Part II — Representation Theory
Definitions

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Linear Algebra and Groups, Rings and Modules are essential

Representations of finite groups
Representations of groups on vector spaces, matrix representations. Equivalence of representations. Invariant subspaces and submodules. Irreducibility and Schur's Lemma. Complete reducibility for finite groups. Irreducible representations of Abelian groups.

Character theory
Determination of a representation by its character. The group algebra, conjugacy classes, and orthogonality relations. Regular representation. Permutation representations and their characters. Induced representations and the Frobenius reciprocity theorem. Mackey's theorem. Frobenius's Theorem. [12]

Arithmetic properties of characters
Divisibility of the order of the group by the degrees of its irreducible characters. Burnside's $p^a q^b$ theorem. [2]

Tensor products
Tensor products of representations and products of characters. The character ring. Tensor, symmetric and exterior algebras. [3]

Representations of S^1 and SU_2
The groups S^1, SU_2 and $\mathrm{SO}(3)$, their irreducible representations, complete reducibility. The Clebsch-Gordan formula. *Compact groups.* [4]

Further worked examples
The characters of one of $\mathrm{GL}_2(F_q), S_n$ or the Heisenberg group. [3]

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0 Introduction

1 Group actions

Notation. $\mathbb{F}$ always represents a field.

Notation. We write V for a vector space over $\mathbb{F}$ — this will always be finite dimensional over $\mathbb{F}$. We write $\mathrm{GL}(V)$ for the group of invertible linear maps $\theta : V \to V$. This is a group with the operation given by composition of maps, with the identity as the identity map (and inverse by inverse).

Notation. Let V be a finite-dimensional vector space over $\mathbb{F}$. We write $\mathrm{End}(V)$ for the endomorphism algebra, the set of all linear maps $V \to V$.

Definition (Symmetric group S_n)**.** The *symmetric group* S_n is the set of all permutations of $X = \{1, \cdots, n\}$, i.e. the set of all bijections $X \to X$. We have $|S_n| = n!$.

Definition (Alternating group A_n)**.** The *alternating group* A_n is the set of products of an even number of transpositions $(i\ j)$ in S_n. We know $|A_n| = \frac{n!}{2}$. So this is a subgroup of index 2 and hence normal.

Definition (Cyclic group C_m)**.** The *cyclic group* of order m, written C_m is

$$C_m = \langle x : x^m = 1\rangle.$$

This also occurs naturally, as $\mathbb{Z}/m\mathbb{Z}$ over addition, and also the group of nth roots of unity in $\mathbb{C}$. We can view this as a subgroup of $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. Alternatively, this is the group of rotation symmetries of a regular m-gon in $\mathbb{R}^2$, and can be viewed as a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

Definition (Dihedral group D_{2m})**.** The *dihedral group* D_{2m} of order $2m$ is

$$D_{2m} = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1}\rangle.$$

This is the symmetry group of a regular m-gon. The x^i are the rotations and $x^i y$ are the reflections. For example, in D_8, x is rotation by $\frac{\pi}{2}$ and y is any reflection.

This group can be viewed as a subgroup of $\mathrm{GL}_2(\mathbb{R})$, but since it also acts on the vertices, it can be viewed as a subgroup of S_m.

Definition (Quaternion group)**.** The *quaternion group* is given by

$$Q_8 = \langle x, y : x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}\rangle.$$

This has order 8, and we write $i = x, j = y$, $k = ij$, $-1 = i^2$, with

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

We can view this as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ via

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad -j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad -k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

.

Definition (Conjugacy class)**.** The *conjugacy class* of $g \in G$ is

$$\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}.$$

Definition (Centralizer)**.** The *centralizer* of $g \in G$ is

$$C_G(g) = \{x \in G : xg = gx\}.$$

Definition (Group action)**.** Let G be a group and X a set. We say G *acts on* X if there is a map $* : G \times X \to X$, written $(g, x) \mapsto g * x = gx$ such that

(i) $1x = x$

(ii) $g(hx) = (gh)x$

Definition (Permutation representation)**.** The *permutation representation* of a group action G on X is the homomorphism $\theta : G \to \mathrm{Sym}(X)$ obtained above.

2 Basic definitions

Definition (Representation). Let V be a finite-dimensional vector space over $\mathbb{F}$. A *(linear) representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \to \mathrm{GL}(V).$$

We sometimes write ρ_g for $\rho_V(g)$, so for each $g \in G$, $\rho_g \in \mathrm{GL}(V)$, and $\rho_g\rho_h = \rho_{gh}$ and $\rho_{g^{-1}} = (\rho_g)^{-1}$ for all $g, h \in G$.

Definition (Dimension or degree of representation). The *dimension* (or *degree*) of a representation $\rho : G \to \mathrm{GL}(V)$ is $\dim_{\mathbb{F}}(V)$.

Definition (Faithful representation). A *faithful* representation is a representation ρ such that $\ker \rho = 1$.

Definition (Linear action). A group G *acts linearly* on a vector space V if it acts on V such that

$$g(\mathbf{v}_1 + \mathbf{v}_2) = g\mathbf{v}_1 + g\mathbf{v}_2, \quad g(\lambda \mathbf{v}_1) = \lambda(g\mathbf{v}_1)$$

for all $g \in G$, $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\lambda \in \mathbb{F}$. We call this a *linear action*.

Definition (G-space/G-module). If there is a linear action G on V, we say V is a *G-space* or *G-module*.

Definition (Group algebra). The *group algebra* $\mathbb{F}G$ is defined to be the algebra (i.e. a vector space with a bilinear multiplication operation) of formal sums

$$\mathbb{F}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in F \right\}$$

with the obvious addition and multiplication.

Definition (Matrix representation). R is a *matrix representation* of G of degree n if R Is a homomorphism $G \to \mathrm{GL}_n(\mathbb{F})$.

Definition (G-homomorphism/intertwine). Fix a group G and a field $\mathbb{F}$. Let V, V' be finite-dimensional vector spaces over $\mathbb{F}$ and $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ be representations of G. The linear map $\varphi : V \to V'$ is a *G-homomorphism* if

$$\varphi \circ \rho(g) = \rho'(g) \circ \varphi \tag{$*$}$$

for all $g \in G$. In other words, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ V' & \xrightarrow{\rho'_g} & V' \end{array}$$

i.e. no matter which way we go form V (top left) to V' (bottom right), we still get the same map.

We say φ *intertwines* ρ and ρ'. We write $\mathrm{Hom}_G(V, V')$ for the $\mathbb{F}$-space of all these maps.

Definition (G-isomorphism)**.** A G-homomorphism is a *G-isomorphism* if φ is bijective.

Definition (Equivalent/isomorphic representations)**.** Two representations ρ, ρ' are *equivalent* or *isomorphic* if there is a G-isomorphism between them.

Definition (G-subspace)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say $W \leq V$ is a *G-subspace* if it is a subspace that is $\rho(G)$-invariant, i.e.

$$\rho_g(W) \leq W$$

for all $g \in G$.

Definition (Irreducible/simple representation)**.** A representation ρ is *irreducible* or *simple* if there are no proper non-zero G-subspaces.

Definition (Subrepresentation)**.** If W is a G-subspace, then the corresponding map $G \to \mathrm{GL}(W)$ given by $g \mapsto \rho(g)|_W$ gives us a new representation of W. This is a *subrepresentation* of ρ.

Definition ((In)decomposable representation)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is *decomposable* if there are proper G-invariant subspaces $U, W \leq V$ with

$$V = U \oplus W.$$

We say ρ is a direct sum $\rho_u \oplus \rho_w$.

If no such decomposition exists, we say that ρ is *indecomposable.*

Definition (Direct sum)**.** Let $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ be representations of G. Then the *direct sum* of ρ, ρ' is the representation

$$\rho \oplus \rho' : G \to \mathrm{GL}(V \oplus V')$$

given by

$$(\rho \oplus \rho')(g)(\mathbf{v} + \mathbf{v}') = \rho(g)\mathbf{v} + \rho'(g)\mathbf{v}'.$$

3 Complete reducibility and Maschke's theorem

Definition (Completely reducible/semisimple representation)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is *completely reducible* or *semisimple* if it is the direct sum of irreducible representations.

Definition (Hermitian inner product)**.** For V a complex space, $\langle \cdot , \cdot \rangle$ is a *Hermitian inner product* if

(i) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (Hermitian)

(ii) $\langle \mathbf{v}, \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 \rangle = \lambda_1 \langle \mathbf{v}, \mathbf{w}_1 \rangle + \lambda_2 \langle \mathbf{v}, \mathbf{w}_2 \rangle$ (sesquilinear)

(iii) $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ if $\mathbf{v} \neq 0$ (positive definite)

Definition (G-invariant inner product)**.** An inner product $\langle \cdot , \cdot \rangle$ is in addition *G-invariant* if

$$\langle g\mathbf{v}, g\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle.$$

Definition (Regular representation and regular module)**.** The *regular representation* of a group G, written ρ_{reg}, is the natural action of G on $\mathbb{F}G$. $\mathbb{F}G$ is called the *regular module*.

Definition (Permutation representation)**.** Let $\mathbb{F}$ be a field, and let G act on a set X. Let $\mathbb{F}X = \langle \mathbf{e}_x : x \in X \rangle$ with a G-action given by

$$g \sum_x a_x \mathbf{e}_x = \sum_x a_x \mathbf{e}_{gx}.$$

So we have a G-space on $\mathbb{F}X$. The representation $G \to \mathrm{GL}(\mathbb{F}X)$ is the corresponding *permutation representation*.

4 Schur's lemma

Definition (Canonical decomposition/decomposition into isotypical components)**.** A decomposition of V as $\bigoplus W_j$, where each W_j is (isomorphic to) n_j copies of the irreducible S_j (with $S_j \not\cong S_i$ for $i \neq j$) is the *canonical decomposition* or *decomposition into isotypical components*.

5 Character theory

Definition (Character)**.** The *character* of a representation $\rho : G \to \mathrm{GL}(V)$, written $\chi_\rho = \chi_v = \chi$, is defined as

$$\chi(g) = \operatorname{tr} \rho(g).$$

We say ρ *affords* the character χ.

Alternatively, the character is $\operatorname{tr} R(g)$, where $R(g)$ is any matrix representing $\rho(g)$ with respect to any basis.

Definition (Degree of character)**.** The *degree* of χ_v is $\dim V$.

Definition (Linear character)**.** We say χ is *linear* if $\dim V = 1$, in which case χ is a homomorphism $G \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$.

Definition (Irreducible character)**.** A character χ is *irreducible* if ρ is *irreducible.*

Definition (Faithful character)**.** A character χ is *faithful* if ρ is *faithful.*

Definition (Trivial/principal character)**.** A character χ is *trivial* or *principal* if ρ is the trivial representation. We write $\chi = 1_G$.

Definition (Space of class functions)**.** Define the *complex space of class functions* of G to be

$$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g) \text{ for all } h, g \in G\}.$$

This is a vector space by $f_1 + f_2 : g \mapsto f_1(g) + f_2(g)$ and $\lambda f : g \mapsto \lambda f(g)$.

Definition (Class number)**.** The *class number* $k = k(G)$ is the number of conjugacy classes of G.

Definition (Character table)**.** The *character table* of G is the $k \times k$ matrix $X = (\chi_i(g_j))$, where $1 = \chi_1, \chi_2, \cdots, \chi_k$ are the irreducible characters of G, and $\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \cdots, \mathcal{C}_k$ are the conjugacy classes with $g_j \in \mathcal{C}_j$.

6 Proof of orthogonality

7 Permutation representations

Definition (Permutation character)**.** The *permutation character* π_X is

$$\pi_X(g) = |\operatorname{fix}(g)| = |\{x \in X : gx = x\}|.$$

Definition (2-transitive)**.** Let G act on X, $|X| > 2$. Then G is 2-*transitive* on X if G has two orbits on $X \times X$, namely $\{(x, x) : x \in X\}$ and $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

8 Normal subgroups and lifting

9 Dual spaces and tensor products of representations

9.1 Dual spaces

Definition (Self-dual representation)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is *self-dual* if there is isomorphism of G-spaces $V \cong V^*$.

9.2 Tensor products

Definition (Tensor product)**.** Let V, W be vector spaces over $\mathbb{F}$. Suppose $\dim V = m$ and $\dim W = n$. We fix a basis $\mathbf{v}_1, \cdots, \mathbf{v}_m$ and $\mathbf{w}_1, \cdots, \mathbf{w}_n$ of V and W respectively.

The *tensor product space* $V \otimes W = V \otimes_{\mathbb{F}} W$ is an nm-dimensional vector space (over $\mathbb{F}$) with basis given by formal symbols

$$\{\mathbf{v}_i \otimes \mathbf{w}_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Thus

$$V \otimes W = \left\{\sum \lambda_{ij} \mathbf{v}_i \otimes \mathbf{w}_j : \lambda_{ij} \in \mathbb{F}\right\},$$

with the "obvious" addition and scalar multiplication.

If

$$\mathbf{v} = \sum \alpha_i \mathbf{v}_i \in V, \quad \mathbf{w} = \sum \beta_j \mathbf{w}_j,$$

we define

$$\mathbf{v} \otimes \mathbf{w} = \sum_{i,j} \alpha_i \beta_j (\mathbf{v}_i \otimes \mathbf{w}_j).$$

9.3 Powers of characters

Notation.

$$V^{\otimes 2} = V \otimes V.$$

Definition (Symmetric and exterior square)**.** We define the *symmetric square* and *exterior square* of V to be, respectively,

$$S^2 V = \{\mathbf{x} \in V^{\otimes 2} : \tau(\mathbf{x}) = \mathbf{x}\}$$
$$\Lambda^2 V = \{\mathbf{x} \in V^{\otimes 2} : \tau(\mathbf{x}) = -\mathbf{x}\}.$$

The exterior square is also known as the *anti-symmetric square* and *wedge power*.

9.4 Characters of $G \times H$

9.5 Symmetric and exterior powers

Definition (Symmetric and exterior power)**.** For a G-space V, define

(i) The nth symmetric power of V is

$$S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \text{ for all } \sigma \in S_n\}.$$

(ii) The nth exterior power of V is

$$\Lambda^n V = \{x \in V^{\otimes n} : \sigma(x) = (\operatorname{sgn} \sigma) x \text{ for all } \sigma \in S_n\}.$$

Both of these are G-subspaces of $V^{\otimes n}$.

9.6 Tensor algebra

Definition (Tensor algebra)**.** Let $T^n V = V^{\otimes n}$. Then the *tensor algebra* of V is

$$T^\bullet(V) = T(V) = \bigoplus_{n \geq 0} T^n V,$$

with $T^0 V = \mathbb{F}$ by convention.

This is a vector space over $\mathbb{F}$ with the obvious addition and multiplication by scalars. $T(V)$ is also a (non-commutative) (graded) ring with product $x \cdot y = x \otimes y$. This is graded in the sense that if $x \in T^n V$ and $y \in T^m V$, $x \cdot y = x \otimes y \in T^{n+m} V$.

Definition (Symmetric and exterior algebra)**.** We define the *symmetric algebra* of V to be

$$S(V) = T(V)/(\text{ideal of } T(V) \text{ generated by all } u \otimes v - v \otimes u).$$

The *exterior algebra* of V is

$$\Lambda(V) = T(V)/(\text{ideal of } T(V) \text{ generated by all } v \otimes v).$$

Note that v and u are not elements of V, but arbitrary elements of $T^n V$ for some n.

9.7 Character ring

Definition (Character ring)**.** The *character ring* of G is the $\mathbb{Z}$-submodule of $\mathcal{C}(G)$ spanned by the irreducible characters and is denoted $R(G)$.

Definition (Generalized/virtual characters)**.** The elements of $R(G)$ are called *generalized or virtual characters.* These are class functions of the form

$$\psi = \sum_\chi n_\chi \chi,$$

summing over all irreducibles χ, and $n_\chi \in \mathbb{Z}$.

10 Induction and restriction

Definition (Restriction)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation affording χ. We can think of V as an H-space by restricting ρ's attention to $h \in H$. We get a representation $\mathrm{Res}^G_H \rho = \rho_H = \rho\downarrow_H$, the *restriction* of ρ to H. It affords the character $\mathrm{Res}^G_H \chi = \chi_H = \chi\downarrow_H$.

Definition (Induced class function)**.** Let $\psi \in \mathcal{C}(H)$. We define the *induced class function* $\mathrm{Ind}^G_H \psi = \psi \uparrow^G = \psi^G$ by

$$\mathrm{Ind}^G_H \psi(g) = \frac{1}{|H|}\sum_{x \in G} \mathring{\psi}(x^{-1}gx),$$

where

$$\mathring{\psi}(y) = \begin{cases} \psi(y) & y \in H \\ 0 & y \notin H \end{cases}.$$

Definition (Induced representation)**.** Let $H \leq G$ have index n, and $1 = t_1, \cdots, t_n$ be a left transversal. Let W be a H-space. Define the *induced representation* to be the vector space

$$\mathrm{Ind}^G_H W = W \oplus t_2 \otimes W \oplus \cdots \oplus t_n \otimes W,$$

with the G-action

$$g : t_i\mathbf{w} \mapsto t_j(t_j^{-1}gt_i)W,$$

where t_j is the unique element (among $t_1, \cdots, t_n$) such that $t_j^{-1}gt_i \in H$.

11 Frobenius groups

Definition (Frobenius group and Frobenius complement)**.** A *Frobenius group* is a group G having a subgroup H such that $H \cap gHg^{-1} = 1$ for all $g \notin H$. We say H is a *Frobenius complement* of G.

Definition (Frobenius kernel)**.** The *Frobenius kernel* of a Frobenius group G is the K obtained from Frobenius' theorem.

12 Mackey theory

13 Integrality in the group algebra

Definition (Algebraic integers)**.** A complex number $a \in \mathbb{C}$ is an *algebraic integer* if a is a root of a monic polynomial with integer coefficients. Equivalently, a is such that the subring of $\mathbb{C}$ given by

$$\mathbb{Z}[a] = \{f(a) : f \in \mathbb{Z}[X]\}$$

is finitely generated. Equivalently, a is the eigenvalue of a matrix, all of whose entries lie in $\mathbb{Z}$.

Definition (Class sum)**.** The *class sum* of a conjugacy class $\mathcal{C}_j$ of a group G is

$$C_j = \sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G.$$

Definition (Class algebra/structure constants)**.** The constants $a_{ij\ell}$ as defined above are the *class algebra constants* or *structure constants*.

Definition (Representation of algebra)**.** Let A be an algebra. A *representation of A* is a homomorphism of algebras $\rho : A \to \operatorname{End} V$.

14 Burnside's theorem

15 Representations of compact groups

Definition (Topological group)**.** A *topological group* is a group G which is also a topological space, and for which multiplication $G \times G \to G$ ($(h, g) \mapsto hg$) and inverse $G \to G$ ($g \mapsto g^{-1}$) are continuous maps.

Definition (Compact group)**.** A topological group is a *compact group* if it is compact as a topological space.

Definition (Representation of topological group)**.** A *representation* of a topological group on a finite-dimensional space V is a continuous group homomorphism $G \to \mathrm{GL}(V)$.

Definition (Haar measure)**.** Let G be a topological group, and let

$$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f \text{ is continuous}, f(gxg^{-1}) = f(x) \text{ for all } g, x \in G\}.$$

A non-trivial linear functional $\int_G : \mathcal{C}(G) \to C$, written as

$$\int_G f = \int_G f(g)\, \mathrm{d}g$$

is called a *Haar measure* if

(i) It satisfies the normalization condition

$$\int_G 1\, \mathrm{d}g = 1$$

so that the "total volume" is 1.

(ii) It is left and right translation invariant, i.e.

$$\int_G f(xg)\, \mathrm{d}g = \int_G f(g)\, \mathrm{d}g = \int_G f(gx)\, \mathrm{d}g$$

for all $x \in G$.

Definition (Character)**.** If $\rho : G \to \mathrm{GL}(V)$ is a representation, then the *character* $\chi_\rho = \operatorname{tr} \rho$ is a continuous class function, since each component $\rho(g)_{ij}$ is continuous.

15.1 Representations of SU(2)

Notation. We write $\mathbb{N}[z, z^{-1}]$ for the set of all Laurent polynomials, i.e.

$$\mathbb{N}[z, z^{-1}] = \left\{\sum a_n z^n : a_n \in \mathbb{N} : \text{only finitely many } a_n \text{ non-zero}\right\}.$$

We further write

$$\mathbb{N}[z, z^{-1}]_{\mathrm{ev}} = \{f \in \mathbb{N}[z, z^{-1}] : f(z) = f(z^{-1})\}.$$

15.2 Representations of SO(3), SU(2) and $U(2)$