Part II — Representation Theory
Theorems

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Linear Algebra and Groups, Rings and Modules are essential

Representations of finite groups
Representations of groups on vector spaces, matrix representations. Equivalence of representations. Invariant subspaces and submodules. Irreducibility and Schur's Lemma. Complete reducibility for finite groups. Irreducible representations of Abelian groups.

Character theory
Determination of a representation by its character. The group algebra, conjugacy classes, and orthogonality relations. Regular representation. Permutation representations and their characters. Induced representations and the Frobenius reciprocity theorem. Mackey's theorem. Frobenius's Theorem. [12]

Arithmetic properties of characters
Divisibility of the order of the group by the degrees of its irreducible characters. Burnside's $p^a q^b$ theorem. [2]

Tensor products
Tensor products of representations and products of characters. The character ring. Tensor, symmetric and exterior algebras. [3]

Representations of S^1 and SU_2
The groups S^1, SU_2 and $\mathrm{SO}(3)$, their irreducible representations, complete reducibility. The Clebsch-Gordan formula. *Compact groups.* [4]

Further worked examples
The characters of one of $\mathrm{GL}_2(F_q), S_n$ or the Heisenberg group. [3]

Contents

0 Introduction

1 Group actions

Proposition. As groups, $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$, with the isomorphism given by $\theta \mapsto A_\theta$.

Proposition. Matrices A_1, A_2 represent the same element of $\mathrm{GL}(V)$ with respect to different bases if and only if they are *conjugate*, namely there is some $X \in \mathrm{GL}_n(\mathbb{F})$ such that

$$A_2 = XA_1X^{-1}.$$

Proposition.

$$\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A).$$

Proposition. Let $\alpha \in \mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$ and $\alpha^m = \mathrm{id}$ for some positive integer m. Then α is diagonalizable.

Proposition. Let V be a finite-dimensional vector space over $\mathbb{C}$, and $\alpha \in \mathrm{End}(V)$, not necessarily invertible. Then α is diagonalizable if and only if there is a polynomial f with distinct linear factors such that $f(\alpha) = 0$.

Proposition. A finite family of individually diagonalizable endomorphisms of a vector space over $\mathbb{C}$ can be simultaneously diagonalized if and only if they commute.

Lemma. Given an action of G on X, we obtain a homomorphism $\theta : G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the set of all permutations of X.

2 Basic definitions

Lemma. The relation of "being isomorphic" is an equivalence relation on the set of all linear representations of G over $\mathbb{F}$.

Lemma. If ρ, ρ' are isomorphic representations, then they have the same dimension.

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and W be a G-subspace of V. If $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis containing a basis $\mathcal{B}_1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ of W (with $0 < m < n$), then the matrix of $\rho(g)$ with respect to $\mathcal{B}$ has the block upper triangular form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

for each $g \in G$.

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a decomposable representation with G-invariant decomposition $V = U \oplus W$. Let $\mathcal{B}_1 = \{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \cdots, \mathbf{w}_\ell\}$ be bases for U and W, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ be the corresponding basis for V. Then with respect to $\mathcal{B}$, we have

$$[\rho(g)]_{\mathcal{B}} = \begin{pmatrix} [\rho_u(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_u(g)]_{\mathcal{B}_2} \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Theorem. Every finite-dimensional representation V of a finite group over a field of characteristic 0 is completely reducible, namely, $V \cong V_1 \oplus \cdots \oplus V_r$ is a direct sum of irreducible representations.

Theorem (Maschke's theorem)**.** Let G be a finite group, and $\rho : G \to \mathrm{GL}(V)$ a representation over a finite-dimensional vector space V over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} = 0$. If W is a G-subspace of V, then there exists a G-subspace U of V such that $V = W \oplus U$.

Proposition. Let W be G-invariant subspace of V, and V have a G-invariant inner product. Then $W^\perp$ is also G-invariant.

Theorem (Weyl's unitary trick)**.** Let ρ be a complex representation of a finite group G on the complex vector space V. Then there is a G-invariant Hermitian inner product on V.

Corollary. Every finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{U}(n)$.

Proposition. Let ρ be an irreducible representation of the finite group G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.

4 Schur's lemma

Theorem (Schur's lemma)**.**

(i) Assume V and W are irreducible G-spaces over a field $\mathbb{F}$. Then any G-homomorphism $\theta : V \to W$ is either zero or an isomorphism.

(ii) If $\mathbb{F}$ is algebraically closed, and V is an irreducible G-space, then any G-endomorphism $V \to V$ is a scalar multiple of the identity map ι_V.

Corollary. If V, W are irreducible complex G-spaces, then

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \begin{cases} 1 & V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Corollary. If G is a finite group and has a faithful complex irreducible representation, then its center $Z(G)$ is cyclic.

Corollary. The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proposition. The finite abelian group $G = C_{n_1} \times \cdots \times C_{n_r}$ has precisely $|G|$ irreducible representations over $\mathbb{C}$.

Lemma. Let V, V_1, V_2 be G-vector spaces over $\mathbb{F}$. Then

(i) $\operatorname{Hom}_G(V, V_1 \oplus V_2) \cong \operatorname{Hom}_G(V, V_1) \oplus \operatorname{Hom}_G(V, V_2)$

(ii) $\operatorname{Hom}_G(V_1 \oplus V_2, V) \cong \operatorname{Hom}_G(V_1, V) \oplus \operatorname{Hom}_G(V_2, V)$.

Lemma. Let $\mathbb{F}$ be an algebraically closed field, and V be a representation of G. Suppose $V = \bigoplus_{i=1}^n V_i$ is its decomposition into irreducible components. Then for each irreducible representation S of G,

$$|\{j : V_j \cong S\}| = \dim \operatorname{Hom}_G(S, V).$$

5 Character theory

Theorem.

(i) $\chi_V(1) = \dim V$.

(ii) χ_V is a *class function*, namely it is conjugation invariant, i.e.

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

for all $g, h \in G$. Thus χ_V is constant on conjugacy classes.

(iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(iv) For two representations V, W, we have

$$\chi_{V \oplus W} = \chi_V + \chi_W.$$

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation affording the character χ. Then

$$|\chi(g)| \leq \chi(1),$$

with equality if and only if $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ if and only if $g \in \ker \rho$.

Lemma.

(i) If χ is a complex (irreducible) character of G, then so is $\bar{\chi}$.

(ii) If χ is a complex (irreducible) character of G, then so is $\varepsilon\chi$ for any linear (1-dimensional) character ε.

Theorem (Completeness of characters)**.** The complex irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$, namely

(i) If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are two complex irreducible representations affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho \text{ and } \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

This is the (row) orthogonality of characters.

(ii) Each class function of G can be expressed as a linear combination of irreducible characters of G.

Corollary. Complex representations of finite groups are characterised by their characters.

Corollary (Irreducibility criterion)**.** If $\rho : G \to \mathrm{GL}(V)$ is a complex representation of G affording the character χ, then ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Theorem. Let $\rho_1, \cdots, \rho_k$ be the irreducible complex representations of G, and let their dimensions be $n_1, \cdots, n_k$. Then

$$|G| = \sum n_i^2.$$

Corollary. The number of irreducible characters of G (up to equivalence) is k, the number of conjugacy classes.

Corollary. Two elements g_1, g_2 are conjugate if and only if $\chi(g_1) = \chi(g_2)$ for all irreducible characters χ of G.

6 Proof of orthogonality

Theorem (Row orthogonality relations)**.** If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are two complex irreducible representations affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho \text{ and } \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}.$$

Theorem (Column orthogonality relations)**.** We have

$$\sum_{i=1}^{k} \overline{\chi_i(g_j)} \chi_i(g_\ell) = \delta_{j\ell} |C_G(g_\ell)|.$$

Corollary.

$$|G| = \sum_{i=1}^{k} \chi_i^2(1).$$

Theorem. Each class function of G can be expressed as a linear combination of irreducible characters of G.

7 Permutation representations

Lemma. π_X always contains the trivial character 1_G (when decomposed in the basis of irreducible characters). In particular, $\mathrm{span}\{\mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_n}\}$ is a trivial G-subspace of $\mathbb{C}X$, with G-invariant complement $\{\sum_x a_x \mathbf{e}_x : \sum a_x = 0\}$.

Lemma. $\langle \pi_X, 1\rangle$, which is the multiplicity of 1 in π_X, is the number of orbits of G on X.

Lemma. Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ by

$$g(x_1, x_2) = (gx_1, gx_2).$$

Then the character

$$\pi_{X_1 \times X_2} = \pi_{X_1}\pi_{X_2},$$

and so

$$\langle \pi_{X_1}, \pi_{X_2}\rangle = \text{number of orbits of } G \text{ on } X_1 \times X_2.$$

Lemma. Let G act on X, with $|X| > 2$. Then

$$\pi_X = 1_G + \chi,$$

with χ irreducible if and only if G is 2-transitive on X.

Lemma. Let $g \in A_n$, $n > 1$. If g commutes with some odd permutation in S_n, then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$. Otherwise, $\mathcal{C}_{S_n}$ splits into two conjugacy classes in A_n of equal size.

8 Normal subgroups and lifting

Lemma. Let $N \lhd G$. Let $\tilde{\rho} : G/N \to \mathrm{GL}(V)$ be a representation of G/N. Then the composition

$$\rho : G \xrightarrow{\text{natural}} G/N \xrightarrow{\tilde{\rho}} \mathrm{GL}(V)$$

is a representation of G, where $\rho(g) = \tilde{\rho}(gN)$. Moreover,

(i) ρ is irreducible if and only if $\tilde{\rho}$ is irreducible.

(ii) The corresponding characters satisfy $\chi(g) = \tilde{\chi}(gN)$.

(iii) $\deg \chi = \deg \tilde{\chi}$.

(iv) The lifting operation $\tilde{\chi} \mapsto \chi$ is a bijection

$$\{\text{irreducibles of } G/N\} \longleftrightarrow \{\text{irreducibles of } G \text{ with } N \text{ in their kernel}\}.$$

We say $\tilde{\chi}$ *lifts to* χ.

Lemma. Given a group G, the *derived subgroup* or *commutator subgroup*

$$G' = \langle [a, b] : a, b \in G \rangle,$$

where $[a, b] = aba^{-1}b^{-1}$, is the unique minimal normal subgroup of G such that G/G' is abelian. So if G/N is abelian, then $G' \leq N$.

Moreover, G has precisely $\ell = |G : G'|$ representations of dimension 1, all with kernel containing G', and are obtained by lifting from G/G'.

In particular, by Lagrange's theorem, $\ell \mid G$.

Lemma. G is not simple if and only if $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and some $1 \neq g \in G$. Any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of G, i.e. $N = \bigcap \ker \chi_i$.

9 Dual spaces and tensor products of representations

9.1 Dual spaces

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation over $\mathbb{F}$, and let $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ be the dual space of V. Then V^* is a G-space under

$$(\rho^*(g)\varphi)(\mathbf{v}) = \varphi(\rho(g^{-1})\mathbf{v}).$$

This is the *dual representation* to ρ. Its character is $\chi(\rho^*)(g) = \chi_\rho(g^{-1})$.

9.2 Tensor products

Lemma.

(i) For $\mathbf{v} \in V$, $\mathbf{w} \in W$ and $\lambda \in \mathbb{F}$, we have

$$(\lambda\mathbf{v}) \otimes \mathbf{w} = \lambda(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \otimes (\lambda\mathbf{w}).$$

(ii) If $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in V$ and $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in W$, then

$$(\mathbf{x}_1 + \mathbf{x}_2) \otimes \mathbf{y} = (\mathbf{x}_1 \otimes \mathbf{y}) + (\mathbf{x}_2 \otimes \mathbf{y})$$
$$\mathbf{x} \otimes (\mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x} \otimes \mathbf{y}_1) + (\mathbf{x} \otimes \mathbf{y}_2).$$

Lemma. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be any other basis of V, and $\{\mathbf{f}_1, \cdots, \mathbf{f}_m\}$ be another basis of W. Then

$$\{\mathbf{e}_i \otimes \mathbf{f}_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

is a basis of $V \otimes W$.

Proposition. Let $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$. We define

$$\rho \otimes \rho' : G \to \mathrm{GL}(V \otimes V')$$

by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij}\mathbf{v}_i \otimes \mathbf{w}_j \mapsto \sum \lambda_{ij}(\rho(g)\mathbf{v}_i) \otimes (\rho'(g)\mathbf{w}_j).$$

Then $\rho \otimes \rho'$ is a representation of g, with character

$$\chi_{\rho\otimes\rho'}(g) = \chi_\rho(g)\chi_{\rho'}(g)$$

for all $g \in G$.

9.3 Powers of characters

Lemma. For any G-space V, S^2V and $\Lambda^2 V$ are G-subspaces of $V^{\otimes 2}$, and

$$V^{\otimes 2} = S^2V \oplus \Lambda^2 V.$$

The space S^2V has basis

$$\{\mathbf{v}_i\mathbf{v}_j = \mathbf{v}_i \otimes \mathbf{v}_j + \mathbf{v}_j \otimes \mathbf{v}_i : 1 \leq i \leq j \leq n\},$$

while $\Lambda^2 V$ has basis

$$\{\mathbf{v}_i \wedge \mathbf{v}_j = \mathbf{v}_i \otimes \mathbf{v}_j - \mathbf{v}_j \otimes \mathbf{v}_i : 1 \leq i < j \leq n\}.$$

Note that we have a strict inequality for $i < j$, since $\mathbf{v}_i \otimes \mathbf{v}_j - \mathbf{v}_j \otimes \mathbf{v}_i = 0$ if $i = j$. Hence

$$\dim S^2 V = \frac{1}{2} n(n+1), \quad \dim \Lambda^2 V = \frac{1}{2} n(n-1).$$

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation affording the character χ. Then $\chi^2 = \chi_S + \chi_\Lambda$ where $\chi_S = S^2\chi$ is the character of G in the subrepresentation on $S^2 V$, and $\chi_\Lambda = \Lambda^2 \chi$ the character of G in the subrepresentation on $\Lambda^2 V$. Moreover, for $g \in G$,

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)), \quad \chi_\Lambda(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

9.4 Characters of $G \times H$

Proposition. Let G and H be two finite groups with irreducible characters $\chi_1, \cdots, \chi_k$ and $\psi_1, \cdots, \psi_r$ respectively. Then the irreducible characters of the direct product $G \times H$ are precisely

$$\{\chi_i \psi_j : 1 \leq i \leq k, 1 \leq j \leq r\},$$

where

$$(\chi_i \psi_j)(g, h) = \chi_i(g) \psi_j(h).$$

9.5 Symmetric and exterior powers

9.6 Tensor algebra

9.7 Character ring

Lemma. Suppose α is a generalized character and $\langle \alpha, \alpha \rangle = 1$ and $\alpha(1) > 0$. Then α is actually a character of an irreducible representation of G.

10 Induction and restriction

Lemma. Let $H \leq G$. If ψ is any non-zero irreducible character of H, then there exists an irreducible character χ of G such that ψ is a constituent of $\operatorname{Res}^G_H \chi$, i.e.

$$\langle \operatorname{Res}^G_H \chi, \psi \rangle \neq 0.$$

Lemma. Let χ be an irreducible character of G, and let

$$\operatorname{Res}^G_H \chi = \sum_i c_i \chi_i,$$

with χ_i irreducible characters of H, and c_i non-negative integers. Then

$$\sum c_i^2 \leq |G : H|,$$

with equality iff $\chi(g) = 0$ for all $g \in G \setminus H$.

Lemma. Let $\psi \in \mathcal{C}_H$. Then $\operatorname{Ind}^G_H \psi \in \mathcal{C}(G)$, and $\operatorname{Ind}^G_H \psi(1) = |G : H| \psi(1)$.

Lemma. Given a (left) transversal $t_1, \cdots, t_n$ of H, we have

$$\operatorname{Ind}^G_H \psi(g) = \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i).$$

Theorem (Frobenius reciprocity)**.** Let $\psi \in \mathcal{C}(H)$ and $\varphi \in \mathcal{C}(G)$. Then

$$\langle \operatorname{Res}^G_H \varphi, \psi \rangle_H = \langle \varphi, \operatorname{Ind}^G_H \psi \rangle_G.$$

Corollary. Let ψ be a character of H. Then $\operatorname{Ind}^G_H \psi$ is a character of G.

Proposition. Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$\mathcal{C}_G(g) \cap H = \bigcup_{i=1}^m \mathcal{C}_H(x_i),$$

where the x_i are the representatives of the H conjugacy classes of elements of H conjugate to g. If $m = 0$, then $\operatorname{Ind}^G_H \psi(g) = 0$. Otherwise,

$$\operatorname{Ind}^G_H \psi(g) = |C_G(g)| \sum_{i=1}^m \frac{\psi(x_i)}{|C_H(x_i)|}.$$

Lemma. Let $\psi = 1_H$, the trivial character of H. Then $\operatorname{Ind}^G_H 1_H = \pi_X$, the permutation character of G on the set X, where $X = G/H$ is the set of left cosets of H.

11 Frobenius groups

Theorem (Frobenius' theorem (1891))**.** Let G be a transitive permutation group on a finite set X, with $|X| = n$. Assume that each non-identity element of G fixes *at most one* element of X. Then the set of fixed point-free elements ("derangements")

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \text{ for all } \alpha \in X\}$$

is a normal subgroup of G with order n.

Proposition. The left action of any finite Frobenius group on the cosets of the Frobenius complement satisfies the hypothesis of Frobenius' theorem.

12 Mackey theory

Proposition. Let G be a finite group and $H, K \leq G$. Let $g_1, \cdots, g_k$ be the representatives of the double cosets $K\backslash G/H$. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G 1_H \cong \bigoplus_{i=1}^k \operatorname{Ind}_{g_i H g_i^{-1} \cap K}^K 1.$$

Theorem (Mackey's restriction formula)**.** In general, for $K, H \leq G$, we let $\mathcal{S} = \{1, g_1, \cdots, g_r\}$ be a set of double coset representatives, so that

$$G = \bigcup K g_i H.$$

We write $H_g = gHg^{-1} \cap K \leq G$. We let (ρ, W) be a representation of H. For each $g \in G$, we define (ρ_g, W_g) to be a representation of H_g, with the same underlying vector space W, but now the action of H_g is

$$\rho_g(x) = \rho(g^{-1}xg),$$

where $h = g^{-1}xg \in H$ by construction.

This is clearly well-defined. Since $H_g \leq K$, we obtain an induced representation $\operatorname{Ind}_{H_g}^K W_g$.

Let G be finite, $H, K \leq G$, and W be a H-space. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K W_g.$$

Corollary. Let ψ be a character of a representation of H. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \psi = \sum_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K \psi_g,$$

where ψ_g is the class function (and a character) on H_g given by

$$\psi_g(x) = \psi(g^{-1}xg).$$

Corollary (Mackey's irreducibility criterion)**.** Let $H \leq G$ and W be a H-space. Then $V = \operatorname{Ind}_H^G W$ is irreducible if and only if

(i) W is irreducible; and

(ii) For each $g \in \mathcal{S}\backslash H$, the two H_g spaces W_g and $\operatorname{Res}_{H_g}^H W$ have no irreducible constituents in common, where $H_g = gHg^{-1} \cap H$.

Corollary. Let $H \lhd G$, and suppose ψ is an irreducible character of H. Then $\operatorname{Ind}_H^G \psi$ is irreducible if and only if ψ is distinct from all its conjugates ψ_g for $g \in G \setminus H$ (where $\psi_g(h) = \psi(g^{-1}hg)$ as before).

Theorem (Mackey's restriction formula)**.** In general, for $K, H \leq G$, we let $\mathcal{S} = \{1, g_1, \cdots, g_r\}$ be a set of double coset representatives, so that

$$G = \bigcup K g_i H.$$

We write $H_g = gHg^{-1} \cap K \leq G$. We let (ρ, W) be a representation of H. For each $g \in G$, we define (ρ_g, W_g) to be a representation of H_g, with the same underlying vector space W, but now the action of H_g is

$$\rho_g(x) = \rho(g^{-1}xg),$$

where $h = g^{-1}xg \in H$ by construction.

This is clearly well-defined. Since $H_g \leq K$, we obtain an induced representation $\operatorname{Ind}_{H_g}^K W_g$.

Let G be finite, $H, K \leq G$, and W be a H-space. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K W_g.$$

13 Integrality in the group algebra

Proposition.

(i) The algebraic integers form a subring of $\mathbb{C}$.

(ii) If $a \in \mathbb{C}$ is both an algebraic integer and rational, then a is in fact an integer.

(iii) Any subring of $\mathbb{C}$ which is a finitely generated $\mathbb{Z}$-module consists of algebraic integers.

Proposition. If χ is a character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.

Proposition. The class sums $C_1, \cdots, C_k$ form a basis of $Z(\mathbb{C}G)$. There exists non-negative *integers* $a_{ij\ell}$ (with $1 \leq i, j, \ell \leq k$) with

$$C_i C_j = \sum_{\ell=1}^{k} a_{ij\ell} C_\ell.$$

Lemma. The values of

$$\omega_\chi(C_i) = \frac{\chi(g)}{\chi(1)} |\mathcal{C}_i|$$

are algebraic integers.

Theorem. The degree of any irreducible character of G divides $|G|$, i.e.

$$\chi_j(1) \mid |G|$$

for each irreducible χ_j.

14 Burnside's theorem

Theorem (Burside's $p^a q^b$ theorem)**.** Let p, q be primes, and let $|G| = p^a q^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then G is not simple.

Lemma. Suppose

$$\alpha = \frac{1}{m} \sum_{j=1}^{m} \lambda_j,$$

is an algebraic integer, where $\lambda_j^n = 1$ for all j and some n. Then either $\alpha = 0$ or $|\alpha| = 1$.

Lemma. Suppose χ is an irreducible character of G, and $\mathcal{C}$ is a conjugacy class in G such that $\chi(1)$ and $|\mathcal{C}|$ are coprime. Then for $g \in \mathcal{C}$, we have

$$|\chi(g)| = \chi(1) \text{ or } 0.$$

Proposition. If in a finite group, the number of elements in a conjugacy class $\mathcal{C}$ is of (non-trivial) prime power order, then G is not non-abelian simple.

Theorem (Burside's $p^a q^b$ theorem)**.** Let p, q be primes, and let $|G| = p^a q^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then G is not simple.

15 Representations of compact groups

Theorem. Every one-dimensional (continuous) representation S^1 is of the form

$$\rho : z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

Lemma. If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continuous group homomorphism, then there exists a $c \in \mathbb{R}$ such that

$$\psi(x) = cx$$

for all $x \in \mathbb{R}$.

Lemma. Continuous homomorphisms $\varphi : (\mathbb{R}, +) \to S^1$ are of the form

$$\varphi(x) = e^{icx}$$

for some $c \in \mathbb{R}$.

Theorem. Every one-dimensional (continuous) representation S^1 is of the form

$$\rho : z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

Theorem. Let G be a compact Hausdorff topological group. Then there exists a unique Haar measure on G.

Corollary (Weyl's unitary trick)**.** Let G be a compact group. Then every representation (ρ, V) has a G-invariant Hermitian inner product.

Theorem (Maschke's theoerm)**.** Let G be compact group. Then every representation of G is completely reducible.

Theorem (Orthogonality)**.** Let G be a compact group, and V and W be irreducible representations of G. Then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}.$$

15.1 Representations of SU(2)

Lemma (SU(2)-conjugacy classes)**.**

(i) Let $t \in T$. Then $sts^{-1} = t^{-1}$.

(ii) $s^2 = -I \in Z(\mathrm{SU}(2))$.

(iii) The normalizer

$$N_G(T) = T \cup sT = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} : a \in \mathbb{C}, |a| = 1 \right\}.$$

(iv) Every conjugacy class $\mathcal{C}$ of SU(2) contains an element of T, i.e. $\mathcal{C} \cap T \neq \emptyset$.

(v) In fact,

$$\mathcal{C} \cap T = \{t, t^{-1}\}$$

for some $t \in T$, and $t = t^{-1}$ if and only if $t = \pm I$, in which case $\mathcal{C} = \{t\}$.

(vi) There is a bijection

$$\{\text{conjugacy classes in SU}(2)\} \leftrightarrow [-1, 1],$$

given by

$$A \mapsto \frac{1}{2} \operatorname{tr} A.$$

We can see that if

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix},$$

then

$$\frac{1}{2} \operatorname{tr} A = \frac{1}{2}(\lambda + \bar{\lambda}) = \operatorname{Re}(\lambda).$$

Proposition. For $t \in (-1, 1)$, the class $\mathcal{C}_t \cong S^2$ as topological spaces.

Lemma. A continuous class function $f : G \to \mathbb{C}$ is determined by its restriction to T, and $F|_T$ is even, i.e.

$$f\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) = f\left(\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}\right).$$

Lemma. If χ is a character of a representation of SU(2), then its restriction $\chi|_T$ is a Laurent polynomial, i.e. a finite $\mathbb{N}$-linear combination of functions

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mapsto \lambda^n$$

for $n \in \mathbb{Z}$.

Theorem. The representations $\rho_n : \text{SU}(2) \to \text{GL}(V_n)$ of dimension $n + 1$ are irreducible for $n \in \mathbb{Z}_{\geq 0}$.

Theorem. Every finite-dimensional continuous irreducible representation of G is one of the $\rho_n : G \to \text{GL}(V_n)$ as defined above.

Proposition. Let $G = \text{SU}(2)$ or $G = S^1$, and V, W are representations of G. Then

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Proposition (Clebsch-Gordon rule)**.** For $n, m \in \mathbb{N}$, we have

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

15.2 Representations of SO(3), SU(2) and $U(2)$

Proposition. There are isomorphisms of topological groups:

(i) $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\} = \mathrm{PSU}(2)$

(ii) $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm(I, I)\}$

(iii) $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SU}(2)/\{\pm(I, I)\}$

All maps are group isomorphisms, but in fact also homeomorphisms. To show this, we can use the fact that a continuous bijection from a Hausdorff space to a compact space is automatically a homeomorphism.

Corollary. Every irreducible representation of SO(3) has the following form:

$$\rho_{2m} : \mathrm{SO}(3) \to \mathrm{GL}(V_{2m}),$$

for some $m \geq 0$, where V_n are the irreducible representations of SU(2).

Proposition. $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\}$.

Proposition. The complete list of irreducible representations of SO(4) is $\rho_m \times \rho_n$, where $m, n > 0$ and $m \equiv n \pmod 2$.

Proposition. The complete list of irreducible representations of U(2) is

$$\det{}^{\otimes m} \otimes \rho_n,$$

where $m, n \in \mathbb{Z}$ and $n \geq 0$, and det is the obvious one-dimensional representation.