Part II — Number Fields

Theorems

Based on lectures by I. Grojnowski

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Lent 2016

Part IB Groups, Rings and Modules is essential and Part II Galois Theory is desirable

Definition of algebraic number fields, their integers and units. Norms, bases and discriminants. [3]

Ideals, principal and prime ideals, unique factorisation. Norms of ideals. [3]

Minkowski's theorem on convex bodies. Statement of Dirichlet's unit theorem. Determination of units in quadratic fields. [2]

Ideal classes, finiteness of the class group. Calculation of class numbers using statement of the Minkowski bound. [3]

Dedekind's theorem on the factorisation of primes. Application to quadratic fields. [2]

Discussion of the cyclotomic field and the Fermat equation or some other topic chosen by the lecturer. [3]

Contents

0 Introduction

1 Number fields

Lemma. $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$, i.e. $\alpha \in \mathbb{Q}$ is an algebraic integer if and only if $\alpha \in \mathbb{Z}$.

Theorem. $\mathcal{O}_L$ is a ring, i.e. if $\alpha, \beta \in \mathcal{O}_L$, then so is $\alpha \pm \beta$ and $\alpha\beta$.

Proposition.

(i) Let $R \subseteq S$ be rings. If $S = R[s]$ and s is integral over R, then S is finitely-generated over R.

(ii) If $S = R[s_1, \cdots, s_n]$ with s_i integral over R, then S is finitely-generated over R.

Theorem. If S is finitely-generated over R, then S is integral over R.

Corollary. Let $L \supseteq \mathbb{Q}$ be a number field. Then $\mathcal{O}_L$ is a ring.

Corollary. If $A \subseteq B \subseteq C$ be ring extensions such that B over A and C over B are integral extensions. Then C is integral over A.

Lemma. If $f \in K[x]$ with $f(\alpha) = 0$, then $p_\alpha \mid f$.

Proposition. Let L be a number field. Then $\alpha \in \mathcal{O}_L$ if and only if the minimal polynomial $p_\alpha(x) \in \mathbb{Q}[x]$ for the field extension $\mathbb{Q} \subseteq L$ is in fact in $\mathbb{Z}[x]$.

Lemma. We have

$$\operatorname{Frac} \mathcal{O}_L = \left\{ \frac{\alpha}{\beta} : \alpha, \beta \in \mathcal{O}_L, \beta \neq 0 \right\} = L.$$

In fact, for any $\alpha \in L$, there is some $n \in \mathbb{Z}$ such that $n\alpha \in \mathcal{O}_L$.

2 Norm, trace, discriminant, numbers

Proposition. For a field extension L/K and $a, b \in L$, we have $N(ab) = N(a)N(b)$ and $\mathrm{tr}(a+b) = \mathrm{tr}(a) + \mathrm{tr}(b)$.

Proposition. Let $p_\alpha \in K[x]$ be the minimal polynomial of α. Then the characteristic polynomial of m_α is

$$\det(xI - m_\alpha) = p_\alpha^{[L:K(\alpha)]}$$

Hence if $p_\alpha(x)$ splits in some field $L' \supseteq K(\alpha)$, say

$$p_\alpha(x) = (x - \alpha_1)\cdots(x - \alpha_r),$$

then

$$N_{K(\alpha)/K}(\alpha) = \prod \alpha_i, \quad \mathrm{tr}_{K(\alpha)/K}(\alpha) = \sum \alpha_i,$$

and hence

$$N_{L/K}(\alpha) = \left(\prod \alpha_i\right)^{[L:K(\alpha)]}, \quad \mathrm{tr}_{L/K} = [L : K(\alpha)]\left(\sum \alpha_i\right).$$

Corollary. Let $L \supseteq \mathbb{Q}$ be a number field. Then the following are equivalent:

(i) $\alpha \in \mathcal{O}_L$.

(ii) The minimal polynomial p_α is in $\mathbb{Z}[x]$

(iii) The characteristic polynomial of m_α is in $\mathbb{Z}[x]$.

This in particular implies $N_{L/\mathbb{Q}}(\alpha) \in \mathbb{Z}$ and $\mathrm{tr}_{L/\mathbb{Q}}(\alpha) \in \mathbb{Z}$.

Lemma. Let $L = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is not $0, 1$ and is square-free. Then

$$\mathcal{O}_L = \begin{cases} \mathbb{Z}[\sqrt{d}] & d \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1}{2}(1+\sqrt{d})\right] & d \equiv 1 \pmod 4 \end{cases}$$

Theorem (Primitive element theorem)**.** Let $K \subseteq L$ be a separable field extension. Then there exists an $\alpha \in L$ such that $K(\alpha) = L$.

Lemma. The degree $[L : \mathbb{Q}] = n$ of a number field is the number of field embeddings $L \hookrightarrow \mathbb{C}$.

Corollary. Let $L/\mathbb{Q}$ be a number field. If $\sigma_1, \cdots, \sigma_n : L \to \mathbb{C}$ are the different field embeddings and $\beta \in L$, then

$$\mathrm{tr}_{L/\mathbb{Q}}(\beta) = \sum \sigma_i(\beta), \quad N_{L/\mathbb{Q}}(\beta) = \prod_i \sigma_i(\beta).$$

We call $\sigma_1(\beta), \cdots, \sigma_n(\beta)$ the *conjugates* of β in $\mathbb{C}$.

Lemma. Let $x \in \mathcal{O}_L$. Then x is a unit if and only if $N_{L/\mathbb{Q}}(x) = \pm 1$.

Corollary. If $x \in \mathcal{O}_L$ is such that $N(x)$ is prime, then x is irreducible.

Proposition. Let L/K be a separable extension. Then a K-bilinear form $L \times L \to K$ defined by $(x, y) \mapsto \mathrm{tr}_{L/K}(xy)$ is non-degenerate. Equivalent, if $\alpha_1, \cdots, \alpha_n$ are a K-basis for L, the Gram matrix $(\mathrm{tr}(\alpha_i\alpha_j))_{i,j=1,\cdots,n}$ has non-zero determinant.

Theorem. Let $\mathbb{Q}/L$ be a number field. Then there exists an integral basis for $\mathcal{O}_L$. In particular, $\mathcal{O}_L \cong \mathbb{Z}^n$ with $n = [L : \mathbb{Q}]$.

3 Multiplicative structure of ideals

Proposition. Let $L/\mathbb{Q}$ be a number field, and $\mathcal{O}_L$ be its ring of integers. Then $\mathcal{O}_L$ is a Dedekind domain.

Lemma. Let $\mathfrak{a} \lhd \mathcal{O}_L$ be a non-zero ideal. Then $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$ and $\mathcal{O}_L/\mathfrak{a}$ is finite.

Lemma. Let $\mathfrak{p}$ be a prime ideal in a ring R. Then for $\mathfrak{a}, \mathfrak{b} \lhd R$ ideals, then $\mathfrak{ab} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

Lemma. Let $0 \neq \mathfrak{a} \lhd \mathcal{O}_L$ a non-zero ideal. Then there is a subset of $\mathfrak{a}$ that is a product of prime ideals.

Proposition.

(i) Let $0 \neq \mathfrak{a} \lhd \mathcal{O}_L$ be an ideal. If $x \in L$ has $x\mathfrak{a} \subseteq \mathfrak{a}$, then $x \in \mathcal{O}_L$.

(ii) Let $0 \neq \mathfrak{a} \lhd \mathcal{O}_L$ be a *proper* ideal. Then

$$\{y \in L : y\mathfrak{a} \leq \mathcal{O}_L\}$$

contains elements that are not in $\mathcal{O}_L$. In other words,

$$\frac{\{y \in L : y\mathfrak{a} \leq \mathcal{O}_L\}}{\mathcal{O}_L} \neq 0.$$

Lemma. An $\mathcal{O}_L$ module $\mathfrak{q} \subseteq L$ is a fractional ideal if and only if there is some $c \in L^\times$ such that $c\mathfrak{q}$ is an ideal in $\mathcal{O}_L$. Moreover, we can pick c such that $c \in \mathbb{Z}$.

Corollary. Let $\mathfrak{q}$ be a fractional ideal. Then as an abelian group, $\mathfrak{q} \cong \mathbb{Z}^n$, where $n = [L : \mathbb{Q}]$.

Corollary. Let $\mathfrak{a} \leq \mathcal{O}_L$ be a proper ideal. Then $\{x \in L : x\mathfrak{a} \leq \mathcal{O}_L\}$ is a fractional ideal.

Proposition. Every non-zero fractional ideal is invertible. The inverse of $\mathfrak{q}$ is

$$\{x \in L : x\mathfrak{q} \subseteq \mathcal{O}_L\}.$$

Corollary. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \lhd \mathcal{O}_L$ be ideals, $\mathfrak{c} \neq 0$. Then

(i) $\mathfrak{b} \subseteq \mathfrak{a}$ if and only if $\mathfrak{bc} \subseteq \mathfrak{ac}$

(ii) $\mathfrak{a} \mid \mathfrak{b}$ if and only if $\mathfrak{ac} \mid \mathfrak{bc}$

(iii) $\mathfrak{a} \mid \mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$.

Theorem. Let $\mathfrak{a} \lhd \mathcal{O}_L$ be an ideal, $\mathfrak{a} \neq 0$. Then $\mathfrak{a}$ can be written uniquely as a product of prime ideals.

Corollary. The non-zero fractional ideals form a group under multiplication. We denote this I_L. This is a free abelian group generated by the prime ideals, i.e. any fractional ideal $\mathfrak{q}$ can be written uniquely as $\mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{q_r}$, with $\mathfrak{p}_i$ distinct prime ideals and $a_i \in \mathbb{Z}$.

Moreover, if $\mathfrak{q}$ is an integral ideal, i.e. $\mathfrak{q} \lhd \mathcal{O}_L$, then $a_i, \cdots, a_r \geq 0$.

Theorem. The following are equivalent:

(i) $\mathcal{O}_L$ is a principal ideal domain

(ii) $\mathcal{O}_L$ is a unique factorization domain

(iii) cl_L is trivial.

4 Norms of ideals

Proposition. For any ideal $\mathfrak{a}$, we have $N(\mathfrak{a}) \in \mathfrak{a} \cap \mathbb{Z}$.

Proposition. Let $\mathfrak{a}, \mathfrak{b} \lhd \mathcal{O}_L$ be ideals. Then $N(\mathfrak{ab}) = N(\mathfrak{a})N(\mathfrak{b})$.

Proposition. Let $\mathfrak{a} \lhd \mathcal{O}_L$ be an ideal, $n = [L : \mathbb{Q}]$. Then

(i) There exists $\alpha_1, \cdots, \alpha_n \in \mathfrak{a}$ such that

$$\mathfrak{a} = \left\{\sum r_i\alpha_i : r_i \in \mathbb{Z}\right\} = \bigoplus_1^n \alpha_i\mathbb{Z},$$

and $\alpha_1, \cdots, \alpha_n$ are a basis of L over $\mathbb{Q}$. In particular, $\mathfrak{a}$ is a free $\mathbb{Z}$-module of n generators.

(ii) For any such $\alpha_1, \cdots, \alpha_n$,

$$\Delta(\alpha_1, \cdots, \alpha_n) = N(\mathfrak{a})^2 D_L.$$

Lemma. Let M be a $\mathbb{Z}$-module (i.e. abelian group), and suppose $M \leq \mathbb{Z}^n$. Then $M \cong \mathbb{Z}^r$ for some $0 \leq r \leq n$.

Moreover, if $r = n$, then we can choose a basis $v_1, \cdots, v_n$ of M such that the change of basis matrix $A = (a_{ij}) \in M_{n\times n}(\mathbb{Z})$ is upper triangular, where

$$v_j = \sum a_{ij}e_i,$$

where $e_1, \cdots, e_n$ is the standard basis of $\mathbb{Z}^n$.

In particular,

$$|\mathbb{Z}^n/M| = |a_{11}a_{12}\cdots a_{nn}| = |\det A|.$$

Corollary. Suppose $\mathfrak{a} \lhd \mathcal{O}_L$ has basis $\alpha_1, \cdots, \alpha_n$, and $\Delta(\alpha_1, \cdots, \alpha_n)$ is square-free. Then $\mathfrak{a} = \mathcal{O}_L$ (and D_L is square-free).

Lemma. If $\alpha \in \mathcal{O}_L$, then

$$N(\langle\alpha\rangle) = |N_{L/\mathbb{Q}}(\alpha)|.$$

5 Structure of prime ideals

Lemma. Let $\mathfrak{p} \lhd \mathcal{O}_L$ be a prime ideal. Then there exists a unique $p \in \mathbb{Z}$, p prime, with $\mathfrak{p} \mid \langle p \rangle$. Moreover, $N(\mathfrak{p}) = p^f$ for some $1 \leq f \leq n$.

Theorem (Dedekind's criterion)**.** Let $\alpha \in \mathcal{O}_L$ and $g(x) \in \mathbb{Z}[x]$ be its minimal polynomial. Suppose $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_L$ has finite index, coprime to p (i.e. $p \nmid |\mathcal{O}_L/\mathbb{Z}[\alpha]|$). We write

$$\bar{g}(x) = g(x) \pmod{p},$$

so $\bar{g}(x) \in \mathbb{F}_p[x]$. We factor

$$\bar{g}(x) = \varphi_1^{e_1} \cdots \varphi_m^{e_m}$$

into distinct irreducibles in $\mathbb{F}_p[x]$. We define the ideal

$$\mathfrak{p}_i = \langle p, \tilde{\varphi}_i(\alpha) \rangle \lhd \mathcal{O}_L,$$

generated by p and $\tilde{\varphi}_i$, where $\tilde{\varphi}_i$ is any polynomial in $\mathbb{Z}[x]$ such that $\tilde{\varphi}_i \bmod p = \varphi_i$. Notice that if $\tilde{\varphi}'$ is another such polynomial, then $p \mid (\tilde{\varphi}_i - \varphi_i)$, so $\langle p, \tilde{\varphi}'(\alpha) \rangle = \langle p, \tilde{\varphi}(\alpha) \rangle$.

Then the $\mathfrak{p}_i$ are prime, and

$$\langle p \rangle = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}.$$

Moreover, $f_i = \deg \varphi_i$, so $N(\mathfrak{a}) = p^{\deg \varphi_i}$.

Lemma. In $L = \mathbb{Q}(\sqrt{d})$,

(i) 2 splits in L if and only if $d \equiv 1 \pmod 8$;

(ii) 2 is inert in L if and only if $d \equiv 5 \pmod 8$;

(iii) 2 ramifies in L if $d \equiv 2, 3 \pmod 4$.

Corollary. If p is prime and $p < n = [L : \mathbb{Q}]$, and $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_L$ has finite index coprime to p, then p does *not* split completely in $\mathcal{O}_L$.

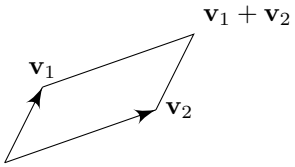
Theorem. $p \mid D_L$ if and only if p ramifies in $\mathcal{O}_L$.

6 Minkowski bound and finiteness of class group

Lemma (Minskowski's lemma)**.** Let $\Lambda = \mathbb{Z}\mathbf{v}_1 + \mathbb{Z}\mathbf{v}_2 \subseteq \mathbb{R}^2$ be a lattice, with $\mathbf{v}_1, \mathbf{v}_2$ linearly independent in $\mathbb{R}$ (i.e. $\mathbb{R}\mathbf{v}_1 + \mathbb{R}\mathbf{v}_2 = \mathbb{R}^2$). We write $\mathbf{v}_i = a_i\mathbf{e}_1 + b_i\mathbf{e}_2$. Then let

$$A(\Lambda) = \text{area of fundamental parallelogram} = \left|\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right|,$$

where the fundamental parallelogram is the following:



Then a closed disc S around 0 contains a non-zero point of Λ if

$$\text{area}(S) \geq 4A(\Lambda).$$

In particular, there exists an $\alpha \in \Lambda$ with $\alpha \neq 0$, such that

$$0 < |\alpha|^2 \leq \frac{4A(\Lambda)}{\pi}.$$

Proposition.

(i) If $\alpha = a + b\sqrt{\lambda}$, then as a complex number,

$$|\alpha|^2 = (a + b\sqrt{\lambda})(a - b\sqrt{\lambda}) = N(\alpha).$$

(ii) For $\mathcal{O}_L$, we have

$$A(\mathcal{O}_L) = \frac{1}{2}\sqrt{|D_L|}.$$

(iii) In general, we have

$$A(\mathfrak{a}) = \frac{1}{2}\sqrt{|\Delta(\alpha_1, \alpha_2)|},$$

where α_1, α_2 are the integral basis of $\mathfrak{a}$.

(iv) We have

$$A(\mathfrak{a}) = N(\mathfrak{a})A(\mathcal{O}_L).$$

Proposition (Minkowski bound)**.** For all $[\mathfrak{a}] \in \text{cl}_L$, there is a representative $\mathfrak{b}$ of $[\mathfrak{a}]$ (i.e. an ideal $\mathfrak{b} \leq \mathcal{O}_L$ such that $[\mathfrak{b}] = [\mathfrak{a}]$) such that

$$N(\mathfrak{b}) \leq c_L = \frac{2\sqrt{|D_L|}}{\pi}.$$

Lemma. For every $n \in \mathbb{Z}$, there are only finitely many ideals $\mathfrak{a} \leq \mathcal{O}_L$ with $N(\mathfrak{a}) = m$.

Theorem. The class group cl_L is a finite group, and the divisors of ideals of the form $\langle p \rangle$ for $p \in \mathbb{Z}$, p a prime, and $0 < p < c_L$, collectively generate cl_L.

Theorem. Let $L = \mathbb{Q}(\sqrt{d})$ with $d < 0$. Then $\mathcal{O}_L$ is a UFD if

$$-d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

Moreover, this is actually an "if and only if".

Proposition. Suppose $\Lambda \subseteq \mathbb{R}^n$ is a subgroup. Then Λ is a discrete subgroup of $(\mathbb{R}^n, +)$ if and only if

$$\Lambda = \left\{ \sum_1^m n_i \mathbf{x}_i : n_i \in \mathbb{Z} \right\}$$

for some $\mathbf{x}_1, \cdots, \mathbf{x}_m$ linearly independent over $\mathbb{R}$.

Theorem (Minkowski's theorem)**.** Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, and P be a fundamental domain. We let $S \subseteq \mathbb{R}^n$ be a measurable set, i.e. one for which $\mathrm{vol}(S)$ is defined.

(i) Suppose $\mathrm{vol}(S) > \mathrm{covol}(\Lambda)$. Then there exists distinct $\mathbf{x}, \mathbf{y} \in S$ such that $\mathbf{x} - \mathbf{y} \in \Lambda$.

(ii) Suppose $\mathbf{0} \in S$, and S is symmetric around 0, i.e. $\mathbf{s} \in S$ if and only if $-\mathbf{s} \in S$, and S is convex, i.e. for all $\mathbf{x}, \mathbf{y} \in S$ and $\lambda \in [0, 1]$, then

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in S.$$

Then suppose either

(a) $\mathrm{vol}(S) > 2^n \,\mathrm{covol}(\Lambda)$; or

(b) $\mathrm{vol}(S) \geq 2^n \,\mathrm{covol}(\Lambda)$ and S is closed.

Then S contains a $\gamma \in \Lambda$ with $\gamma \neq 0$.

Lemma.

(i) $\sigma(\mathcal{O}_L)$ is a lattice in $\mathbb{R}^n$ of covolume $2^{-s}|D_L|^{\frac{1}{2}}$.

(ii) More generally, if $\mathfrak{a} \lhd \mathcal{O}_L$ is an ideal, then $\sigma(\mathfrak{a})$ is a lattice and the covolume

$$\mathrm{covol}(\sigma(\mathfrak{a})) = 2^{-s}|D_L|^{\frac{1}{2}} N(\mathfrak{a}).$$

Proposition. Let $\mathfrak{a} \lhd \mathcal{O}_L$ be an ideal. Then there exists an $\alpha \in \mathfrak{a}$ with $\alpha \neq 0$ such that

$$|N(\alpha)| \leq c_L N(\mathfrak{a}),$$

where

$$c_L = \left(\frac{4}{\pi}\right)^s \frac{n!}{n^n} |D_L|^{\frac{1}{2}}.$$

Corollary. Every $[\mathfrak{a}] \in \mathrm{cl}_L$ has a representative $\mathfrak{a} \in \mathcal{O}_L$ with $N(\mathfrak{a}) \leq c_L$.

Theorem (Dirichlet)**.** The class group cl_L is finite, and is generated by prime ideals of norm $\leq c_L$.

7 Dirichlet's unit theorem

Theorem (Dirichlet unit theorem)**.** We have the isomorphism

$$\mathcal{O}_L^\times \cong \mu_L \times \mathbb{Z}^{r+s-1},$$

where

$$\mu_L = \{\alpha \in L : \alpha^N = 1 \text{ for some } N > 0\}$$

is the group of roots of unity in L, and is a finite cyclic group.

Theorem (Pell's equation)**.** There are infinitely many $x + y\sqrt{d} \in \mathcal{O}_L$ such that $x^2 - dy^2 = \pm 1$.

Theorem (Dirchlet's unit theorem for real quadratic fields)**.** Let $L = \mathbb{Q}(\sqrt{d})$. Then there is some $\varepsilon_0 \in \mathcal{O}_L^\times$ such that

$$\mathcal{O}_L^\times = \{\pm\varepsilon_0^n : n \in \mathbb{Z}\}.$$

We call such an ε_0 a *fundamental unit* (which is not unique). So

$$\mathcal{O}_L^\times \cong \{\pm 1\} \times \mathbb{Z}.$$

Theorem (Dirichlet unit theorem)**.** We have the isomorphism

$$\mathcal{O}_L^\times \cong \mu_L \times \mathbb{Z}^{r+s-1},$$

where

$$\mu_L = \{\alpha \in L : \alpha^N = 1 \text{ for some } N > 0\}$$

is the group of roots of unity in L, and is a finite cyclic group.

Lemma.

(i) If $d = -1$, then $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\} = \mathbb{Z}/4\mathbb{Z}$.

(ii) If $d = -3$, then let $\omega = \frac{1}{2}(1 + \sqrt{d})$, and we have $\omega^6 = 1$. So $\mathbb{Z}[\omega]^\times = \{1, \omega, \cdots, \omega^5\} \cong \mathbb{Z}/6\mathbb{Z}$.

(iii) For any other $d < 0$, we have $\mathcal{O}_L^\times = \{\pm 1\}$.

8 L-functions, Dirichlet series*

Theorem (Euclid)**.** There are infinitely many primes.

Theorem (Dirichlet's theorem)**.** Let $a, q \in \mathbb{Z}$ be coprime. Then there exists infinitely many primes in the sequence

$$a, a+q, a+2q, \cdots,$$

i.e. there are infinitely many primes in any such arithmetic progression.

Proposition.

(i) The Riemann zeta function $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$.

(ii) The function
$$\zeta(s) - \frac{1}{s-1}$$
extends to a holomorphic function when $\mathrm{Re}(s) > 0$.

In other words, $\zeta(s)$ extends to a meromorphic function on $\mathrm{Re}(s) > 0$ with a simple pole at 1 with residue 1.

(iii) We have the expression
$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$
for $\mathrm{Re}(s) > 1$, and the product is absolutely convergent. This is the *Euler product*.

Lemma. If there is a real number $r \in \mathbb{R}$ such that
$$a_1 + \cdots + a_N = O(N^r),$$
then
$$\sum a_n n^{-s}$$
converges for $\mathrm{Re}(s) > r$, and is a holomorphic function there.

Theorem.

(i) $\zeta_L(s)$ converges to a holomorphic function if $\mathrm{Re}(s) > 1$.

(ii) *Analytic class number formula*: $\zeta_L(s)$ is a meromorphic function if $\mathrm{Re}(s) > 1 - \frac{1}{n}$ and has a simple pole at $s = 1$ with residue
$$\frac{|\operatorname{cl}_L| 2^r (2\pi)^s R_L}{|D_L|^{1/2} |\mu_L|},$$
where cl_L is the class group, r and s are the number of real and complex embeddings, you know what π is, R_L is the regulator, D_L is the discriminant and μ_L is the roots of unity in L.

(iii)
$$\zeta_L(s) = \prod_{\mathfrak{p} \lhd \mathcal{O}_L \text{ prime ideal}} (1 - N(\mathfrak{p})^{-s})^{-1}.$$
This is again known as the *Euler product*.

Proposition. χ_D, as defined for $L = \mathbb{Q}(\sqrt{d})$ is a Dirichlet character of modulus D.

Lemma. Let χ be any non-trivial Dirichlet character. Then $L(\chi, s)$ is holomorphic for $\text{Re}(s) > 0$.

Corollary. For quadratic characters χ_D, we have
$$L(\chi_D, 1) \neq 0.$$

Proposition.

(i) We have $[L : \mathbb{Q}] = \varphi(q)$, where
$$\varphi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times|.$$

(ii) $L \supseteq \mathbb{Q}$ is a Galois extension, with
$$\text{Gal}(L/\mathbb{Q}) = (\mathbb{Z}/q\mathbb{Z})^\times,$$
where if $r \in (\mathbb{Z}/q\mathbb{Z})^\times$, then r acts on $\mathbb{Q}(w_Q)$ by sending $\omega_q \mapsto \omega_q^r$. This is what plays the role of quadratic reciprocity for cyclotomic fields.

(iii) The ring of integers is
$$\mathcal{O}_L = \mathbb{Z}[\omega_q] = \mathbb{Z}[x]/\Phi_q(x),$$
where
$$\Phi_q(x) = \frac{x^q - 1}{\prod_{d|q, d\neq q} \Phi_d(x)}$$
is the qth cyclotomic polynomial.

(iv) Let p be a prime. Then p ramifies in $\mathcal{O}_L$ if and only if $p \mid D_L$, if and only if $p \mid q$. So while D might be messy, the prime factors of D are the prime factors of q.

(v) Let p be a prime and $p \nmid q$. Then $\langle p \rangle$ factors as a product of $\varphi(q)/f$ distinct prime ideals, each of norm p^f, where f is the order of p in $(\mathbb{Z}/q\mathbb{Z})^\times$.

Proposition. We have
$$\zeta_{\mathbb{Q}(\omega_q)}(s) = \prod_{i=1}^{\varphi(q)} L(\chi_i, s) \cdot (\text{corr. factor}) = \zeta_{\mathbb{Q}}(s) \prod_{i=2}^{\varphi(q)} L(\chi_i, s) \cdot (\text{corr. factor})$$
where the correction factor is a finite product coming from the primes that divide q.

Corollary. If χ is any non-trivial Dirichlet character, then $L(\chi, 1) \neq 0$.

Theorem (Dirichlet, 1839)**.** Let $a, q \in \mathbb{N}$ be coprime, i.e. $\gcd(a, q) = 1$. Then there are infinitely many primes in the arithmetic progression
$$a, a + q, a + 2q, a + 3q, \cdots .$$