Part III — Differential Geometry
Theorems

Based on lectures by J. A. Ross
Notes taken by Dexter Chua

Michaelmas 2016

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are, for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, smooth maps. Tangent vectors and vector fields, tangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Lie Groups.
- *Vector Bundles.* Structure group. The example of Hopf bundle. Bundle morphisms and automorphisms. Exterior algebra of differential forms. Tensors. Symplectic forms. Orientability of manifolds. Partitions of unity and integration on manifolds, Stokes Theorem; de Rham cohomology. Lie derivative of tensors. Connections on vector bundles and covariant derivatives: covariant exterior derivative, curvature. Bianchi identity.
- *Riemannian Geometry.* Connections on the tangent bundle, torsion. Bianchi's identities for torsion free connections. Riemannian metrics, Levi-Civita connection, Christoffel symbols, geodesics. Riemannian curvature tensor and its symmetries, second Bianchi identity, sectional curvatures.

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor's theorem in several variables). Exposure to some of the ideas of classical differential geometry might also be useful.

Contents

0 Introduction

1 Manifolds

1.1 Manifolds

Lemma. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in some atlas, and $f : M \to \mathbb{R}$, then $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(p)$ if and only if $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$.

Lemma. Let M be a manifold, and $\varphi_1 : U_1 \to \mathbb{R}^n$ and $\varphi_2 : U_2 \to \mathbb{R}^m$ be charts. If $U_1 \cap U_2 \neq \emptyset$, then $n = m$.

1.2 Smooth functions and derivatives

Lemma. $\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$ is a basis of $T_p\mathbb{R}^n$. So these are all the derivations.

Proposition (Chain rule)**.** Let M, N, P be manifolds, and $F \in C^\infty(M, N)$, $G \in C^\infty(N, P)$, and $p \in M, q = F(p)$. Then we have

$$\mathrm{D}(G \circ F)|_p = \mathrm{D}G|_q \circ \mathrm{D}F|_p.$$

Corollary. If F is a diffeomorphism, then $\mathrm{D}F|_p$ is a linear isomorphism, and $(\mathrm{D}F|_p)^{-1} = \mathrm{D}(F^{-1})|_{F(p)}$.

Lemma. We have

$$\mathrm{D}F|_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right) = \sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p)\left.\frac{\partial}{\partial y_j}\right|_q.$$

In other words, $\mathrm{D}F|_p$ has matrix representation

$$\left(\frac{\partial F_j}{\partial x_i}(p)\right)_{ij}.$$

1.3 Bump functions and partitions of unity

Lemma. Suppose $W \subseteq M$ is a coordinate chart with $p \in W$. Then there is an open neighbourhood V of p such that $\bar{V} \subseteq W$ and an $X \in C^\infty(M, \mathbb{R})$ such that $X = 1$ on V and $X = 0$ on $M \setminus W$.

Lemma. Let $p \in W \subseteq U$ and W, U open. Let $f_1, f_2 \in C^\infty(U)$ be such that $f_1 = f_2$ on W. If $X \in \mathrm{Der}_p(C^\infty(U))$, then we have $X(f_1) = X(f_2)$

Theorem. Given any $\{U_\alpha\}$ open cover, there exists a partition of unity subordinate to $\{U_\alpha\}$.

1.4 Submanifolds

Lemma. If S is an embedded submanifold of M, then there exists a unique differential structure on S such that the inclusion map $\iota : S \hookrightarrow M$ is smooth and S inherits the subspace topology.

Proposition. Let S be an embedded submanifold. Then the derivative of the inclusion map $\iota : S \hookrightarrow M$ is injective.

Proposition. Let $F \in C^\infty(M, N)$, and let $c \in N$. Suppose c is a regular value. Then $S = F^{-1}(c)$ is an embedded submanifold of dimension $\dim M - \dim N$.

2 Vector fields

2.1 The tangent bundle

Lemma. The charts actually make TM into a manifold.

Lemma. The map $X \mapsto \mathcal{X}$ is an $\mathbb{R}$-linear isomorphism

$$\Gamma : \mathrm{Vect}(M) \to \mathrm{Der}(C^\infty(M)).$$

2.2 Flows

Theorem (Fundamental theorem on ODEs)**.** Let $U \subseteq \mathbb{R}^n$ be open and $\alpha : U \to \mathbb{R}^n$ smooth. Pick $t_0 \in \mathbb{R}$.

Consider the ODE

$$\begin{aligned} \dot{\gamma}_i(t) &= \alpha_i(\gamma(t)) \\ \gamma_i(t_0) &= c_i, \end{aligned}$$

where $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{R}^n$.

Then there exists an open interval I containing t_0 and an open $U_0 \subseteq U$ such that for every $\mathbf{c} \in U_0$, there is a smooth solution $\gamma_{\mathbf{c}} : I \to U$ satisfying the ODE.

Moreover, any two solutions agree on a common domain, and the function $\Theta : I \times U_0 \to U$ defined by $\Theta(t, \mathbf{c}) = \gamma_{\mathbf{c}}(t)$ is smooth (in both variables).

Theorem (Existence of integral curves)**.** Let $X \in \mathrm{Vect}(M)$ and $p \in M$. Then there exists some open interval $I \subseteq \mathbb{R}$ with $0 \in I$ and an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.

Moreover, if $\tilde{\gamma} : \tilde{I} \to M$ is another integral curve for X, and $\tilde{\gamma}(0) = p$, then $\tilde{\gamma} = \gamma$ on $I \cap \tilde{I}$.

Theorem. Let M be a manifold and X a complete vector field on M. Define $\Theta_t : \mathbb{R} \times M \to M$ by

$$\Theta_t(p) = \gamma_p(t),$$

where γ_p is the maximal integral curve of X through p with $\gamma(0) = p$. Then Θ is a function smooth in p and t, and

$$\Theta_0 = \mathrm{id}, \quad \Theta_t \circ \Theta_s = \Theta_{s+t}$$

Theorem. Let M be a manifold, and $X \in \mathrm{Vect}(M)$. Define

$$D = \{(t, p) \in \mathbb{R} \times M : t \in I_p\}.$$

In other words, this is the set of all (t, p) such that $\gamma_p(t)$ exists. We set

$$\Theta_t(p) = \Theta(t, p) = \gamma_p(t)$$

for all $(t, p) \in D$. Then

(i) D is open and $\Theta : D \to M$ is smooth

(ii) $\Theta(0, p) = p$ for all $p \in M$.

(iii) If $(t, p) \in D$ and $(t, \Theta(s, p)) \in D$, then $(s + t, p) \in D$ and $\Theta(t, \Theta(s, p)) = \Theta(t + s, p)$.

(iv) For any $t \in \mathbb{R}$, the set $M_t : \{p \in M : (t, p) \in D\}$ is open in M, and
$$\Theta_t : M_t \to M_{-t}$$
is a diffeomorphism with inverse Θ_{-t}.

Proposition. Let M be a compact manifold. Then any $X \in \mathrm{Vect}(M)$ is complete.

2.3 Lie derivative

Lemma. $\mathcal{L}_X(g) = X(g)$. In particular, $\mathcal{L}_X(g) \in C^\infty(M, \mathbb{R})$.

Lemma. We have
$$\mathcal{L}_X Y = [X, Y].$$

Corollary. Let $X, Y \in \mathrm{Vect}(M)$ and $f \in C^\infty(M, \mathbb{R})$. Then

(i) $\mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X Y = X(f)Y + f\mathcal{L}_X Y$

(ii) $\mathcal{L}_X Y = -\mathcal{L}_Y X$

(iii) $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

3 Lie groups

Lemma. Given $\xi \in T_e G$, we let

$$X_\xi|_g = \mathrm{D}L_g|_e(\xi) \in T_g(G).$$

Then the map $T_e G \to \mathrm{Vect}^L(G)$ by $\xi \mapsto X_\xi$ is an isomorphism of vector spaces.

Lemma. Let G be an abelian Lie group. Then the bracket of $\mathfrak{g}$ vanishes.

Proposition. Let G be a Lie group and $\xi \in \mathfrak{g}$. Then the integral curve γ for X_ξ through $e \in G$ exists for all time, and $\gamma : \mathbb{R} \to G$ is a Lie group homomorphism.

Proposition.

(i) exp is a smooth map.

(ii) If $F(t) = \exp(t\xi)$, then $F : \mathbb{R} \to G$ is a Lie group homomorphism and $\mathrm{D}F|_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \xi$.

(iii) The derivative

$$\mathrm{D}\exp : T_0\mathfrak{g} \cong \mathfrak{g} \to T_e G \cong \mathfrak{g}$$

is the identity map.

(iv) exp is a local diffeomorphism around $0 \in \mathfrak{g}$, i.e. there exists an open $U \subseteq \mathfrak{g}$ containing 0 such that $\exp : U \to \exp(U)$ is a diffeomorphism.

(v) exp is natural, i.e. if $f : G \to H$ is a Lie group homomorphism, then the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp"] \arrow[d, "\mathrm{D}f|_e"] & G \arrow[d, "f"] \\
\mathfrak{h} \arrow[r, "\exp"] & H
\end{tikzcd}$$

commutes.

Theorem. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then there exists a unique connected Lie subgroup $H \subseteq G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.

Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a (unique) simply-connected Lie group G with Lie algebra $\mathfrak{g}$.

Theorem. Let G, H be Lie groups with G simply connected. Then every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ lifts to a Lie group homomorphism $G \to H$.

4 Vector bundles

4.1 Tensors

Lemma. Tensor products exist (and are unique up to isomorphism) for all pairs of finite-dimensional vector spaces.

Proposition. Given maps $f : V \to W$ and $g : V' \to W'$, we obtain a map $f \otimes g : V \otimes V' \to W \otimes W'$ given by the bilinear map

$$(f \otimes g)(v, w) = f(v) \otimes g(w).$$

Lemma. Given $v, v_i \in V$ and $w, w_i \in W$ and $\lambda_i \in \mathbb{R}$, we have

$$\begin{aligned}(\lambda_1 v_1 + \lambda_2 v_2) \otimes w &= \lambda_1(v_1 \otimes w) + \lambda_2(v_2 \otimes w)\\ v \otimes (\lambda_1 w_1 + \lambda_2 w_2) &= \lambda_1(v \otimes w_1) + \lambda_2(v \otimes w_2).\end{aligned}$$

Lemma. If $v_1, \cdots, v_n$ is a basis for V, and $w_1, \cdots, w_m$ is a basis for W, then

$$\{v_i \otimes w_j : i = 1, \cdots, n; j = 1, \cdots, m\}$$

is a basis for $V \otimes W$. In particular, $\dim V \otimes W = \dim V \times \dim W$.

Proposition. For any vector spaces V, W, U, we have (natural) isomorphisms

(i) $V \otimes W \cong W \otimes V$

(ii) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$

(iii) $(V \otimes W)^* \cong V^* \otimes W^*$

Lemma.

(i) If $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

(ii) If $\dim V = n$ and $p > n$, then we have

$$\dim \Lambda^0 V = 1, \quad \dim \Lambda^n V = 1, \quad \Lambda^p V = \{0\}.$$

(iii) The multilinear map $\det : V \times \cdots \times V \to \mathbb{R}$ spans $\Lambda^n V$.

(iv) If $v_1, \cdots, v_n$ is a basis for V, then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : i_1 < \cdots < i_p\}$$

is a basis for $\Lambda^p V$.

Lemma. Let $F : V \to V$ be a linear map. Then $\Lambda^n F : \Lambda^n V \to \Lambda^n V$ is multiplication by $\det F$.

4.2 Vector bundles

Proposition. We have the following equalities whenever everything is defined:

(i) $\varphi_{\alpha\alpha} = \mathrm{id}$

(ii) $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$

(iii) $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, where $\varphi_{\alpha\beta}\varphi_{\beta\gamma}$ is pointwise matrix multiplication.

These are known as the *cocycle conditions*.

Proposition (Vector bundle construction)**.** Suppose that for each $p \in M$, we have a vector space E_p. We set

$$E = \bigcup_p E_p$$

We let $\pi : E \to M$ be given by $\pi(v_p) = p$ for $v_p \in E_p$. Suppose there is an open cover $\{U_\alpha\}$ of open sets of M such that for each α, we have maps

$$t_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

over U_α that induce fiberwise linear isomorphisms. Suppose the transition functions $\varphi_{\alpha\beta}$ are smooth. Then there exists a unique smooth structure on E making $\pi : E \to M$ a vector bundle such that the t_α are trivializations for E.

5 Differential forms and de Rham cohomology

5.1 Differential forms

Theorem (Exterior derivative)**.** There exists a unique linear map

$$\mathrm{d} = \mathrm{d}_{M,p} : \Omega^p(M) \to \Omega^{p+1}(M)$$

such that

(i) On $\Omega^0(M)$ this is as previously defined, i.e.

$$\mathrm{d}f(X) = X(f) \text{ for all } X \in \mathrm{Vect}(M).$$

(ii) We have

$$\mathrm{d} \circ \mathrm{d} = 0 : \Omega^p(M) \to \Omega^{p+2}(M).$$

(iii) It satisfies the *Leibniz rule*

$$\mathrm{d}(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^p \omega \wedge \mathrm{d}\sigma.$$

It follows from these assumptions that

(iv) d acts locally, i.e. if $\omega, \omega' \in \Omega^p(M)$ satisfy $\omega|_U = \omega'|_U$ for some $U \subseteq M$ open, then $\mathrm{d}\omega|_U = \mathrm{d}\omega'|_U$.

(v) We have

$$\mathrm{d}(\omega|_U) = (\mathrm{d}\omega)|_U$$

for all $U \subseteq M$.

Lemma. Let $F \in C^\infty(M, N)$. Let F^* be the associated pullback map. Then

(i) F^* is a linear map $\Omega^p(N) \to \Omega^p(M)$.

(ii) $F^*(\omega \wedge \sigma) = F^*\omega \wedge F^*\sigma$.

(iii) If $G \in C^\infty(N, P)$, then $(G \circ F)^* = F^* \circ G^*$.

(iv) We have $\mathrm{d}F^* = F^*\mathrm{d}$.

5.2 De Rham cohomology

Proposition.

(i) Let M have k connected components. Then

$$H^0_{\mathrm{dR}}(M) = \mathbb{R}^k.$$

(ii) If $p > \dim M$, then $H^p_{\mathrm{dR}}(M) = 0$.

(iii) If $F \in C^\infty(M, N)$, then this induces a map $F^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ given by

$$F^*[\omega] = [F^*\omega].$$

(iv) $(F \circ G)^* = G^* \circ F^*$.

(v) If $F : M \to N$ is a diffeomorphism, then $F^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ is an isomorphism.

Theorem (Homotopy invariance)**.** Let F_0, F_1 be homotopic maps. Then $F_0^* = F_1^* : H^p_{dR}(N) \to H^p_{\mathrm{dR}}(M)$.

Corollary (Poincaré lemma)**.** Let $U \subseteq \mathbb{R}^n$ be open and star-shaped. Suppose $\omega \in \Omega^p(U)$ is such that $\mathrm{d}\omega = 0$. Then there is some $\sigma \in \Omega^{p-1}(M)$ such that $\omega = \mathrm{d}\sigma$.

Corollary. If M and N are smoothly homotopy equivalent, then $H^p_{\mathrm{dR}}(M) \cong H^p_{\mathrm{dR}}(N)$.

5.3 Homological algebra and Mayer-Vietoris theorem

Proposition. A cochain map induces a well-defined homomorphism on the cohomology groups.

Theorem (Snake lemma)**.** Suppose we have a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{\ i\ } B^\bullet \xrightarrow{\ q\ } C^\bullet \longrightarrow 0\ ,$$

i.e. the i, q are cochain maps and we have a short exact sequence

$$0 \longrightarrow A^p \xrightarrow{\ i^p\ } B^p \xrightarrow{\ q^p\ } C^p \longrightarrow 0\ ,$$

for each p.

Then there are maps

$$\delta : H^p(C^\bullet) \to H^{p+1}(A^\bullet)$$

such that there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(A^\bullet) \arrow[r, "i^*"] & H^p(B^\bullet) \arrow[r, "q^*"] & H^p(C^\bullet) \arrow[dll, "\delta"'] \\
& H^{p+1}(A^\bullet) \arrow[r, "i^*"] & H^{p+1}(B^\bullet) \arrow[r, "q^*"] & H^{p+1}(C^\bullet) \arrow[r] & \cdots
\end{tikzcd}.$$

Theorem (Mayer-Vietoris theorem)**.** Let M be a manifold, and $M = U \cup V$, where U, V are open. We denote the inclusion maps as follows:

$$\begin{tikzcd}
U \cap V \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"] & U \arrow[d, hook, "j_1"] \\
V \arrow[r, hook, "j_2"] & M
\end{tikzcd}$$

Then there exists a natural linear map

$$\delta : H^p_{\mathrm{dR}}(U \cap V) \to H^{p+1}_{\mathrm{dR}}(M)$$

such that the following sequence is exact:

$$\begin{tikzcd}
H^p_{\mathrm{dR}}(M) \arrow[r, "j_1^* \oplus j_2^*"] & H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \arrow[r, "i_1^* - i_2^*"] & H^p_{\mathrm{dR}}(U \cap V) \arrow[dll, "\delta"'] \\
H^{p+1}_{\mathrm{dR}}(M) \arrow[r, "j_1^* \oplus j_2^*"] & H^{p+1}_{\mathrm{dR}}(U) \oplus H^{p+1}_{\mathrm{dR}}(V) \arrow[r, "i_1^* - i_2^*"] & \cdots
\end{tikzcd}$$

6 Integration

6.1 Orientation

6.2 Integration

Lemma. Let $F : D \to E$ be a smooth map between domains of integration in $\mathbb{R}^n$, and assume that $F|_{\mathring{D}} : \mathring{D} \to \mathring{E}$ is an orientation-preserving diffeomorphism. Then

$$\int_E \omega = \int_D F^* \omega.$$

Lemma. This is well-defined, i.e. it is independent of cover and partition of unity.

Theorem. Given a parametrization $\{S_i\}$ of M and an $\omega \in \Omega^n(M)$ with compact support, we have

$$\int_M \omega = \sum_i \int_{D_i} F_i^* \omega.$$

Lemma. Let M be an oriented manifold, and g a Riemannian metric on M. Then there is a unique $\omega \in \Omega^n(M)$ such that for all p, if $e_1, \cdots, e_n$ is an oriented orthonormal basis of T_pM, then

$$\omega(e_1, \cdots, e_n) = 1.$$

We call this the *Riemannian volume form*, written $\mathrm{d}V_g$.

6.3 Stokes Theorem

Proposition. Let M be a manifold with boundary. Then $\mathrm{Int}(M)$ and ∂M are naturally manifolds, with

$$\dim \partial M = \dim \mathrm{Int}\, M - 1.$$

Lemma. Let $p \in \partial M$, say $p \in U \subseteq M$ where (U, φ) is a chart (with boundary). Then

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis for T_pM. In particular, $\dim T_pM = n$.

Theorem (Stokes' theorem)**.** Let M be an oriented manifold with boundary of dimension n. Then if $\omega \in \Omega^{n-1}(M)$ has compact support, then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

In particular, if M has no boundary, then

$$\int_M \mathrm{d}\omega = 0$$

7 De Rham's theorem*

Theorem (de Rham's theorem)**.** There exists a natural isomorphism

$$H^p_{\mathrm{dR}}(M) \cong H^p(M, \mathbb{R}),$$

where $H^p(M, \mathbb{R})$ is the singular cohomology of M, and this is in fact an isomorphism of rings, where $H^p_{\mathrm{dR}}(M)$ has the product given by the wedge, and $H^p(M, \mathbb{R})$ has the cup product.

Theorem. The map $i_* : H_p^\infty(M) \to H_p(M)$ is an isomorphism.

Lemma. I is a well-defined map $H^p_{\mathrm{dR}}(M) \to H^p_\infty(M, \mathbb{R})$.

Lemma. I is functorial and commutes with the boundary map of Mayer-Vietoris. In other words, if $F : M \to N$ is smooth, then the diagram

$$\begin{tikzcd} H^p_{\mathrm{dR}}(M) \arrow[r, "F^*"] \arrow[d, "I"] & H^p_{\mathrm{dR}}(N) \arrow[d, "I"] \\ H^p_\infty(M) \arrow[r, "F^*"] & H^p_\infty(N) \end{tikzcd}.$$

And if $M = U \cup V$ and U, V are open, then the diagram

$$\begin{tikzcd} H^p_{\mathrm{dR}}(U \cap V) \arrow[r, "\delta"] \arrow[d, "I"] & H^{p+1}_{\mathrm{dR}}(U \cup V) \arrow[d, "I"] \\ H^p_\infty(U \cap V, \mathbb{R}) \arrow[r, "\delta"] & H^p(U \cup V, \mathbb{R}) \end{tikzcd}$$

also commutes. Note that the other parts of the Mayer-Vietoris sequence commute because they are induced by maps of manifolds.

Proposition. Let $U \subseteq \mathbb{R}^n$ is convex, then

$$U : H^p_{\mathrm{dR}}(U) \to H^p_\infty(U, \mathbb{R})$$

is an isomorphism for all p.

Proposition. Suppose $\{U, V\}$ is a de Rham cover of $U \cup V$. Then $U \cup V$ is de Rham.

Corollary. If $U_1, \cdots, U_k$ is a finite de Rham cover of $U_1 \cup \cdots \cup U_k = N$, then M is de Rham.

Proposition. The disjoint union of de Rham spaces is de Rham.

Lemma. Let M be a manifold. If it has a de Rham basis, then it is de Rham.

Theorem. Any manifold has a de Rham basis.

8 Connections

8.1 Basic properties of connections

Proposition. For any X, ∇_X is linear in s over $\mathbb{R}$, and linear in X over $C^\infty(M)$. Moreover,

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s$$

for $f \in C^\infty(M)$ and $s \in \Omega^0(E)$.

Lemma. Given a linear connection ∇ and a path $\gamma : I \to M$, there exists a unique map $\mathrm{D}_t : J(\gamma) \to J(\gamma)$ such that

(i) $\mathrm{D}_t(fV) = \dot{f}V + f\mathrm{D}_tV$ for all $f \in C^\infty(I)$

(ii) If U is an open neighbourhood of $\mathrm{im}(\gamma)$ and $\tilde{V}$ is a vector field on U such that $\tilde{V}|_{\gamma(t)} = V_t$ for all $t \in I$, then

$$\mathrm{D}_t(V)|_t = \nabla_{\dot{\gamma}(0)}\tilde{V}.$$

We call D_t the *covariant derivative* along γ.

Lemma. Given a connection ∇ and vector fields $X, Y \in \mathrm{Vect}(M)$, the quantity $\nabla_X Y|_p$ depends only on the values of Y near p and the value of X at p.

Proposition. Any vector bundle admits a connection.

Proposition. The map d_E extends uniquely to $\mathrm{d}_E : \Omega^p(E) \to \Omega^{p+1}(E)$ such that d_E is linear and

$$\mathrm{d}_E(w \otimes s) = \mathrm{d}\omega \otimes s + (-1)^p \omega \wedge \mathrm{d}_E s,$$

for $s \in \Omega^0(E)$ and $\omega \in \Omega^p(M)$. Here $\omega \wedge \mathrm{d}_E s$ means we take the wedge on the form part of $\mathrm{d}_E s$. More generally, we have a wedge product

$$\begin{aligned} \Omega^p(M) \times \Omega^q(E) &\to \Omega^{p+q}(E) \\ (\alpha, \beta \otimes s) &\mapsto (\alpha \wedge \beta) \otimes s. \end{aligned}$$

More generally, the extension satisfies

$$\mathrm{d}_E(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^q \omega \wedge \mathrm{d}_E \xi,$$

where $\xi \in \Omega^p(E)$ and $\omega \in \Omega^q(M)$.

8.2 Geodesics and parallel transport

Theorem. Let ∇ be a linear connection on M, and let $W \in T_pM$. Then there exists a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ for some $\varepsilon > 0$ such that

$$\dot{\gamma}(0) = W.$$

Any two such geodesics agree on their common domain.

Lemma (Parallel transport)**.** Let $t_0 \in I$ and $\xi \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field $V \in J(\gamma)$ such that $V(t_0) = \xi$. We call V the *parallel transport* of ξ along γ.

8.3 Riemannian connections

Lemma. Let ∇ be a connection. Then ∇ is compatible with g if and only if for all $\gamma : I \to M$ and $V, W \in J(\gamma)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(V(t), W(t)) = g(\mathrm{D}_t V(t), W(t)) + g(V(t), \mathrm{D}_t W(t)). \tag{$*$}$$

Corollary. If V, W are parallel along γ, then $g(V(t), W(t))$ is constant with respect to t.

Corollary. If γ is a geodesic, then $|\dot{\gamma}|$ is constant.

Corollary. Parallel transport is an isometry.

Proposition. τ is a tensor of type $(2, 1)$.

Theorem. Let M be a manifold with Riemannian metric g. Then there exists a unique torsion-free linear connection ∇ compatible with g.

8.4 Curvature

Lemma. F_E is a tensor. In particular, $F_E \in \Omega^2(\mathrm{End}(E))$.

Lemma. We have

$$F_E(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

In other words, we have

$$F_E(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Theorem. Let M be a manifold with Riemannian metric g .Then M is flat iff it is locally isometric to $\mathbb{R}^n$.

Proposition. Let $\dim M = n$ and $U \subseteq M$ open. Let $V_1, \cdots, V_n \in \mathrm{Vect}(U)$ be such that

(i) For all $p \in U$, we know $V_1(p), \cdots, V_n(p)$ is a basis for $T_p M$, i.e. the V_i are a frame.

(ii) $[V_i, V_j] = 0$, i.e. the V_i form a frame that pairwise commutes.

Then for all $p \in U$, there exists coordinates $x_1, \cdots, x_n$ on a chart $p \in U_p$ such that

$$V_i = \frac{\partial}{\partial x_i}.$$

Suppose that g is a Riemannian metric on M and the V_i are orthonormal in $T_p M$. Then the map defined above is an isometry.