_\alpha \in F[t]$ but we are not told that $F$ is a subfield of $E$. So we use our $\psi$ to "move" our things to $E$.

We let $M = \psi(F) \subseteq E$, and $q \in M[t]$ be the image of $P_\alpha$ under the homomorphism $F[t] \to M[t]$ induced by $\psi$. As we have previously shown, there is a one-to-one correspondence

$$\operatorname{Root}_q(E) \longleftrightarrow \operatorname{Hom}_M(M[t]/\langle q \rangle, E).$$

What we really want to show is the correspondence between $\operatorname{Root}_q(E)$ and the $K$-homomorphisms $F[t]/\langle P_\alpha \rangle \to E$ that restrict to $\psi$ on $F$. Let's ignore the quotient for the moment and think: what does it mean for $\phi \in \operatorname{Hom}_K(F[t], E)$ to restrict to $\psi$ on $F$? We know that any $\phi \in \mathrm{Hom}_L(F[t], E)$ is uniquely determined by the values it takes on $F$ and $t$. Hence if $\phi|_F = \psi$, then our $\phi$ must send $F$ to $\psi(F) = M$, and can send $t$ to anything in $E$. This corresponds exactly to the $M$-homomorphisms $M[t] \to E$ that does nothing to $M$ and sends $t$ to that "anything" in $E$.

The situation does not change when we put back the quotient. Changing from $M[t] \to E$ to $M[t]/\langle q\rangle \to E$ just requires that the image of $t$ must be a root of $q$. On the other hand, using $F[t]/\langle P_\alpha\rangle$ instead of $F[t]$ requires that $\phi(P_\alpha(t)) = 0$. But we know that $\phi(P_\alpha) = \psi(P_\alpha) = q$. So this just requires $q(t) = 0$ as well. So we get the one-to-one correspondence

$$\mathrm{Hom}_M(M[t]/\langle q\rangle, E) \longleftrightarrow \{\phi \in \mathrm{Hom}_K(F[t]/\langle P_\alpha\rangle, E) : \phi|_F = \psi\}.$$

Since $F[t]/\langle P_\alpha\rangle = F(\alpha)$, there is a one-to-one correspondence

$$\mathrm{Root}_q(E) \longleftrightarrow \{\phi \in \mathrm{Hom}_K(F(\alpha), E) : \phi|_F = \psi\}.$$

So done. □

**Theorem.** Let $L/K$ and $E/K$ be field extensions. Then

(i) $|\mathrm{Hom}_K(L, E)| \leq [L : K]$. In particular, $|\mathrm{Aut}_K(L)| \leq [L : K]$.

(ii) If equality holds in (i), then for any intermediate field $K \subseteq F \subseteq L$:

  (a) We also have $|\mathrm{Hom}_K(F, E)| = [F : K]$.

  (b) The map $\mathrm{Hom}_K(L, E) \to \mathrm{Hom}_K(F, E)$ by restriction is surjective.

*Proof.*

(i) We have previously shown we can find a sequence of field extensions

$$K = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = L$$

such that for each $i$, there is some $\alpha_i$ such that $F_i = F_{i-1}(\alpha_i)$. Then by our previous lemma, we have

$$\begin{aligned}
|\mathrm{Hom}_K(L, E)| &\leq [F_n : F_{n-1}]|\mathrm{Hom}_K(F_{n-1}, E)| \\
&\leq [F_n : F_{n-1}][F_{n-1} : F_{n-2}]|\mathrm{Hom}_K(F_{n-2}, E)| \\
&\ \ \vdots \\
&\leq [F_n : F_{n-1}][F_{n-1} : F_{n-2}]\cdots[F_1 : F_0]|\mathrm{Hom}_K(F_0, E)| \\
&= [F_n : F_0] \\
&= [L : K]
\end{aligned}$$

(ii) (a) If equality holds in (i), then every inequality in the proof above has to an equality. Instead of directly decomposing $K \subseteq L$ as a chain above, we can first decompose $K \subseteq F$, then $F \subseteq L$, then join them together. Then we can assume that $F = F_i$ for some $i$. Then we get

$$|\mathrm{Hom}_K(L, E)| = [L : F]|\mathrm{Hom}_K(F, E)| = [L : K].$$

Then the tower law says

$$|\mathrm{Hom}_K(F, E)| = [F : K].$$

(b) By the proof of the lemma, for each $\psi \in \mathrm{Hom}_K(F, E)$, we know that

$$\{\phi : \mathrm{Hom}_K(L, E) : \phi|_F = \psi\} \leq [L : F]. \qquad (*)$$

As we know that

$$|\mathrm{Hom}_K(F, E)| = [F : K], \quad |\mathrm{Hom}_K(L, E)| = [L : K]$$

we must have had equality in $(*)$, or else we won't have enough elements. So in particular $\{\phi : \mathrm{Hom}_K(L, E) : \phi|_F = \psi\} \geq 1$. So the map is surjective. □

**Theorem.** Let $L/K$ be a finite field extension. Then the following are equivalent:

(i) There is some extension $E$ of $K$ such that $|\mathrm{Hom}_K(L, E)| = [L : K]$.

(ii) $L/K$ is separable.

(iii) $L = K(\alpha_1, \cdots, \alpha_n)$ such that $P_{\alpha_i}$, the minimal polynomial of $\alpha_i$ over $K$, is separable for all $i$.

(iv) $L = K(\alpha_1, \cdots, \alpha_n)$ such that $R_{\alpha_i}$, the minimal polynomial of $\alpha_i$ over $K(\alpha_1, \cdots, \alpha_{i-1})$ is separable for all $i$ for all $i$.

*Proof.*

- (i) ⇒ (ii): For all $\alpha \in L$, if $P_\alpha$ is the minimal polynomial of $\alpha$ over $K$, then since $K(\alpha)$ is a subfield of $L$, by our previous theorem, we have

$$|\mathrm{Hom}_K(K(\alpha), E)| = [K(\alpha) : K].$$

We also know that $|\mathrm{Root}_{P_\alpha}(E)| = |\mathrm{Hom}_K(K(\alpha), E)|$, and that $[K(\alpha) : K] = \deg P_\alpha$. So we know that $P_\alpha$ has no repeated roots in any splitting field. So $P_\alpha$ is a separable. So $L/K$ is a separable extension.

- (ii) ⇒ (iii): Obvious from definition

- (iii) ⇒ (iv): Since $R_{\alpha_i}$ is a minimal polynomial in $K(\alpha_1, \cdots, \alpha_{i-1})$, we know that $R_{\alpha_i} \mid P_{\alpha_i}$. So $R_{\alpha_i}$ is separable as $P_{\alpha_i}$ is separable.

- (iv) ⇒ (i): Let $E$ be the splitting field of $P_{\alpha_1}, \cdots, P_{\alpha_n}$. We do induction on $n$ to show that this satisfies the properties we want. If $n = 1$, then $L = K(\alpha_1)$. Then we have

$$|\mathrm{Hom}_K(L, E)| = |\mathrm{Root}_{P_{\alpha_i}}(E)| = \deg P_{\alpha_1} = [K(\alpha_1) : K] = [L : K].$$

We now induct on $n$. So we can assume that (iv) ⇒ (i) holds for smaller number of generators. For convenience, we write $K_i = K(\alpha_1, \cdots, \alpha_i)$. Then we have

$$|\mathrm{Hom}_K(K_{n-1}, E)| = [K_{n-1} : K].$$

We also know that

$$|\mathrm{Hom}_K(K_n, E)| \leq [K_n : K_{n-1}]|\mathrm{Hom}_K(K_{n-1}, E)|.$$

What we actually want is equality. We now re-do (parts of) the proof of this result, and see that separability guarantees that equality holds. If we pick $\psi \in \mathrm{Hom}_K(K_{n-1}, E)$, then there is a one-to-one correspondence between $\{\phi \in \mathrm{Hom}_K(K_n, E) : \phi|_{K_{n-1}} = \psi\}$ and $\mathrm{Root}_q(E)$, where $q \in M[t]$ is defined as the image of $R_{\alpha_n}$ under $K_{n-1}[t] \to M[t]$, and $M$ is the image of $\psi$.

Since $P_{\alpha_n} \in K[t]$ and $R_{\alpha_n} \mid P_{\alpha_n}$, then $q \mid P_{\alpha_n}$. So $q$ splits over $E$. By separability assumption , we get that

$$|\mathrm{Root}_q(E)| = \deg q = \deg R_{\alpha_n} = [K_n : K_{n-1}].$$

Hence we know that

$$\begin{aligned} |\mathrm{Hom}_K(L, E)| &= [K_n : K_{n-1}]|\mathrm{Hom}_K(K_{n-1}, E)| \\ &= [K_n : K_{n-1}][K_{n-1} : K] \\ &= [K_n : K]. \end{aligned}$$

So done. □

**Lemma.** Let $L$ be a field, $L^* \setminus \{0\}$ be the multiplicative group of $L$. If $G$ is a finite subgroup of $L^*$, then $G$ is cyclic.

*Proof.* Since $L^*$ is abelian, $G$ is also abelian. Then by the structure theorem on finite abelian groups,

$$G \cong \frac{\mathbb{Z}}{\langle n_1 \rangle} \times \cdots \times \frac{\mathbb{Z}}{\langle n_r \rangle},$$

for some $n_i \in \mathbb{N}$. Let $m$ be the least common multiple of $n_1, \cdots, n_r$, and let $f = t^m - 1$.

If $\alpha \in G$, then $\alpha^m = 1$. So $f(\alpha) = 0$ for all $\alpha \in G$. Therefore

$$|G| = n_1 \cdots n_r \leq |\mathrm{Root}_f(L)| \leq \deg f = m.$$

Since $m$ is the least common multiple of $n_1, \cdots, n_r$, we must have $m = n_1 \cdots n_r$ and thus $(n_i, n_j) = 1$ for all $i \neq j$. Then by the Chinese remainder theorem, we have

$$G \cong \frac{Z}{\langle n_1 \rangle} \times \cdots \times \frac{\mathbb{Z}}{\langle n_r \rangle} = \frac{\mathbb{Z}}{\langle n_1 \cdots n_r \rangle}.$$

So $G$ is cyclic. □

**Theorem** (Primitive element theorem)**.** Assume $L/K$ is a finite and separable extension. Then $L/K$ is simple, i.e. there is some $\alpha \in L$ such that $L = K(\alpha)$.

*Proof.* At some point in our proof, we will require that $L$ is infinite. So we first do the finite case first. If $K$ is finite, then $L$ is also finite, which in turns implies $L^*$ is finite too. So by the lemma, $L^*$ is a cyclic group (since it is a finite subgroup of itself). So there is some $\alpha \in L^*$ such that every element in $L^*$ is a power of $\alpha$. So $L = K(\alpha)$.

So focus on the case where $K$ is infinite. Also, assume $K \neq L$. Then since $L/K$ is a finite extension, there is some intermediate field $K \subseteq F \subsetneq L$ such that $L = F(\beta)$ for some $\beta$. Now $L/K$ is separable. So $F/K$ is also separable, and $[F : K] < [L : K]$. Then by induction on degree of extension, we can assume $F/K$ is simple. In other words, there is some $\lambda \in F$ such that $F = K(\lambda)$. Now $L = K(\lambda, \beta)$. In the rest of the proof, we will try to replace the two generators $\lambda, \beta$ with just a single generator.

Unsurprisingly, the generator of $L$ will be chosen to be a linear combination of $\beta$ and $\lambda$. We set

$$\alpha = \beta + a\lambda$$

for some $a \in K$ to be chosen later. We will show that $K(\alpha) = L$. Actually, almost any choice of $a$ will do, but at the end of the proof, we will see which ones are the bad ones.

Let $P_\beta$ and $P_\lambda$ be the minimal polynomial of $\beta$ and $\lambda$ over $K$ respectively. Consider the polynomial $f = P_\beta(\alpha - at) \in K(\alpha)[t]$. Then we have

$$f(\lambda) = P_\beta(\alpha - a\lambda) = P_\beta(\beta) = 0.$$

On the other hand, $P_\lambda(\lambda) = 0$. So $\lambda$ is a common root of $P_\lambda$ and $f$.

We now want to pick an $a$ such that $\lambda$ is the *only* common root of $f$ and $P_\lambda$ (in $E$). If so, then the gcd of $f$ and $P_\alpha$ in $K(\alpha)$ must only have $\lambda$ as a root. But since $P_\lambda$ is separable, it has no double roots. So the gcd must be $t - \lambda$. In particular, we must have $\lambda \in K(\alpha)$. Since $\alpha = \beta + a\lambda$, it follows that $\beta \in K(\alpha)$ as well, and so $K(\alpha) = L$.

Thus, it remains to choose an $a$ such that there are no other common roots. We work in a splitting field of $P_\beta P_\lambda$, and write

$$\begin{aligned} P_\beta &= (t - \beta_1) \cdots (t - \beta_m) \\ P_\lambda &= (t - \lambda_1) \cdots (t - \lambda_n). \end{aligned}$$

We wlog $\beta_1 = \beta$ and $\lambda_1 = \lambda$.

Now suppose $\theta$ is a common root of $f$ and $P_\lambda$. Then

$$\begin{cases} f(\theta) = 0 \\ P_\lambda(\theta) = 0 \end{cases} \Rightarrow \begin{cases} P_\beta(\alpha - a\theta) = 0 \\ P_\lambda(\theta) = 0 \end{cases} \Rightarrow \begin{cases} \alpha - a\theta = \beta_i \\ \theta = \lambda_j \end{cases}$$

for some $i, j$. Then we know that

$$\alpha = \beta_i + a\lambda_j.$$

However, by definition, we also know that

$$\alpha = \beta + a\lambda$$

Now we see how we need to choose $a$. We need to choose $a$ such that the elements

$$\beta + a\lambda \neq \beta_i + a\lambda_j$$

for all $i, j$. But if they were equal, then we have

$$a = \frac{\lambda - \lambda_j}{\beta_i - \beta},$$

and there are only finitely many elements of this form. So we just have to pick an $a$ *not* in this list. □

**Corollary.** Any finite extension $L/K$ of field of characteristic 0 is simple, i.e. $L = K(\alpha)$ for some $\alpha \in L$.

*Proof.* This follows from the fact that all extensions of fields of characteristic zero are separable. □

**Proposition.** Let $L/K$ be an extension of finite fields. Then the extension is separable.

*Proof.* Let the characteristic of the fields be $p$. Suppose the extension were not separable. Then there is some non-separable element $\alpha \in L$. Then its minimal polynomial must be of the form $P_\alpha = \sum a_i t^{pi}$.

Now note that the map $K \to K$ given by $x \mapsto x^p$ is injective, hence surjective. So we can write $a_i = b_i^p$ for all $i$. Then we have

$$P_\alpha = \sum a_i t^{pi} = \left(\sum b_i t^i\right)^p,$$

and so $P_\alpha$ is not irreducible, which is a contradiction. □

## 2.7 Normal extensions

**Lemma.** Let $L/F/K$ be finite extensions, and $\bar{K}$ is the algebraic closure of $K$. Then any $\psi \in \mathrm{Hom}_K(F, \bar{K})$ extends to some $\phi \in \mathrm{Hom}_K(L, \bar{K})$.

*Proof.* Let $\psi \in \mathrm{Hom}_K(F, \bar{K})$. If $F = L$, then the statement is trivial. So assume $L \neq F$.

Pick $\alpha \in L \setminus F$. Let $q_\alpha \in F[t]$ be the minimal polynomial of $\alpha$ over $F$. Consider $\psi(q_\alpha) \in \bar{K}[t]$. Let $\beta$ be any root of $q_\alpha$, which exists since $\bar{K}$ is algebraically closed. Then as before, we can extend $\psi$ to $F(\alpha)$ by sending $\alpha$ to $\beta$. More explicitly, we send

$$\sum_{i=0}^{N} a_i \alpha^i \mapsto \sum \psi(a_i)\beta^i,$$

which is well-defined since any polynomial relation satisfied by $\alpha$ in $F$ is also satisfied by $\beta$.

Repeat this process finitely many times to get some element in $\mathrm{Hom}_K(L, \bar{K})$. □

**Theorem.** Let $L/K$ be a finite extension. Then $L/K$ is a normal extension if and only if $L$ is the splitting field of some $f \in K[t]$.

*Proof.* Suppose $L/K$ is normal. Since $L$ is finite, let $L = K(\alpha_1, \cdots, \alpha_n)$ for some $\alpha_i \in L$. Let $P_{\alpha_i}$ be the minimal polynomial of $\alpha_i$ over $K$. Take $f = P_{\alpha_1} \cdots P_{\alpha_n}$. Since $L/K$ is normal, each $P_{\alpha_i}$ splits over $L$. So $f$ splits over $L$, and $L$ is a splitting field of $f$.

For the other direction, suppose that $L$ is the splitting field of some $f \in K[t]$. First we wlog assume $L \subseteq \bar{K}$. This is possible since the natural injection $K \hookrightarrow \bar{K}$ extends to some $\phi : L \to \bar{K}$ by our previous lemma, and we can replace $L$ with $\phi(L)$.

Now suppose $\beta \in L$, and let $P_\beta$ be its minimal polynomial. Let $\beta'$ be another root. We want to show it lives in $L$.

Now consider $K(\beta)$. By the proof of the lemma, we can produce an embedding $\iota : K(\beta) \to \bar{K}$ that sends $\beta$ to $\beta'$. By the lemma again, this extends to an embedding of $L$ into $\bar{K}$. But any such embedding must send a root of $f$ to a root of $f$. So it must send $L$ to $L$. In particular, $\iota(\beta) = \beta' \in L$. So $P_\beta$ splits over $L$. ☐

**Theorem.** Let $L/K$ be a finite extension. Then the following are equivalent:

(i) $L/K$ is a Galois extension.

(ii) $L/K$ is separable and normal.

(iii) $L = K(\alpha_1, \cdots, \alpha_n)$ and $P_{\alpha_i}$, the minimal polynomial of $\alpha_i$ over $K$, is separable and splits over $L$ for all $i$.

*Proof.*

– (i) ⇒ (ii): Suppose $L/K$ is a Galois extension. Then by definition, this means
$$|\operatorname{Hom}_K(L, L)| = |\operatorname{Aut}_K(L)| = [L : K].$$
To show that $L/K$ is separable, recall that we proved that an extension is separable if and only if there is some $E$ such that $|\operatorname{Hom}_K(L, E)| = [L : K]$. In this case, just pick $E = L$. Then we know that the extension is separable.

To check normality, let $\alpha \in L$, and let $P_\alpha$ be its minimal polynomial over $K$. We know that
$$|\operatorname{Root}_{P_\alpha}(L)| = |\operatorname{Hom}_K(K[t]/\langle P_\alpha\rangle, L)| = |\operatorname{Hom}_K(K(\alpha), L)|.$$
But since $|\operatorname{Hom}_K(L, L)| = [L : K]$ and $K(\alpha)$ is a subfield of $L$, this implies
$$|\operatorname{Hom}_K(K(\alpha), L)| = [K(\alpha) : K] = \deg P_\alpha.$$
Hence we know that
$$|\operatorname{Root}_{P_\alpha}(L)| = \deg P_\alpha.$$
So $P_\alpha$ splits over $L$.

– (ii) ⇒ (iii): Just pick $\alpha_1, \cdots, \alpha_n$ such that $L = K(\alpha_1, \cdots, \alpha_n)$. Then these polynomials are separable since the extension is separable, and they split since $L/K$ is normal. In fact, by the primitive element theorem, we can pick these such that $n = 1$.

– (iii) ⇒ (i): Since $L = K(\alpha_1, \cdots, \alpha_n)$ and the minimal polynomials $P_{\alpha_i}$ over $K$ are separable, by a previous theorem, there are some extension $E$ of $K$ such that
$$|\operatorname{Hom}_K(L, E)| = [L : K].$$
To simplify notation, we first replace $L$ with its image inside $E$ under some $K$-homomorphism $L \to E$, which exists since $|\operatorname{Hom}_K(L, E)| = [L : K] > 0$. So we can assume $L \subseteq E$.

We now claim that the inclusion
$$\operatorname{Hom}_K(L, L) \to \operatorname{Hom}_K(L, E)$$
is a surjection, hence a bijection. Indeed, if $\phi : L \to E$, then $\phi$ takes $\alpha_i$ to $\phi(\alpha_i)$, which is a root of $P_{\alpha_i}$. Since $P_{\alpha_i}$ splits over $L$, we know $\phi(\alpha_i) \in L$ for all $i$. Since $L$ is generated by these $\alpha_i$, it follows that $\phi(L) \subseteq L$.

Thus, we have

$$[L : K] = |\operatorname{Hom}_K(L, E)| = |\operatorname{Hom}_K(L, L)|,$$

and the extension is Galois. □

**Corollary.** Let $K$ be a field and $f \in K[t]$ be a separable polynomial. Then the splitting field of $f$ is Galois.

## 2.8 The fundamental theorem of Galois theory

**Lemma** (Artin's lemma)**.** Let $L/K$ be a field extension and $H \leq \operatorname{Aut}_K(L)$ a finite subgroup. Then $L/L^H$ is a Galois extension with $\operatorname{Aut}_{L^H}(L) = H$.

*Proof.* Pick any $\alpha \in L$. We set

$$\{\alpha_1, \cdots, \alpha_n\} = \{\phi(\alpha) : \phi \in H\},$$

where $\alpha_i$ are distinct. Here we are allowing for the possibility that $\phi(\alpha) = \psi(\alpha)$ for some distinct $\phi, \psi \in H$.

By definition, we clearly have $n < |H|$. Let

$$f = \prod_1^n (t - \alpha_i) \in L[t].$$

We know that any $\phi \in H$ gives an homomorphism $L[t] \to L[t]$, and any such map fixes $f$ because $\phi$ just permutes the $\alpha_i$. Thus, the coefficients of $f$ are in $L^H$, and thus $f \in L^H[t]$.

Since $\mathrm{id} \in H$, we know that $f(\alpha) = 0$. So $\alpha$ is algebraic over $L^H$. Moreover, if $q_\alpha$ is the minimal polynomial of $\alpha$ over $L^H$, then $q_\alpha \mid f$ in $L^H[t]$. Hence

$$[L^H(\alpha) : L^H] = \deg q_\alpha \leq \deg f \leq |H|.$$

Further, we know that $f$ has distinct roots. So $q_\alpha$ is separable, and so $\alpha$ is separable. So it follows that $L/L^H$ is a separable extension.

We next show that $L/L^H$ is simple. This doesn't immediately follow from the primitive element theorem, because we don't know it is a finite extension yet, but we can still apply the theorem cleverly.

Pick $\alpha \in L$ such that $[L^H(\alpha) : L^H]$ is maximal. This is possible since $[L^H(\alpha) : L^H]$ is bounded by $|H|$. The claim is that $L = L^H(\alpha)$.

We pick an arbitrary $\beta \in L$, and will show that this is in $L^H(\alpha)$. By the above arguments, $L^H \subseteq L^H(\alpha, \beta)$ is a finite separable extension. So by the primitive element theorem, there is some $\lambda \in L$ such that $L^H(\alpha, \beta) = L^H(\lambda)$. Note that we must have

$$[L^H(\lambda) : L^H] \geq [L^H(\alpha) : L^H].$$

By maximality of $[L^H(\alpha) : L^H]$, we must have equality. So $L^H(\lambda) = L^H(\alpha)$. So $\beta \in L^H(\alpha)$. So $L = L^H(\alpha)$.

Finally, we show it is a Galois extension. Let $L = L^H(\alpha)$. Then

$$[L : L^H] = [L^H(\alpha) : L^H] \leq |H| \leq |\operatorname{Aut}_{L^H}(L)|$$

Recall that we have previously shown that for any extension $L/L^H$, we have $|\operatorname{Aut}_{L^H}(L)| \leq [L : L^H]$. Hence we must have equality above. So

$$[L : L^H] = |\operatorname{Aut}_{L^H}(L)|.$$

So the extension is Galois. Also, since we know that $H \subseteq \operatorname{Aut}_{L^H}(L)$, we must have $H = \operatorname{Aut}_{L^H}(L)$. □

**Theorem.** Let $L/K$ be a finite field extension. Then $L/K$ is Galois if and only if $L^H = K$, where $H = \operatorname{Aut}_K(L)$.

*Proof.* ($\Rightarrow$) Suppose $L/K$ is a Galois extension. We want to show $L^H = K$. Using Artin's lemma (and the definition of $H$), we have

$$[L : K] = |\operatorname{Aut}_K(L)| = |H| = |\operatorname{Aut}_{L^H}(L)| = [L : L^H]$$

So $[L : K] = [L : L^H]$. So we must have $L^H = K$.

($\Leftarrow$) By the lemma, $K = L^H \subseteq L$ is Galois. □

**Theorem** (Fundamental theorem of Galois theory)**.** Assume $L/K$ is a (finite) Galois extension. Then

(i) There is a one-to-one correspondence

$$H \leq \operatorname{Aut}_K(L) \longleftrightarrow \text{intermediate fields } K \subseteq F \subseteq L.$$

This is given by the maps $H \mapsto L^H$ and $F \mapsto \operatorname{Aut}_F(L)$ respectively. Moreover, $|\operatorname{Aut}_K(L) : H| = [L^H : K]$.

(ii) $H \leq \operatorname{Aut}_K(L)$ is normal (as a subgroup) if and only if $L^H/K$ is a normal extension if and only if $L^H/K$ is a Galois extension.

(iii) If $H \lhd \operatorname{Aut}_K(L)$, then the map $\operatorname{Aut}_K(L) \to \operatorname{Aut}_K(L^H)$ by the restriction map is well-defined and surjective with kernel isomorphic to $H$, i.e.

$$\frac{\operatorname{Aut}_K(L)}{H} = \operatorname{Aut}_K(L^H).$$

*Proof.* Note that since $L/K$ is a Galois extension, we know

$$|\operatorname{Aut}_K(L)| = |\operatorname{Hom}_K(L, L)| = [L : K],$$

By a previous theorem, for any intermediate field $K \subseteq F \subseteq L$, we know $|\operatorname{Hom}_K(F, L)| = [F : K]$ and the restriction map $\operatorname{Hom}_K(L, L) \to \operatorname{Hom}_K(F, L)$ is surjective.

(i) The maps are already well-defined, so we just have to show that the maps are inverses to each other. By Artin's lemma, we know that $H = \operatorname{Aut}_{L^H}(L)$, and since $L/F$ is a Galois extension, the previous theorem tells that $L^{\operatorname{Aut}_F(L)} = F$. So they are indeed inverses. The formula relating the index and the degree follows from Artin's lemma.

(ii) Note that for every $\phi \in \mathrm{Aut}_K(L)$, we have that $L^{\phi H \phi^{-1}} = \phi L^H$, since $\alpha \in L^{\phi H \phi^{-1}}$ iff $\phi(\psi(\phi^{-1}(\alpha))) = \alpha$ for all $\psi \in H$ iff $\psi(\phi^{-1}(\alpha)) = \phi^{-1}(\alpha)$ for all $\psi \in H$ iff $\alpha \in \phi L^H$. Hence $H$ is a normal subgroup if and only if

$$\phi(L^H) = L^H \text{ for all } \phi \in \mathrm{Aut}_K(L). \tag{$*$}$$

Assume $(*)$. We want to first show that $\mathrm{Hom}_K(L^H, L^H) = \mathrm{Hom}_K(L^H, L)$. Let $\psi \in \mathrm{Hom}_K(L^H, L)$. Then by the surjectivity of the restriction map $\mathrm{Hom}_K(L, L) \to \mathrm{Hom}_K(L^H, L)$, $\psi$ must be the restriction of some $\tilde{\psi} \in \mathrm{Hom}_K(L, L)$. So $\tilde{\psi}$ fixes $L^H$ by $(*)$. So $\psi$ sends $L^H$ to $L^H$. So $\psi \in \mathrm{Hom}_K(L^H, L^H)$. So we have

$$|\mathrm{Aut}_K(L^H)| = |\mathrm{Hom}_K(L^H, L^H)| = |\mathrm{Hom}_K(L^H, L)| = [L^H : K].$$

So $L^H/K$ is Galois, and hence normal.

Now suppose $L^H/K$ is a normal extension. We want to show this implies $(*)$. Pick any $\alpha \in L^H$ and $\phi \in \mathrm{Aut}_K(L)$. Let $P_\alpha$ be the minimal polynomial of $\alpha$ over $K$. So $\phi(\alpha)$ is a root of $P_\alpha$ (since $\phi$ fixes $P_\alpha \in K$, and hence maps roots to roots). Since $L^H/K$ is normal, $P_\alpha$ splits over $L^H$. This implies that $\phi(\alpha) \in L^H$. So $\phi(L^H) = L^H$.

Hence, $H$ is a normal subgroup if and only if $\phi(L^H) = L^H$ if and only if $L^H/K$ is a Galois extension.

(iii) Suppose $H$ is normal. We know that $\mathrm{Aut}_K(L) = \mathrm{Hom}_K(L, L)$ restricts to $\mathrm{Hom}_K(L^H, L)$ surjectively. To show that we in fact have restriction to $\mathrm{Aut}_K(L^H)$, by the proof above, we know that $\phi(L^H) = L^H$ for all $\phi \in \mathrm{Aut}_K(L^H)$. So this does restrict to an automorphism of $L^H$. In other words, the map $\mathrm{Aut}_K(L) \to \mathrm{Aut}_K(L^H)$ is well-defined. It is easy to see this is a group homomorphism.

Finally, we have to calculate the kernel of this homomorphism. Let $E$ be the kernel. Then by definition, $E \supseteq H$. So it suffices to show that $|E| = |H|$. By surjectivity of the map and the first isomorphism theorem of groups, we have

$$\frac{|\mathrm{Aut}_K(L)|}{|E|} = |\mathrm{Aut}_K(L^H)| = [L^H : K] = \frac{[L : K]}{[L : L^H]} = \frac{|\mathrm{Aut}_K(L)|}{|H|},$$

noting that $L^H/K$ and $L/K$ are both Galois extensions, and $|H| = [L^H : K]$ by Artin's lemma. So $|E| = |H|$. So we must have $E = H$. □

## 2.9 Finite fields

**Lemma.** Let $K$ be a finite field with $q = |K|$ element. Then

(i) $q = p^d$ for some $d \in \mathbb{N}$, where $p = \mathrm{char}\, K > 0$.

(ii) Let $f = t^q - t$. Then $f(\alpha) = 0$ for all $\alpha \in K$. Moreover, $K$ is the splitting field of $f$ over $\mathbb{F}_p$.

*Proof.*

(i) Consider the set $\{m \cdot 1_K\}_{m \in \mathbb{Z}}$, where $1_K$ is the unit in $K$ and $m\cdot$ represents repeated addition. We can identify this with $\mathbb{F}_p$. So we have the extension $\mathbb{F}_p \subseteq K$. Let $d = [K : \mathbb{F}_p]$. Then $q = |K| = p^d$.

(ii) Note that $K^* = K \setminus \{0\}$ is a finite multiplicative group with order $q - 1$. Then by Lagrange's theorem, $\alpha^{q-1} = 1$ for all $\alpha \in K^*$. So $\alpha^q - \alpha = 0$ for all $\alpha \neq 0$. The $\alpha = 0$ case is trivial.

Now every element in $K$ is a root of $f$. So we need to check that all roots of $f$ are in $K$. Note that the derivative $f' = qt^{q-1} - 1 = -1$ (since $q$ is a power of the characteristic). So $f'(\alpha) = -1 \neq 0$ for all $\alpha \in K$. So $f$ and $f'$ have no common roots. So $f$ has no repeated roots. So $K$ contains $q$ distinct roots of $f$. So $K$ is a splitting field. □

**Lemma.** Let $q = p^d$, $q' = p^{d'}$, where $d, d' \in \mathbb{N}$. Then

(i) There is a finite field $K$ with exactly $q$ elements, which is unique up to isomorphism. We write this as $\mathbb{F}_q$.

(ii) We can embed $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$ iff $d \mid d'$.

*Proof.*

(i) Let $f = t^q - t$, and let $K$ be a splitting field of $f$ over $\mathbb{F}_p$. Let $L = \mathrm{Root}_f(K)$. The objective is to show that $L = K$. Then we will have $|K| = |L| = |\mathrm{Root}_f(K)| = \deg f = q$, because the proof of the previous lemma shows that $f$ has no repeated roots.

To show that $L = K$, by definition, we have $L \subseteq K$. So we need to show every element in $K$ is in $L$. We do so by showing that $L$ itself is a field. Then since $L$ contains all the roots of $f$ and is a subfield of the splitting field $K$, we must have $K = L$.

It is straightforward to show that $L$ is a field: if $\alpha, \beta \in L$, then

$$(\alpha + \beta)^q = \alpha^q + \beta^q = \alpha + \beta.$$

So $\alpha + \beta \in L$. Similarly, we have

$$(\alpha\beta)^q = \alpha^q \beta^q = \alpha\beta.$$

So $\alpha\beta \in L$. Also, we have

$$(\alpha^{-1})^q = (\alpha^q)^{-1} = \alpha^{-1}.$$

So $\alpha^{-1} \in L$. So $L$ is in fact a field.

Since any field of size $q$ is a splitting field of $f$, and splitting fields are unique to isomorphism, we know that $K$ is unique.

(ii) Suppose $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$. Then let $n = [\mathbb{F}_{q'} : \mathbb{F}_q]$. So $q' = q^n$. So $d' = nd$. So $d \mid d'$.

On the other hand, suppose $d \mid d'$. Let $d' = dn$. We let $f = t^{q'} - t$. Then for any $\alpha \in \mathbb{F}_q$, we have

$$f(\alpha) = \alpha^{q'} - \alpha = \alpha^{q^n} - \alpha = (\cdots((\alpha^q)^q)^q \cdots)^q - \alpha = \alpha - \alpha = 0.$$

Since $\mathbb{F}_{q'}$ is the splitting field of $f$, all roots of $f$ are in $\mathbb{F}_{q'}$. So we know that $\mathbb{F}_q \subseteq \mathbb{F}_q'$. □

**Theorem.** Consider $\mathbb{F}_{q^n}/\mathbb{F}_q$. Then $\mathrm{Fr}_q$ is an element of order $n$ as an element of $\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$.

*Proof.* For all $\alpha \in \mathbb{F}_{q^n}$, we have $\mathrm{Fr}_q^n(\alpha) = \alpha^{q^n} = \alpha$. So the order of $\mathrm{Fr}_q$ divides $n$.

If $m \mid n$, then the set

$$\{\alpha \in \mathbb{F}_{q^n} : \mathrm{Fr}_q^m(\alpha) = \alpha\} = \{\alpha \in \mathbb{F}_{q^n} : \alpha^{q^m} = \alpha\} = \mathbb{F}_{q^m}.$$

So if $m$ is the order of $\mathrm{Fr}_q$, then $\mathbb{F}_{q^m} = \mathbb{F}_{q^n}$. So $m = n$. □

**Theorem.** The extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is Galois with Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong \mathbb{Z}/n\mathbb{Z}$, generated by $\mathrm{Fr}_q$.

*Proof.* The multiplicative group $\mathbb{F}_{q^n}^* = \mathbb{F}_{q^n} \setminus \{0\}$ is finite. We have previously seen that multiplicative groups of finite fields are cyclic. So let $\alpha$ be a generator of this group. Then $\mathbb{F}_{q^n} = \mathbb{F}_q(\alpha)$. Let $P_\alpha$ be the minimal polynomial of $\alpha$ over $\mathbb{F}_q$. Then since $\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$ has an element of order $n$, we get

$$n \leq |\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n})| = |\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q(\alpha), \mathbb{F}_{q^n})|.$$

Since $F_q(\alpha)$ is generated by one element, we know

$$|\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q(\alpha), \mathbb{F}_{q^n})| = |\mathrm{Root}_{P_\alpha}(\mathbb{F}_{q^n})|$$

So we have

$$n \leq |\mathrm{Root}_{P_\alpha}(\mathbb{F}_{q^n})| \leq \deg P_\alpha = [\mathbb{F}_{q^n} : \mathbb{F}_q] = n.$$

So we know that

$$|\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n})| = [\mathbb{F}_{q^n} : \mathbb{F}_q] = n.$$

So $\mathbb{F}_{q^n}/\mathbb{F}_q$ is a Galois extension.

Since $|\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^n})|$, it has to be generated by $\mathrm{Fr}_q$, since this has order $n$. In particular, this group is cyclic. □

# 3 Solutions to polynomial equations

## 3.1 Cyclotomic extensions

**Theorem.** For each $d \in \mathbb{N}$, there exists a *dth cyclotomic monic polynomial* $\phi_d \in \mathbb{Z}[t]$ satisfying:

(i) For each $n \in \mathbb{N}$, we have

$$t^n - 1 = \prod_{d \mid n} \phi_d.$$

(ii) Assume $\operatorname{char} K = 0$ or $0 < \operatorname{char} K \nmid n$. Then

$$\mathrm{Root}_{\phi_n}(L) = \{n\text{th primitive roots of unity}\}.$$

Note that here we have an abuse of notation, since $\phi_n$ is a polynomial in $\mathbb{Z}[t]$, not $K[t]$, but we can just use the canonical map $\mathbb{Z}[t] \to K[t]$ mapping 1 to 1 and $t$ to $t$.

*Proof.* We do induction on $n$ to construct $\phi_n$. When $n = 1$, let $\phi_1 = t - 1$. Then (i) and (ii) hold in this case, trivially.

Assume now that (i) and (ii) hold for smaller values of $n$. Let

$$f = \prod_{d \mid n, d < n} \phi_d.$$

By induction, $f \in \mathbb{Z}[t]$. Moreover, if $d \mid n$ and $d < n$, then $\phi_d \mid (t^n - 1)$ because $(t^d - 1) \mid (t^n - 1)$. We would like to say that $f$ also divides $t^n - 1$. However, we have to be careful, since to make this conclusion, we need to show that $\phi_d$ and $\phi_{d'}$ have no common roots for distinct $d, d' \mid n$ (and $d, , d' < n$).

Indeed, by induction, $\phi_d$ and $\phi'_d$ have no common roots because

$$\mathrm{Root}_{\phi_d}(L) = \{d\text{th primitive roots of unity}\},$$
$$\mathrm{Root}_{\phi_{d'}}(L) = \{d'\text{th primitive roots of unity}\},$$

and these two sets are disjoint (or else the roots would not be *primitive*). Therefore $\phi_d$ and $\phi_{d'}$ have no common irreducible factors. Hence $f \mid t^n - 1$. So we can write

$$t^n - 1 = f\phi_n,$$

where $\phi_n \in \mathbb{Q}[t]$. Since $f$ is monic, $\phi_n$ has integer coefficients. So indeed $\phi_n \in \mathbb{Z}[t]$. So the first part is proven.

To prove the second part, note that by induction,

$$\mathrm{Root}_f(L) = \{\text{non-primitive } n\text{th roots of unit}\},$$

since all $n$th roots of unity are $d$th primitive roots of unity for some smaller $d$.

Since $f\phi_n = t^n - 1$, $\phi_n$ contains the remaining, primitive $n$th roots of unit. Since $t^n - 1$ has no repeated roots, we know that $\phi_n$ does not contain any extra roots. So

$$\mathrm{Root}_{\phi_n}(L) = \{n\text{th primitive roots of unity}\}.$$

□

**Theorem.** Let $K$ be a field with $\operatorname{char} K = 0$ or $0 < \operatorname{char} K \nmid n$. Let $L$ be the $n$th cyclotomic extension of $K$. Then $L/K$ is a Galois extension, and there is an injective homomorphism $\theta : \mathrm{Gal}(L/K) \to (\mathbb{Z}/n\mathbb{Z})^\times$.

In addition, every irreducible factor of $\phi_n$ (in $K[t]$) has degree $[L : K]$.

*Proof.* Let $\mu$ be an $n$th primitive root of unity. Then

$$\mathrm{Root}_{t^n-1}(L) = \{1, \mu, \mu^2, \cdots, \mu^{n-1}\}$$

is a cyclic group of order $n$ generated by $\mu$. We first construct the homomorphism $\theta : \mathrm{Aut}_K(L) \to (\mathbb{Z}/n\mathbb{Z})^\times$ as follows: for each $\phi \in \mathrm{Aut}_K(L)$, $\phi$ is completely determined by the value of $\phi(\mu)$ since $L = K(\mu)$. Since $\phi$ is an automorphism, it must take an $n$th primitive root of unity to another $n$th primitive root of unity. So $\phi(\mu) = \mu^i$ for some $i$ such that $(i, n) = 1$. Now let $\theta(\phi) = \bar{i} \in (\mathbb{Z}/n\mathbb{Z})^\times$. Note that this is well-defined since if $\mu^i = \mu^j$, then $i - j$ has to be a multiple of $n$.

Now it is easy to see that if $\phi, \psi \in \mathrm{Aut}_K(L)$ are given by $\phi(\mu) = \mu^i$, and $\psi(\mu) = \mu^j$, then $\phi \circ \psi(\mu) = \phi(\mu^j) = \mu^{ij}$. So $\theta(\phi\psi) = \bar{i}\bar{j} = \theta(\phi)\theta(\psi)$. So $\theta$ is a group homomorphism.

Now we check that $\theta$ is injective. If $\theta(\phi) = \bar{1}$ (note that $(\mathbb{Z}/n\mathbb{Z})^\times$ is a multiplicative group with unit 1), then $\phi(\mu) = \mu$. So $\phi = \mathrm{id}$.

Now we show that $L/K$ is Galois. Recall that $L = K(\mu)$, and let $P_\mu$ be a minimal polynomial of $\mu$ over $K$. Since $\mu$ is a root of $t^n - 1$, we know that $P_\mu \mid t^n - 1$. Since $t^n - 1$ has no repeated roots, $P_\mu$ has no repeated roots. So $P_\mu$ is separable. Moreover, $P_\mu$ splits over $L$ as $t^n - 1$ splits over $L$. So the extension is separable and normal, and hence Galois.

Applying the previous theorem, each irreducible factor $g$ of $\phi_n$ is a minimal polynomial of some $n$th primitive root of unity, say $\lambda$. Then $L = K(\lambda)$. So

$$\deg g = \deg P_\lambda = [K(\lambda) : K] = [L : K]. \qquad \square$$

**Lemma.** Under the notation and assumptions of the previous theorem, $\phi_n$ is irreducible in $K[t]$ if and only if $\theta$ is an isomorphism.

*Proof.* ($\Rightarrow$) Suppose $\phi_n$ is irreducible. Recall that $\mathrm{Root}_{\phi_n}(L)$ is exactly the $n$th primitive roots of unity. So if $\mu$ is an $n$th primitive root of unity, then $P_\mu$, the minimal polynomial of $\mu$ over $K$ is $\phi_n$. In particular, if $\lambda$ is also an $n$th primitive root of unity, then $P_\mu = P_\lambda$. This implies that there is some $\phi_\lambda \in \mathrm{Aut}_K(L)$ such that $\phi_\lambda(\mu) = \lambda$.

Now if $\bar{i} \in (\mathbb{Z}/n\mathbb{Z})^\times$, then taking $\lambda = \mu^i$, this shows that we have $\phi_\lambda \in \mathrm{Aut}_K(L)$ such that $\theta(\phi_\lambda) = \bar{i}$. So $\theta$ is surjective, and hence an isomorphism.

($\Leftrightarrow$) Suppose that $\theta$ is an isomorphism. We will reverse the above argument and show that all roots have the same minimal polynomial. Let $\mu$ be a $n$th primitive root of unity, and pick $\bar{i} \in (\mathbb{Z}/n\mathbb{Z})^\times$, and let $\lambda = \mu^i$. Since $\theta$ is an isomorphism, there is some $\phi_\lambda \in \mathrm{Aut}_K(L)$ such that $\theta(\phi_\lambda) = \bar{i}$, i.e. $\phi_\lambda(\mu) = \mu^i = \lambda$. Then we must have $P_\mu = P_\lambda$.

Since every $n$th primitive root of unity is of the form $\mu^i$ (with $(i, n) = 1$), this implies that all $n$th primitive roots have the same minimal polynomial. Since the roots of $\phi_n$ are all the $n$th primitive roots of unity, its irreducible factors are exactly the minimal polynomials of the primitive roots. Moreover, $\phi_n$ does not have repeated roots. So $\phi_n = P_\mu$. In particular, $\phi_n$ is irreducible. $\square$

**Theorem.** $\phi_n$ is irreducible in $\mathbb{Q}[t]$. In particular, it is also irreducible in $\mathbb{Z}[t]$.

*Proof.* As before, this can be achieved by showing that all $n$th primitive roots have the same minimal polynomial. Moreover, let $\mu$ be our favorite $n$th primitive root. Then all other primitive roots $\lambda$ are of the form $\lambda = \mu^i$, where $(i, n) = 1$. By the fundamental theorem of arithmetic, we can write $i$ as a product $i = q_1 \cdots q_n$. Hence it suffices to show that for all primes $q \nmid n$, we have $P_\mu = P_{\mu^q}$. Noting that $\mu^q$ is also an $n$th primitive root, this gives

$$P_\mu = P_{\mu^{q_1}} = P_{(\mu^{q_1})^{q_2}} = P_{\mu^{q_1 q_2}} = \cdots = P_{\mu^{q_1 \cdots q_r}} = P_{\mu^i}.$$

So we now let $\mu$ be an $n$th primitive root, $P_\mu$ be its minimal polynomial. Since $\mu$ is a root of $\phi_n$, we can write $P_\mu \mid \phi_n$ inside $\mathbb{Q}[t]$. So we can write

$$\phi_n = P_\mu R,$$

Since $\phi_n$ and $P_\mu$ are monic, $R$ is also monic. By Gauss' lemma, we must have $P_\mu, R \in \mathbb{Z}[t]$.

Note that showing $P_\mu = P_{\mu^q}$ is the same as showing $\mu^q$ is a root of $P_\mu$, since $\deg P_\mu = \deg P_{\mu^q}$. So suppose it's not. Since $\mu^q$ is an $n$th primitive root of unity, it is a root of $\phi_n$. So $\mu^q$ must be a root of $R$. Now let $S = R(t^q)$. Then $\mu$ is a root of $S$, and so $P_\mu \mid S$.

We now reduce mod $q$. For any polynomial $f \in \mathbb{Z}[t]$, we write the result of reducing the coefficients mod $q$ as $\bar{f}$. Then we have $\bar{S} = \overline{R(t^q)} = \overline{R(t)}^q$. Since $\bar{P}_\mu$ divides $\bar{S}$ (by Gauss' lemma), we know $\bar{P}_\mu$ and $\overline{R(t)}$ have common roots. But $\bar{\phi}_n = \bar{P}_\mu \bar{R}$, and so this implies $\bar{\phi}_n$ has repeated roots. This is impossible since $\bar{\phi}^n$ divides $t^n - 1$, and since $q \nmid n$, we know the derivative of $t^n - 1$ does not vanish at the roots. So we are done. □

**Corollary.** Let $K = \mathbb{Q}$ and $L$ be the $n$th cyclotomic extension of $\mathbb{Q}$. Then the injection $\theta : \mathrm{Gal}(L/\mathbb{Q}) \to (\mathbb{Z}/n\mathbb{Z})^\times$ is an isomorphism.

## 3.2 Kummer extensions

**Theorem.** Let $K$ be a field, $\lambda \in K$ non-zero, $n \in \mathbb{N}$, $\operatorname{char} K = 0$ or $0 < \operatorname{char} K \nmid n$. Let $L$ be the splitting field of $t^n - \lambda$. Then

(i) $L$ contains an $n$th primitive root of unity, say $\mu$.

(ii) $L/K(\mu)$ is a cyclic (and in particular Galois) extension with degree $[L : K(\mu)] \mid n$.

(iii) $[L : K(\mu)] = n$ if and only if $t^n - \lambda$ is irreducible in $K(\mu)[t]$.

*Proof.*

(i) Under our assumptions, $t^n - \lambda$ and $(t^n - \lambda)' = nt^{n-1}$ have no common roots in $L$. So $t^n - \lambda$ has distinct roots in $L$, say $\alpha_1, \cdots, \alpha_n \in L$.

It then follows by direct computation that $\alpha_1\alpha_1^{-1}, \alpha_2\alpha_1^{-1}, \cdots, \alpha_n\alpha_1^{-1}$ are distinct roots of unity, i.e. roots of $t^n - 1$. Then one of these, say $\mu$ must be an $n$th primitive root of unity.

(ii) We know $L/K(\mu)$ is a Galois extension because it is the splitting field of the separable polynomial $t^n - \lambda$.

To understand the Galois group, we need to know how this field exactly looks like. We let $\alpha$ be any root of $t^n - \lambda$. Then the set of all roots can be written as

$$\{\alpha, \mu\alpha, \mu^2\alpha, \cdots, \mu^{n-1}\alpha\}$$

Then

$$L = K(\alpha_1, \cdots, \alpha_n) = K(\mu, \alpha) = K(\mu)(\alpha).$$

Thus, any element of $\mathrm{Gal}(L/K(\mu))$ is uniquely determined by what it sends $\alpha$ to, and any homomorphism must send $\alpha$ to one of the other roots of $t^n - \lambda$, namely $\mu^i\alpha$ for some $i$.

Define a homomorphism $\sigma : \mathrm{Gal}(L/K(\mu)) \to \mathbb{Z}/n\mathbb{Z}$ that sends $\phi$ to the corresponding $i$ (as an element of $\mathbb{Z}/n\mathbb{Z}$, so that it is well-defined).

It is easy to see that $\sigma$ is an injective group homomorphism. So we know $\mathrm{Gal}(L/K(\mu))$ is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z}$. Since the subgroup of any cyclic group is cyclic, we know that $\mathrm{Gal}(L/K(\mu))$ is cyclic, and its size is a factor of $n$ by Lagrange's theorem. Since $|\mathrm{Gal}(L/K(\mu))| = [L : K(\mu)]$ by definition of a Galois extension, it follows that $[L : K(\mu)]$ divides $n$.

(iii) We know that $[L : K(\mu)] = [K(\mu, \alpha) : K(\mu)] = \deg q_\alpha$. So $[L : K(\mu)] = n$ if and only if $\deg q_\alpha = n$. Since $q_\alpha$ is a factor of $t^n - \lambda$, $\deg q_\alpha = n$ if and only if $q_\alpha = t^n - \lambda$. This is true if and only if $t^n - \lambda$ is irreducible $K(\mu)[t]$. So done. □

**Lemma.** Assume $L/K$ is a field extension. Then $\mathrm{Hom}_K(L, L)$ is linearly independent. More concretely, let $\lambda_1, \cdots, \lambda_n \in L$ and $\phi_1, \cdots, \phi_n \in \mathrm{Hom}_K(L, L)$ distinct. Suppose for all $\alpha \in L$, we have

$$\lambda_1\phi_1(\alpha) + \cdots + \lambda_n\phi_n(\alpha) = 0.$$

Then $\lambda_i = 0$ for all $i$.

*Proof.* We perform induction on $n$.

Suppose we have some $\lambda_i \in L$ and $\phi_i \in \mathrm{Hom}_K(L, L)$ such that

$$\lambda_1\phi_1(\alpha) + \cdots + \lambda_n\phi_n(\alpha) = 0.$$

The $n = 1$ case is trivial, since $\lambda_1\phi_1 = 0$ implies $\lambda_1 = 0$ (the zero homomorphism does not fix $K$).

Otherwise, since the homomorphisms are distinct, pick $\beta \in L$ such that $\phi_1(\beta) \neq \phi_n(\beta)$. Then we know that

$$\lambda_1\phi_1(\alpha\beta) + \cdots + \lambda_n\phi_n(\alpha\beta) = 0$$

for all $\alpha \in L$. Since $\phi_i$ are homomorphisms, we can write this as

$$\lambda_1\phi_1(\alpha)\phi_1(\beta) + \cdots + \lambda_n\phi_n(\alpha)\phi_n(\beta) = 0.$$

On the other hand, by just multiplying the original equation by $\phi_n(\beta)$, we get

$$\lambda_1\phi_1(\alpha)\phi_n(\beta) + \cdots + \lambda_n\phi_n(\alpha)\phi_n(\beta) = 0.$$

Subtracting the equations gives

$$\lambda_1\phi_1(\alpha)(\phi_1(\beta) - \phi_n(\beta)) + \cdots + \lambda_{n-1}\phi_{n-1}(\alpha)(\phi_{n-1}(\beta) - \phi_n(\beta)) = 0$$

for all $\alpha \in L$. By induction, $\lambda_i(\phi_i(\beta) - \phi_n(\beta)) = 0$ for all $1 \leq i \leq n-1$. In particular, since $\phi_1(\beta) - \phi_n(\beta) \neq 0$, we have $\lambda_1 = 0$. Then we are left with

$$\lambda_2\phi_2(\alpha) + \cdots + \lambda_n\phi_n(\alpha) = 0.$$

Then by induction again, we know that all coefficients are zero. □

**Theorem.** Let $K$ be a field, $n \in \mathbb{N}$, $\operatorname{char} K = 0$ or $0 < \operatorname{char} K \nmid n$. Suppose $K$ contains an $n$th primitive root of unity, and $L/K$ is a cyclic extension of degree $[L : K] = n$. Then $L/K$ is a Kummer extension.

*Proof.* Our objective here is to find a clever $\lambda \in K$ such that $L$ is the splitting field of $t^n - \lambda$. To do so, we will have to hunt for a root $\beta$ of $t^n - \lambda$ in $L$.

Pick $\phi$ a generator of $\operatorname{Gal}(L/K)$. We know that if $\beta$ were a root of $t^n - \lambda$, then $\phi(\beta) = \mu^{-1}\beta$ for some primitive $n$th root of unity $\mu$. Thus, we want to find an element that satisfies such a property.

By the previous lemma, we can find some $\alpha \in L$ such that

$$\beta = \alpha + \mu\phi(\alpha) + \mu^2\phi^2(\alpha) + \cdots + \mu^{n-1}\phi^{n-1}(\alpha) \neq 0.$$

Then, noting that $\phi^n$ is the identity and $\phi$ fixes $\mu \in K$, we see that $\beta$ trivially satisfies

$$\phi(\beta) = \phi(\alpha) + \mu\phi^2\alpha + \cdots + \mu^{n-1}\phi^n(\alpha) = \mu^{-1}\beta,$$

In particular, we know that $\phi(\beta) \in K(\beta)$.

Now pick $\lambda = \beta^n$. Then $\phi(\beta^n) = \mu^{-n}\beta^n = \beta^n$. So $\phi$ fixes $\beta^n$. Since $\phi$ generates $\operatorname{Gal}(L/K)$, we know all automorphisms of $L/K$ fixes $\beta^n$. So $\beta^n \in K$.

Now the roots of $t^n - \lambda$ are $\beta, \mu\beta, \cdots, \mu^{n-1}\beta$. Since these are all in $\beta$, we know $K(\beta)$ is the splitting field of $t^n - \lambda$.

Finally, to show that $K(\beta) = L$, we observe that $\mathrm{id}, \phi|_{K(\beta)}, \ldots, \phi^n|_{K(\beta)}$ are distinct elements of $\operatorname{Aut}_K(K(\beta))$ since they do different things to $\beta$. Recall our previous theorem that

$$[K(\beta) : K] \geq |\operatorname{Aut}_K(K(\beta))|.$$

So we know that $n = [L : K] = [K(\beta) : K]$. So $L = K(\beta)$. So done. □

## 3.3 Radical extensions

**Lemma.** Let $L/K$ be a Galois extension, $\operatorname{char} K = 0$, $\gamma \in L$ and $F$ the splitting field of $t^n - \gamma$ over $L$. Then there exists a further extension $E/F$ such that $E/L$ is radical and $E/K$ is Galois.

*Proof.* Since we know that $L/K$ is Galois, we would rather work in $K$ than in $L$. However, our $\gamma$ is in $L$, not $K$. Hence we will employ a trick we've used before, where we introduce a new polynomial $f$, and show that its coefficients are fixed by $\operatorname{Gal}(L/K)$, and hence in $K$. Then we can look at the splitting field of $f$ or its close relatives.

Let

$$f = \prod_{\phi \in \operatorname{Gal}(L/K)} (t^n - \phi(\gamma)).$$

Each $\phi \in \mathrm{Gal}(L/K)$ induces a homomorphism $L[t] \to L[t]$. Since each $\phi \in \mathrm{Gal}(L/K)$ just rotates the roots of $f$ around, we know that this induced homomorphism fixes $f$. Since all automorphisms in $\mathrm{Gal}(L/K)$ fix the coefficients of $f$, the coefficients must all be in $K$. So $f \in K[t]$.

Now since $L/K$ is Galois, we know that $L/K$ is normal. So $L$ is the splitting field of some $g \in K[t]$. Let $E$ be the splitting field of $fg$ over $K$. Then $K \subseteq E$ is normal. Since the characteristic is zero, this is automatically separable. So the extension $K \subseteq E$ is Galois.

We have to show that $L \subseteq E$ is a radical extension. We pick our fields as follows:

- $E_0 = L$
- $E_1 =$ splitting field of $t^n - 1$ over $E_0$
- $E_2 =$ splitting field of $t^n - \gamma$ over $E_1$
- $E_3 =$ splitting field of $t^n - \phi_1(\gamma)$ over $E_2$
- $\cdots$
- $E_r = E$,

where we enumerate $\mathrm{Gal}(L/K)$ as $\{\mathrm{id}, \phi_1, \phi_2, \cdots\}$.

We then have the sequence of extensions

$$L = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_r$$

Here $E_0 \subseteq E_1$ is a cyclotomic extension, and $E_1 \subseteq E_2$, $E_2 \subseteq E_3$ etc. are Kummer extensions since they contain enough roots of unity and are cyclic. By construction, $F \subseteq E_2$. So $F \subseteq E$. □

**Theorem.** Suppose $L/K$ is a radical extension and $\mathrm{char}\, K = 0$. Then there is an extension $E/L$ such that $E/K$ is Galois and there is a sequence

$$K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E,$$

where $E_i \subseteq E_{i+1}$ is cyclotomic or Kummer.

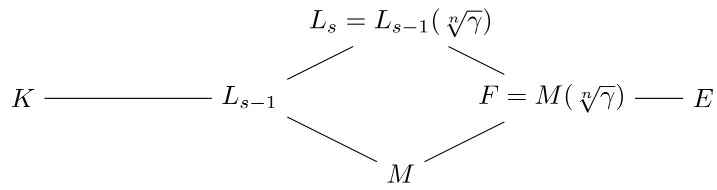
*Proof.* Note that this is equivalent to proving the following statement: Let

$$K = L_0 \subseteq L_1 \subseteq \cdots L_s$$

be a sequence of cyclotomic or Kummer extensions. Then there exists an extension $L_s \subseteq E$ such that $K \subseteq E$ is Galois and can be written as a sequence of cyclotomic or Kummer extensions.

We perform induction on $s$. The $s = 0$ case is trivial.

If $s > 0$, then by induction, there is an extension $M/L_{s-1}$ such that $M/K$ is Galois and is a sequence of cyclotomic and Kummer extensions. Now $L_s$ is a splitting field of $t^n - \gamma$ over $L_{s-1}$ for some $\gamma \in L_{s-1}$. Let $F$ be the splitting field of $t^n - \gamma$ over $M$. Then by the lemma and its proof, there exists an extension $E/M$ that is a sequence of cyclotomic or Kummer extensions, and $E/K$ is Galois.




However, we already know that $M/K$ is a sequence of cyclotomic and Kummer extensions. So $E/K$ is a sequence of cyclotomic and Kummer extension. So done. $\square$

## 3.4 Solubility of groups, extensions and polynomials

**Lemma.** Let $G$ be a finite group. Then

(i) If $G$ is soluble, then any subgroup of $G$ is soluble.

(ii) If $A \lhd G$ is a normal subgroup, then $G$ is soluble if and only if $A$ and $G/A$ are both soluble.

*Proof.*

(i) If $G$ is soluble, then by definition, there is a sequence

$$G_r = \{1\} \lhd \cdots \lhd G_1 \lhd G_0 = G,$$

such that $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is cyclic.

Let $H_i = H \cap G_i$. Note that $H_{i+1}$ is just the kernel of the obvious homomorphism $H_i \to G_i/G_{i+1}$. So $H_{i+1} \lhd H_i$. Also, by the first isomorphism theorem, this gives an injective homomorphism $H_i/H_{i+1} \to G_i/G_{i+1}$. So $H_i/H_{i+1}$ is cyclic. So $H$ is soluble.

(ii) ($\Rightarrow$) By (i), we know that $A$ is solvable. To show the quotient is soluble, by assumption, we have the sequence

$$G_r = \{1\} \lhd \cdots \lhd G_1 \lhd G_0 = G,$$

such that $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is cyclic. We construct the sequence for the quotient in the obvious way. We want to define $E_i$ as the quotient $G_i/A$, but since $A$ is not necessarily a subgroup of $E$, we instead define $E_i$ to be the image of quotient map $G_i \to G/A$. Then we have a sequence

$$E_r = \{1\} \lhd \cdots \lhd E_0 = G/A.$$

The quotient map induces a surjective homomorphism $G_i/G_{i+1} \to E_i/E_{i+1}$, showing that $E_i/E_{i+1}$ are cyclic.

($\Leftarrow$) From the assumptions, we get the sequences

$$A_m = \{1\} \lhd \cdots \lhd A_0 = A$$

$$F_n = A \lhd \cdots \lhd F_0 = G$$

where each quotient is cyclic. So we get a sequence

$$A_m = \{1\} \lhd A_1 \lhd \cdots \lhd A_0 = F_n \lhd F_{n-1} \lhd \cdots \lhd F_0 = G,$$

and each quotient is cyclic. So done. $\square$

**Lemma.** Let $L/K$ be a Galois extension. Then $L/K$ is soluble if and only if $\mathrm{Gal}(L/K)$ is soluble.

*Proof.* ($\Leftarrow$) is clear from definition.

($\Rightarrow$) By definition, there is some $E \subseteq L$ such that $E/K$ is Galois and $\mathrm{Gal}(E/K)$ is soluble. By the fundamental theorem of Galois theory, $\mathrm{Gal}(L/K)$ is a quotient of $\mathrm{Gal}(E/K)$. So by our previous lemma, $\mathrm{Gal}(L/K)$ is also soluble. $\square$

**Theorem.** Let $K$ be a field with $\operatorname{char} K = 0$, and $L/K$ is a radical extension. Then $L/K$ is a soluble extension.

*Proof.* We have already shown that if we have a radical extension $L/K$, then there is a finite extension $K \subseteq E$ such that $K \subseteq E$ is a Galois extension, and there is a sequence of cyclotomic or Kummer extensions

$$E_0 = K \subseteq E_1 \subseteq \cdots \subseteq E_r = E.$$

Let $G_i = \mathrm{Gal}(E/E_i)$. By the fundamental theorem of Galois theory, inclusion of subfields induces an inclusion of subgroups

$$G_0 = \mathrm{Gal}(E/K) \geq G_1 \geq \cdots \geq G_r = \{1\}.$$

In fact, $G_i \rhd G_{i+1}$ because $E_i \subseteq E_{i+1}$ are Galois (since cyclotomic and Kummer extensions are). So in fact we have

$$G_0 = \mathrm{Gal}(E/K) \rhd G_1 \rhd \cdots \rhd G_r = \{1\}.$$

Finally, note that by the fundamental theorem of Galois theory,

$$G_i/G_{i+1} = \mathrm{Gal}(E_{i+1}/E_i).$$

We also know that the Galois groups of cyclotomic and Kummer extensions are abelian. Since abelian groups are soluble, our previous lemma implies that $L/K$ is soluble. $\square$

**Corollary.** Let $K$ be a field with $\operatorname{char} K = 0$, and $f \in K[t]$. If $f$ can be solved by radicals, then $\mathrm{Gal}(L/K)$ is soluble, where $L$ is the splitting field of $f$ over $K$.

*Proof.* We have seen that $L/K$ is a Galois extension. By assumption, $L/K$ is thus a radical extension. By the theorem, $L/K$ is also a soluble extension. So $\mathrm{Gal}(L/K)$ is soluble. $\square$

**Lemma.** Let $K$ be a field, $f \in K[t]$ of degree $n$ with no repeated roots. Let $L$ be the splitting field of $f$ over $K$. Then $L/K$ is Galois and there exist an injective group homomorphism

$$\mathrm{Gal}(L/K) \to S_n.$$

*Proof.* Let $\mathrm{Root}_f(L) = \{\alpha_1, \cdots, \alpha_n\}$. Let $P_{\alpha_i}$ be the minimal polynomial of $\alpha_i$ over $K$. Then $P_{\alpha_i} \mid f$ implies that $P_{\alpha_i}$ is separable and splits over $L$. So $L/K$ is Galois.

Now each $\phi \in \mathrm{Gal}(L/K)$ permutes the $\alpha_i$, which gives a map $\mathrm{Gal}(L/K) \to S_n$. It is easy to show this is an injective group homomorphism. $\square$

**Lemma.** Let $p$ be a prime, and $\sigma \in S_p$ have order $p$. Then $\sigma$ is a $p$-cycle.

*Proof.* By IA Groups, we can decompose $\sigma$ into a product of disjoint cycles:

$$\sigma = \sigma_1 \cdots \sigma_r.$$

Let $\sigma_i$ have order $m_i > 1$. Again by IA Groups, we know that

$$p = \text{order of } \sigma = \operatorname{lcm}(m_1, \cdots, m_r).$$

Since $p$ is a prime number, we know that $p = m_i$ for all $i$. Hence we must have $r = 1$, since the cycles are disjoint and there are only $p$ elements. So $\sigma = \sigma_1$. Hence $\sigma$ is indeed an $p$ cycle. □

**Theorem.** Let $f \in \mathbb{Q}[t]$ be irreducible and $\deg f = p$ prime. Let $L \subseteq \mathbb{C}$ be the splitting field of $f$ over $\mathbb{Q}$. Let

$$\operatorname{Root}_f(L) = \{\alpha_1, \alpha_2, \cdots, \alpha_{p-2}, \alpha_{p-1}, \alpha_p\}.$$

Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_{p-2}$ are all real numbers, but $\alpha_{p-1}$ and $\alpha_p$ are not. In particular, $\alpha_{p-1} = \bar{\alpha}_p$. Then the homomorphism $\beta : \operatorname{Gal}(L/\mathbb{Q}) \to S_n$ is an isomorphism.

*Proof.* From IA groups, we know that the cycles $(1\ 2\ \cdots\ p)$ and $(p-1\ p)$ generate the whole of $S_n$. So we show that these two are both in the image of $\beta$.

As $f$ is irreducible, we know that $f = P_{\alpha_1}$, the minimal polynomial of $\alpha_1$ over $\mathbb{Q}$. Then

$$p = \deg P_{\alpha_i} = [\mathbb{Q}(\alpha_1) : \mathbb{Q}].$$

By the tower law, this divides $[L : \mathbb{Q}]$, which is equal to $|\operatorname{Gal}(L/\mathbb{Q})|$ since the extension is Galois. Since $p$ divides the order of $\operatorname{Gal}(L/\mathbb{Q})$, by Cauchy's theorem of groups, there must be an element of $\operatorname{Gal}(L/\mathbb{Q})$ that is of order $p$. This maps to an element $\sigma \in \operatorname{im} \beta$ of order exactly $p$. So $\sigma$ is a $p$-cycle.

On the other hand, the isomorphism $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto \bar{z}$ restricted to $L$ gives an automorphism in $\operatorname{Gal}(L/\mathbb{Q})$. This simply permutes $\alpha_{p-1}$ and $\alpha_p$, since it fixes the real numbers and $\alpha_{p-1}$ and $\alpha_p$ must be complex conjugate pairs. So $\tau = (p-1\ p) \in \operatorname{im} \beta$.

Now for every $1 \leq i < p$, we know that $\sigma^i$ again has order $p$, and hence is a $p$-cycle. So if we change the labels of the roots $\alpha_1, \cdots, \alpha_p$ and replace $\sigma$ with $\sigma^i$, and then waffle something about combinatorics, we can assume $\sigma = (1\ 2\ \cdots\ p-1\ p)$. So done. □

## 3.5 Insolubility of general equations of degree 5 or more

**Theorem** (Symmetric rational function theorem)**.** Let $K$ be a field, $L = K(x_1, \cdots, x_n)$. Let $F$ the field fixed by the automorphisms that permute the $x_i$. Then

(i) $L$ is the splitting field of

$$f = t^n - e_1 t^{n-1} + \cdots + (-1)^n e_n$$

over $F$.

(ii) $F = L^{S_n} \subseteq L$ is a Galois group with $\mathrm{Gal}(L/F)$ isomorphic to $S_n$.

(iii) $F = K(e_1, \cdots, e_n)$.

*Proof.*

(i) In $L[t]$, we have
$$f = (t - x_1) \cdots (t - x_n).$$
So $L$ is the splitting field of $f$ over $F$.

(ii) By Artin's lemma, $L/K$ is Galois and $\mathrm{Gal}(L/F) \cong S_n$.

(iii) Let $E = K(e_1, \cdots, e_n)$. Clearly, $E \subseteq F$. Now $E \subseteq L$ is a Galois extension, since $L$ is the splitting field of $f$ over $E$ and $f$ has no repeated roots.

By the fundamental theorem of Galois theory, since we have the Galois extensions $E \subseteq F \subseteq L$, we have $\mathrm{Gal}(L/F) \leq \mathrm{Gal}(L/E)$. So $S_n \leq \mathrm{Gal}(L/E)$. However, we also know that $\mathrm{Gal}(L/E)$ is a subgroup of $S_n$, we must have $\mathrm{Gal}(L/E) = \mathrm{Gal}(L/F) = S_n$. So we must have $E = F$. □

**Theorem.** Let $K$ be a field with $\operatorname{char} K = 0$. Then the general polynomial polynomial over $K$ of degree $n$ cannot be solved by radicals if $n \geq 5$.

*Proof.* Let
$$f = t^n + u_1 t^{n-1} + \cdots + u_n.$$
be our general polynomial of degree $n \geq 5$. Let $N$ be a splitting field of $f$ over $K(u_1, \cdots, u_n)$. Let
$$\mathrm{Root}_f(N) = \{\alpha_1, \cdots, \alpha_n\}.$$
We know the roots are distinct because $f$ is irreducible and the field has characteristic 0. So we can write
$$f = (t - \alpha_1) \cdots (t - \alpha_n) \in N[t].$$
We can expand this to get
$$\begin{aligned} u_1 &= -(\alpha_1 + \cdots + \alpha_n) \\ u_2 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n \\ &\vdots \\ u_i &= (-1)^i (i\text{th elementary symmetric polynomial in } \alpha_1, \cdots, \alpha_n). \end{aligned}$$
Now let $x_1, \cdots, x_n$ be new variables, and $e_i$ the $i$th elementary symmetric polynomial in $x_1, \cdots, x_n$. Let $L = K(x_1, \cdots, x_n)$, and $F = K(e_1, \cdots, e_n)$. We know that $F \subseteq L$ is a Galois extension with Galois group isomorphic to $S_n$.

We define a ring homomorphism
$$\begin{aligned} \theta : K[u_1, \cdots, u_n] &\to K[e_1, \cdots, e_n] \subseteq K[x_1, \cdots, x_n] \\ u_i &\mapsto (-1)^i e_i. \end{aligned}$$
This is our equations of $u_i$ in terms $\alpha_i$, but with $x_i$ instead of $\alpha_i$.

We want to show that $\theta$ is an isomorphism. Note that since the homomorphism just renames $u_i$ into $e_i$, the fact that $\theta$ is an isomorphism means there are no "hidden relations" between the $e_i$. It is clear that $\theta$ is a surjection. So it suffices to show $\theta$ is injective. Suppose $\theta(h) = 0$. Then

$$h(-e_1, \cdots, (-1)^n e_n) = 0.$$

Since the $x_i$ are just arbitrary variables, we now replace $x_i$ with $\alpha_i$. So we get

$$h(-e_1(\alpha_1, \cdots, \alpha_n), \cdots, (-1)^n(e_n(\alpha_1, \cdots, \alpha_n))) = 0.$$

Using our expressions for $u_i$ in terms of $e_i$, we have

$$h(u_1, \cdots, u_n) = 0,$$

But $h(u_1, \cdots, u_n)$ is just $h$ itself. So $h = 0$. Hence $\theta$ is injective. So it is an isomorphism. This in turns gives an isomorphism between

$$K(u_1, \cdots, u_n) \to K(e_1, \cdots, e_n) = F.$$

We can extend this to their polynomial rings to get isomorphisms between

$$K(u_1, \cdots, u_n)[t] \to F[t].$$

In particular, this map sends our original $f$ to

$$f \mapsto t^n - e_1 t^{n-1} + \cdots + (-1)^n e_n = g.$$

Thus, we get an isomorphism between the splitting field of $f$ over $K(u_1, \cdots, u_n)$ and the splitting field $g$ over $F$.

The splitting field of $f$ over $K(u_1, \cdots, u_n)$ is just $N$ by definition. From the symmetric rational function theorem, we know that the splitting field of $g$ over $F$ is just $L$, and So $N \cong L$. So we have an isomorphism

$$\text{Gal}(N/K(u_1, \cdots, u_n)) \to \text{Gal}(L/F) \cong S_n.$$

Since $S_n$ is not soluble, $f$ is not soluble. □

**Theorem.** Let $K$ be a field with $\operatorname{char} K = 0$. If $L/K$ is a soluble extension, then it is a radical extension.

*Proof.* Let $L \subseteq E$ be such that $K \subseteq E$ is Galois and $\text{Gal}(E/K)$ is soluble. We can replace $L$ with $E$, and assume that in fact $L/K$ is a soluble Galois extension. So there is a sequence of groups

$$\{0\} = G_r \lhd \cdots \lhd G_1 \lhd G_0 = \text{Gal}(L/K)$$

such that $G_i/G_{i+1}$ is cyclic.

By the fundamental theorem of Galois theory, we get a sequence of field extension given by $L_i = L^{G_i}$:

$$K = L_0 \subseteq \cdots \subseteq L_r = L.$$

Moreover, we know that $L_i \subseteq L_{i+1}$ is a Galois extension with Galois group $\text{Gal}(L_{i+1}/L_i) \cong G_i/G_{i+1}$. So $\text{Gal}(L_{i+1}/L_i)$ is cyclic.

Let $n = [L : K]$. Recall that we proved a previous theorem that if $\text{Gal}(L_{i+1}/L_i)$ is cyclic, and $L_i$ contains a primitive $k$th root of unity (with

$k = [L_{i+1} : L_i]$), then $L_i \subseteq L_{i+1}$ is a Kummer extension. However, we do not know of $L_i$ contains the right root of unity. Hence, the trick here is to add an $n$th primitive root of unity to each field in the sequence.

Let $\mu$ an $n$th primitive root of unity. Then if we add the $n$th primitive root to each item of the sequence, we have

$$\begin{array}{ccccccccccc} L_0(\mu) & \subseteq & \cdots & \subseteq & L_i(\mu) & \subseteq & L_{i+1}(\mu) & \subseteq & \cdots & \subseteq & L_r(\mu) \\ \cup\!\vert & & & & \cup\!\vert & & \cup\!\vert & & & & \cup\!\vert \\ K = L_0 & \subseteq & \cdots & \subseteq & L_i & \subseteq & L_{i+1} & \subseteq & \cdots & \subseteq & L_r = L \end{array}$$

We know that $L_0 \subseteq L_0(\mu)$ is a cyclotomic extension by definition. We will now show that $L_i(\mu) \subseteq L_{i+1}(\mu)$ is a Kummer extension for all $i$. Then $L/K$ is radical since $L \subseteq L_r(\mu)$.

Before we do anything, we have to show $L_i(\mu) \subseteq L_{i+1}(\mu)$ is a Galois extension. To show this, it suffices to show $L_i \subseteq L_{i+1}(\mu)$ is a Galois extension.

Since $L_i \subseteq L_{i+1}$ is Galois, $L_i \subseteq L_{i+1}$ is normal. So $L_{i+1}$ is the splitting of some $h$ over $L_i$. Then $L_{i+1}(\mu)$ is just the splitting field of $(t^n - 1)h$. So $L_i \subseteq L_{i+1}(\mu)$ is normal. Also, $L_i \subseteq L_{i+1}(\mu)$ is separable since $\operatorname{char} K = \operatorname{char} L_i = 0$. Hence $L_i \subseteq L_{i+1}(\mu)$ is Galois, which implies that $L_i(\mu) \subseteq L_{i+1}(\mu)$ is Galois.

We define a homomorphism of groups

$$\operatorname{Gal}(L_{i+1}(\mu)/L_i(\mu)) \to \operatorname{Gal}(L_{i+1}/L_i)$$

by restriction. This is well-defined because $L_{i+1}$ is the splitting field of some $h$ over $L_i$, and hence any automorphism of $L_{i+1}(\mu)$ must send roots of $h$ to roots of $h$, i.e. $L_{i+1}$ to $L_{i+1}$.

Moreover, we can see that this homomorphism is injective. If $\phi \mapsto \phi|_{L_{i+1}} = \mathrm{id}$, then it fixes everything in $L_{i+1}$. Also, since it is in $\operatorname{Gal}(L_{i+1}(\mu)/L_i(\mu))$, it fixes $L_i(\mu)$. In particular, it fixes $\mu$. So $\phi$ must fix the whole of $L_{i+1}(\mu)$. So $\phi = \mathrm{id}$.

By injectivity, we know that $\operatorname{Gal}(L_{i+1}(\mu)/L_i(\mu))$ is isomorphic to a subgroup of $\operatorname{Gal}(L_{i+1}/L_i)$. Hence it is cyclic. By our previous theorem, it follows that $L_i(\mu) \subseteq L_{i+1}(\mu)$ is a Kummer extension. So $L/K$ is radical. □

**Corollary.** Let $K$ be a field with $\operatorname{char} K = 0$ and $h \in K[t]$. Let $L$ be the splitting of $h$ over $K$. Then $h$ can be solved by radicals if and only if $\operatorname{Gal}(L/K)$ is soluble.

*Proof.* (⇒) Proved before.

(⇐) Since $L/K$ is a Galois extension, $L/K$ is a soluble extension. So it is a radial extension. So $h$ can be solved by radicals. □

**Corollary.** Let $K$ be a field with $\operatorname{char} K = 0$. Let $f \in K[t]$ have $\deg f \leq 4$. Then $f$ can be solved by radicals.

*Proof.* Exercise. □

# 4 Computational techniques

## 4.1 Reduction mod $p$

**Theorem.**

$$G = \{\lambda \in S_n : \lambda \text{ preserves the irreducible factor corresponding to } G\}. \quad (\dagger)$$

**Theorem.** Let $f \in \mathbb{Z}[t]$ be monic with no repeated roots. Let $E$ be the splitting field of $f$ over $\mathbb{Q}$, and take $\bar{f} \in \mathbb{F}_p[t]$ be the obvious polynomial obtained by reducing the coefficients of $f$ mod $p$. We also assume this has no repeated roots, and let $\bar{E}$ be the splitting field of $\bar{f}$.

Then there is an injective homomorphism

$$\bar{G} = \mathrm{Gal}(\bar{E}/\mathbb{F}_p) \hookrightarrow G = \mathrm{Gal}(E/\mathbb{Q}).$$

Moreover, if $\bar{f}$ factors as a product of irreducibles of length $n_1, n_2, \cdots, n_r$, then $\mathrm{Gal}(f)$ contains an element of cycle type $(n_1, \cdots, n_r)$.

*Proof.* We apply the previous theorem twice. First, we take $K = \mathbb{Q}$. Then

$$\theta(R) \in \mathbb{Z}[u_1, \cdots, u_n, t].$$

Let $P$ be the irreducible factor of $\theta(R)$ corresponding to the Galois group $G$. Applying Gauss' lemma, we know $P$ has integer coefficients.

Applying the theorem again, taking $K = \mathbb{F}_p$. Denote the ring homomorphism as $\bar{\theta}$. Then $\bar{\theta}(R) \in \mathbb{F}_p[u_1, \cdots, u_n, t]$. Now let $Q$ be the irreducible factor $\bar{\theta}(R)$ corresponding to $\bar{G}$.

Now note that $\theta(R_{(1)}) \mid P$ and $\bar{\theta}(R_{(1)}) \mid Q$, since the identity is in $G$ and $\bar{G}$. Also, note that $\bar{\theta}(R) = \overline{\theta(R)}$, where the bar again denotes reduction mod $p$. So $Q \mid \bar{P}$.

Considering the second action of $S_n$ (i.e. permuting the $u_i$), we can show $\bar{G} \subseteq G$, using the characterization $(\dagger)$. Details are left as an exercise. □

## 4.2 Trace, norm and discriminant

**Lemma.** Let $L/F/K$ be finite field extensions. Then

$$\mathrm{tr}_{L/K} = \mathrm{tr}_{F/K} \circ \mathrm{tr}_{L/F}, \quad N_{L/K} = N_{F/K} \circ N_{L/F}.$$

**Lemma.** Let $F/K$ be a field extension, and $V$ an $F$-vector space. Let $T : V \to V$ be an $F$-linear map. Then it is in particular a $K$-linear map. Then

$$\det_K T = N_{F/K}(\det_F T), \quad \mathrm{tr}_K T = \mathrm{tr}_{F/K}(\mathrm{tr}_F T).$$

*Proof.* For $\alpha \in F$, we will write $m_\alpha : F \to F$ for multiplication by $\alpha$ map viewed as a $K$-linear map.

By IB Groups, Rings and Modules, there exists a basis $\{e_i\}$ such that $T$ is in rational canonical form, i.e. such that $T$ is block diagonal with each diagonal looking like

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{r-1} \end{pmatrix}.$$

Since the norm is multiplicative and trace is additive, and

$$\det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \det B, \quad \operatorname{tr}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{tr} A + \operatorname{tr} B,$$

we may wlog $T$ is represented by a single block as above.

From the rational canonical form, we can read off

$$\det_F T = (-1)^{r-1} a_0, \quad \operatorname{tr}_F T = a_{r-1}.$$

We now pick a basis $\{f_j\}$ of $F$ over $K$, and then $\{e_i f_j\}$ is a basis for $V$ over $K$. Then in this basis, the matrix of $T$ over $K$ is given by

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & m_{a_0} \\ \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & m_{a_1} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} & m_{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} & m_{a_{r-1}} \end{pmatrix}.$$

It is clear that this has trace

$$\operatorname{tr}_K(m_{a_{r-1}}) = \operatorname{tr}_{F/K}(a_{r-1}) = \operatorname{tr}_{F/K}(\operatorname{tr}_F T).$$

Moreover, writing $n = [L : K]$, we have

$$\begin{aligned} \det_K \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & m_{a_0} \\ \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & m_{a_1} \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} & m_{a_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} & m_{a_{r-1}} \end{pmatrix} &= (-1)^{n(r-1)} \det_K \begin{pmatrix} m_{a_0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ m_{a_1} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ m_{a_2} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{a_{r-1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{pmatrix} \\ &= (-1)^{n(r-1)} \det_K(m_{a_0}) \\ &= \det_K((-1)^{r-1} m_{a_0}) \\ &= N_{F/K}((-1)^{r-1} a_0) \\ &= N_{F/K}(\det_F T). \end{aligned}$$

So the result follows. □

**Corollary.** Let $L/K$ be a finite field extension, and $\alpha \in L$. Let $r = [L : K(\alpha)]$ and let $P_\alpha$ be the minimal polynomial of $\alpha$ over $K$, say

$$P_\alpha = t^n + a_{n-1}t^{n-1} + \cdots + a_0.$$

with $a_i \in K$. Then

$$\operatorname{tr}_{L/K}(\alpha) = -r a_{n-1}$$

and

$$N_{L/K}(\alpha) = (-1)^{nr} a_0^r.$$

*Proof.* We first consider the case $r = 1$. Write $m_\alpha$ for the matrix representing multiplication by $\alpha$. Then $P_\alpha$ is the minimal polynomial of $m_\alpha$. But since $\deg P_\alpha = n = \dim_K K(\alpha)$, it follows that this is also the characteristic polynomial. So the result follows.

Now if $r \neq 1$, we can consider the tower of extensions $L/K(\alpha)/K$. Then we have

$$\begin{aligned} N_{L/K}(\alpha) = N_{K(\alpha)/K}(N_{L/K(\alpha)}(\alpha)) = N_{K(\alpha)/K}(\alpha^r) \\ = (N_{K(\alpha)/K}(\alpha))^r = (-1)^{nr} a_0^r. \end{aligned}$$

The computation for trace is similar. □

**Theorem.** Let $L/K$ be a finite but not separable extension. Then $\mathrm{tr}_{L/K}(\alpha) = 0$ for all $\alpha \in L$.

*Proof.* Pick $\beta \in L$ such that $P_\beta$, the minimal polynomial of $\beta$ over $K$, is not separable. Then by the previous characterization of separable polynomials, we know $p = \operatorname{char} K > 0$ with $P_\beta = q(t^p)$ for some $q \in K[t]$.

Now consider

$$K \subseteq K(\beta^p) \subseteq K(\beta) \subseteq L.$$

To show $\mathrm{tr}_{L/K} = 0$, by the previous proposition, it suffices to show $\mathrm{tr}_{K(\beta)/K(\beta^p)} = 0$.

Note that the minimal polynomial of $\beta^p$ over $K$ is $q$ because $q(\beta^p) = 0$ and $q$ is irreducible. Then $[K(\beta) : K] = \deg P_\beta = p \deg q$ and $\deg[K(\beta^p) : K] = \deg q$. So $[K(\beta) : K(\beta^p)] = p$.

Now $\{1, \beta, \beta^2, \cdots, \beta^{p-1}\}$ is a basis of $K(\beta)$ over $K(\beta^p)$. Let $R_{\beta^i}$ be the minimal polynomial of $\beta^i$ over $K(\beta^p)$. Then

$$R_{\beta^i} = \begin{cases} t - 1 & i = 0 \\ t^p - \beta^{ir} & i \neq 0 \end{cases},$$

We get the second case using the fact that $p$ is a prime number, and hence $K(\beta^p)(\beta^i) = K(\beta)$ if $1 \leq i < p$. So $[K(\beta^p)(\beta^i) : K(\beta^p)] = p$ and hence the minimal polynomial has degree $p$. Hence $\mathrm{tr}_{K(\beta)/K(\beta^p)}(\beta^i) = 0$ for all $i$.

Thus, $\mathrm{tr}_{K(\beta)/K(\beta^p)} = 0$. Hence

$$\mathrm{tr}_{L/K} = \mathrm{tr}_{K(\beta^p)/K} \circ \mathrm{tr}_{K(\beta)/K(\beta^p)} \circ \mathrm{tr}_{L/K(\beta)} = 0.$$ □

**Theorem.** Let $L/K$ be a finite separable extension. Pick a further extension $E/L$ such that $E/K$ is normal and

$$|\operatorname{Hom}_K(L, E)| = [L : K].$$

Write $\operatorname{Hom}_K(L, E) = \{\varphi_1, \cdots, \varphi_n\}$. Then

$$\mathrm{tr}_{L/K}(\alpha) = \sum_{i=1}^n \varphi_i(\alpha), \quad N_{L/K}(\alpha) = \prod_{i=1}^n \varphi_i(\alpha)$$

for all $\alpha \in L$.

*Proof.* Let $\alpha \in L$. Let $P_\alpha$ be the minimal polynomial of $\alpha$ over $K$. Then there is a one-to-one correspondence between

$$\operatorname{Hom}_K(K(\alpha), E) \longleftrightarrow \operatorname{Root}_{P_\alpha}(E) = \{\alpha_1, \cdots, \alpha_d\}.$$

wlog we let $\alpha = \alpha_1$.

Also, since

$$|\operatorname{Hom}_K(L, E)| = [L : K],$$

we get

$$|\operatorname{Hom}_K(K(\alpha), E)| = [K(\alpha) : K] = \deg P_\alpha.$$

Moreover, the restriction map $\operatorname{Hom}_K(L, E) \to \operatorname{Hom}_K(K(\alpha), E)$ (defined by $\varphi \mapsto \varphi|_{K(\alpha)}$) is surjective and sends exactly $[K(\alpha) : K]$ elements to any particular element in $\operatorname{Hom}_K(K(\alpha), E)$.

Therefore

$$\sum \varphi_i(\alpha) = [L : K(\alpha)] \sum_{\psi \in \operatorname{Hom}_K(K(\alpha), E)} \psi(\alpha) = [L : K(\alpha)] \sum_{i=1}^{d} \alpha_i.$$

Moreover, we can read the sum of roots of a polynomial is the (negative of the) coefficient of $t^{d-1}$, where

$$P_\alpha = t^d + a_{d-1}t^{d-1} + \cdots + a_0.$$

So

$$\sum \varphi_i(\alpha) = [L : K(\alpha)](-a_{d-1}) = \operatorname{tr}_{L/K}(\alpha).$$

Similarly, we have

$$\begin{aligned} \prod \varphi_i(\alpha) &= \left( \prod_{\psi \in \operatorname{Hom}_K(K(\alpha), E)} \psi(\alpha) \right)^{[L:K(\alpha)]} \\ &= \left( \prod_{i=1}^{d} \alpha_i \right)^{[L:K(\alpha)]} \\ &= ((-1)^d a_0)^{[L:K(\alpha)]} \\ &= N_{L/K}(\alpha). \end{aligned}$$

□

**Corollary.** Let $L/K$ be a finite separable extension. Then there is some $\alpha \in L$ such that $\operatorname{tr}_{L/K}(\alpha) \neq 0$.

*Proof.* Using the notation of the previous theorem, we have

$$\operatorname{tr}_{L/K}(\alpha) = \sum \varphi_i(\alpha).$$

Similar to a previous lemma, we can show that $\varphi_1, \cdots, \varphi_n$ are "linearly independent" over $E$, and hence $\sum \varphi_i$ cannot be identically zero. Hence there is some $\alpha$ such that

$$\operatorname{tr}_{L/K}(\alpha) = \sum \varphi_i(\alpha) \neq 0.$$

□

**Theorem.** Let $K$ be a field and $f \in K[t]$, $L$ is the splitting field of $f$ over $K$. Suppose $D_f \neq 0$ and $\operatorname{char} K \neq 2$. Then

(i) $D_f \in K$.

(ii) Let $G = \operatorname{Gal}(L/K)$, and $\theta : G \to S_n$ be the embedding given by the permutation of the roots. Then $\operatorname{im} \theta \subseteq A_n$ if and only if $\Delta_f \in K$ (if and only if $D_f$ is a square in $K$).

*Proof.*

(i) It is clear that $D_f$ is fixed by $\mathrm{Gal}(L/K)$ since it only permutes the roots.

(ii) Consider a permutation $\sigma \in S_n$ of the form $\sigma = (\ell\ m)$, and let it act on the roots. Then we claim that

$$\sigma(\Delta_f) = -\Delta_f. \tag{†}$$

So in general, odd elements in $S_n$ negate $\Delta_f$ while even elements fix it. Thus, $\Delta_f \in K$ iff $\Delta_f$ is fixed by $\mathrm{Gal}(L/K)$ iff every element of $\mathrm{Gal}(L/K)$ is even.

To prove (†), we have to painstakingly check all terms in the product. We wlog $\ell < m$. If $k < \ell, m$. Then this swaps $(\alpha_k - \alpha_\ell)$ with $\alpha_k - \alpha_m)$, which has no effect. The $k > m$ case is similar. If $\ell < k < m$, then this sends $(\alpha_\ell - \alpha_k) \mapsto (\alpha_m - \alpha_k)$ and $(\alpha_k - \alpha_m) \mapsto (\alpha_\ell - \alpha_m)$. This introduces two negative signs, which has no net effect. Finally, this sends $(\alpha_k - \alpha_m)$ to its negation, and so introduces a negative sign. □

**Theorem.** Let $K$ be a field, and $f \in K[t]$ be an $n$-degree monic irreducible polynomial with no repeated roots. Let $L$ be the splitting field of $f$ over $K$, and let $\alpha \in \mathrm{Root}_F(L)$. Then

$$D_f = (-1)^{n(n-1)/2} N_{K(\alpha)/K}(f'(\alpha)).$$

*Proof.* Let $\mathrm{Hom}_K(K(\alpha), L) = \{\varphi_1, \cdots, \varphi_n\}$. Recall these are in one-to-one correspondence with $\mathrm{Root}_f(L) = \{\alpha_1, \cdots, \alpha_n\}$. Then we can compute

$$\prod_{i \neq j} (\alpha_i - \alpha_j) = \prod_i \prod_{j \neq i} (\alpha_i - \alpha_j).$$

Note that since $f$ is just monic, we have

$$f = (t - \alpha_1) \cdots (t - \alpha_n).$$

Computing the derivative directly, we find

$$\prod_{j \neq i} (\alpha_i - \alpha_j) = f'(\alpha_i).$$

So we have

$$\prod_{i \neq j} (\alpha_i - \alpha_j) = \prod_i f'(\alpha_i).$$

Now since the $\varphi_i$ just maps $\alpha$ to $\alpha_i$, we have

$$\prod_{i \neq j} (\alpha_i - \alpha_j) = \prod_i \varphi_i(f'(\alpha)) = N_{K(\alpha)/K}(f'(\alpha)).$$

Finally, multiplying the factor of $(-1)^{n(n-1)/2}$ gives the desired result. □