These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are, for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.


**Pre-requisites**

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor’s theorem in several variables). Exposure to some of the ideas of classical differential geometry might also be useful.
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0 Introduction
1 Manifolds

1.1 Manifolds

Definition (Chart). A chart \((U, \varphi)\) on a set \(M\) is a bijection \(\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n\), where \(U \subseteq M\) and \(\varphi(U)\) is open.

A chart \((U, \varphi)\) is centered at \(p\) for \(p \in U\) if \(\varphi(p) = 0\).

Definition (Smooth function). Let \((U, \varphi)\) be a chart on \(M\) and \(f : M \to \mathbb{R}\). We say \(f\) is smooth or \(C^\infty\) at \(p \in U\) if \(f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}\) is smooth at \(\varphi(p)\) in the usual sense.

Definition (Atlas). An atlas on a set \(M\) is a collection of charts \(\{(U_\alpha, \varphi_\alpha)\}\) on \(M\) such that

(i) \(M = \bigcup_\alpha U_\alpha\).

(ii) For all \(\alpha, \beta\), we have \(\varphi_\alpha(U_\alpha \cap U_\beta)\) is open in \(\mathbb{R}^n\), and the transition function \(\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)\)

is smooth (in the usual sense).

Definition (Equivalent atlases). Two atlases \(A_1\) and \(A_2\) are equivalent if \(A_1 \cup A_2\) is an atlas.

Definition (Differentiable structure). A differentiable structure on \(M\) is a choice of equivalence class of atlases.

Definition (Manifold). A manifold is a set \(M\) with a choice of differentiable structure whose topology is

(i) Hausdorff, i.e. for all \(x, y \in M\), there are open neighbourhoods \(U_x, U_y \subseteq M\) with \(x \in U_x, y \in U_y\) and \(U_x \cap U_y = \emptyset\).

(ii) Second countable, i.e. there exists a countable collection \((U_n)_{n \in \mathbb{N}}\) of open sets in \(M\) such that for all \(V \subseteq M\) open, and \(p \in V\), there is some \(n\) such that \(p \in U_n \subseteq V\).

Definition (Local coordinates). Let \(M\) be a manifold, and \(\varphi : U \to \varphi(U)\) a chart of \(M\). We can write

\(\varphi = (x_1, \ldots, x_n)\)

where each \(x_i : U \to \mathbb{R}\). We call these the local coordinates.

Definition (Dimension). If \(p \in M\), we say \(M\) has dimension \(n\) at \(p\) if for one (thus all) charts \(\varphi : U \to \mathbb{R}^m\) with \(p \in U\), we have \(m = n\). We say \(M\) has dimension \(n\) if it has dimension \(n\) at all points.
1.2 Smooth functions and derivatives

Definition (Smooth function). A function \( f : M \to N \) is smooth at a point \( p \in M \) if there are charts \((U, \varphi)\) for \( M \) and \((V, \xi)\) for \( N \) with \( p \in U \) and \( f(p) \in V \) such that \( \xi \circ f \circ \varphi^{-1} : \varphi(U) \to \xi(V) \) is smooth at \( \varphi(p) \).

A function is smooth if it is smooth at all points \( p \in M \).

A diffeomorphism is a smooth \( f \) with a smooth inverse.

We write \( C^\infty(M, N) \) for the space of smooth maps \( f : M \to N \). We write \( C^\infty(M) \) for \( C^\infty(M, \mathbb{R}) \), and this has the additional structure of an algebra, i.e. a vector space with multiplication.

Definition (Curve). A curve is a smooth map \( I \to M \), where \( I \) is a non-empty open interval.

Definition (Derivation). A derivation on an open subset \( U \subseteq M \) at \( p \in U \) is a linear map \( X : C^\infty(U) \to \mathbb{R} \) satisfying the Leibniz rule
\[
X(fg) = f(p)X(g) + g(p)X(f).
\]

Definition (Tangent space). Let \( p \in U \subseteq M \), where \( U \) is open. The tangent space of \( M \) at \( p \) is the vector space
\[
T_pM = \{ \text{derivations on } U \text{ at } p \} \equiv \text{Der}_p(C^\infty(U)).
\]

The subscript \( p \) tells us the point at which we are taking the tangent space.

Definition (Derivative). Suppose \( F \in C^\infty(M, N) \), say \( F(p) = q \). We define \( DF|_p : T_pM \to T_qN \) by
\[
DF|_p(X)(g) = X(g \circ F)
\]
for \( X \in T_pM \) and \( g \in C^\infty(V) \) with \( q \in V \subseteq N \).

This is a linear map called the derivative of \( F \) at \( p \).

\[
\begin{array}{ccc}
M & \xrightarrow{F} & N \\
\downarrow{g \circ F} & & \downarrow{g} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

Definition (Derivative). Let \( \gamma : \mathbb{R} \to M \) be a smooth function. Then we write
\[
\frac{d\gamma}{dt}(t) = \dot{\gamma}(t) = D\gamma|_1(1).
\]

Definition (\( \frac{\partial}{\partial x_i} \)). Given a chart \( \varphi : U \to \mathbb{R}^n \) with \( \varphi = (x_1, \ldots, x_n) \), we define
\[
\frac{\partial}{\partial x_i}|_p = (D\varphi|_p)^{-1} \left( \frac{\partial}{\partial x_i}|_{\varphi(p)} \right) \in T_pM.
\]

1.3 Bump functions and partitions of unity

Definition (Partition of unity). Let \( \{U_\alpha\} \) be an open cover of a manifold \( M \). A partition of unity subordinate to \( \{U_\alpha\} \) is a collection \( \varphi_\alpha \in C^\infty(M, \mathbb{R}) \) such that
(i) $0 \leq \varphi_\alpha \leq 1$

(ii) $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$

(iii) For all $p \in M$, all but finitely many $\varphi_\alpha(p)$ are zero.

(iv) $\sum_\alpha \varphi_\alpha = 1$.

## 1.4 Submanifolds

**Definition (Embedded submanifold).** Let $M$ be a manifold with $\dim M = n$, and $S$ be a submanifold of $M$. We say $S$ is an *embedded submanifold* if for all $p \in S$, there are coordinates $x_1, \cdots, x_n$ on some chart $U \subseteq M$ containing $p$ such that

$S \cap U = \{x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$

for some $k$. Such coordinates are known as *slice coordinates* for $S$.

**Definition (Immersed submanifold).** Let $S, M$ be manifolds, and $\iota : S \hookrightarrow M$ be a smooth injective map with $D\iota|_p : T_pS \to T_pM$ injective for all $p \in S$. Then we call $(\iota, S)$ an *immersed submanifold*. By abuse of notation, we identify $S$ and $\iota(S)$.

**Definition (Regular value).** Let $F \in C^\infty(M, N)$ and $c \in N$. Let $S = F^{-1}(c)$. We say $c$ is a *regular value* if for all $p \in S$, the map $DF|_p : T_pM \to T_cN$ is surjective.
2 Vector fields

2.1 The tangent bundle

**Definition (Vector field).** A vector field on some $U \subseteq M$ is a smooth map $X : U \rightarrow TM$ such that for all $p \in U$, we have $X(p) \in T_pM$.

In other words, we have $\pi \circ X = \text{id}$.

**Definition (Vect(U)).** Let $\text{Vect}(U)$ denote the set of all vector fields on $U$. Let $X, Y \in \text{Vect}(U)$, and $f \in C^\infty(U)$. Then we can define $(X + Y)(p) = X(p) + Y(p), \quad (fX)(p) = f(p)X(p)$.

Then we have $X + Y, fX \in \text{Vect}(U)$. So $\text{Vect}(U)$ is a $C^\infty(U)$ module.

Moreover, if $V \subseteq U \subseteq M$ and $X \in \text{Vect}(U)$, then $X|_V \in \text{Vect}(V)$.

Conversely, if $\{V_i\}$ is a cover of $U$, and $X_i \in \text{Vect}(V_i)$ are such that they agree on intersections, then they patch together to give an element of $\text{Vect}(U)$. So we say that $\text{Vect}$ is a sheaf of $C^\infty(M)$ modules.

**Definition (Tangent bundle).** Let $M$ be a manifold, and $TM = \bigcup_{p \in M} T_pM$.

There is a natural projection map $\pi : TM \rightarrow M$ sending $v_p \in T_pM$ to $p$.

Let $x_1, \ldots, x_n$ be coordinates on a chart $(U, \varphi)$. Then for any $p \in U$ and $v_p \in T_pM$, there are some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$v_p = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}|_p.$$

This gives a bijection

$$\pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^n$$

$$v_p \mapsto (x_1(p), \ldots, x_n(p), \alpha_1, \ldots, \alpha_n),$$

These charts make $TM$ into a manifold of dimension $2\dim M$, called the tangent bundle of $M$.

**Definition (F-related).** Let $M, N$ be manifolds, and $X \in \text{Vect}(M)$, $Y \in \text{Vect}(N)$ and $F \in C^\infty(M, N)$. We say they are $F$-related if

$$Y_q = DF|_p(X_p)$$

for all $p \in M$ and $F(p) = q$. In other words, if the following diagram commutes:

$$\begin{array}{ccc}
TM & \xrightarrow{DF} & TN \\
X & \downarrow & \uparrow Y \\
M & \xrightarrow{F} & N
\end{array}$$
Definition (Der($C^\infty(M)$)). Let Der($C^\infty(M)$) be the set of all $\mathbb{R}$-linear maps $\mathcal{X} : C^\infty(M) \to C^\infty(M)$ that satisfy

$$\mathcal{X}(fg) = f\mathcal{X}(g) + \mathcal{X}(f)g.$$ 

This is an $\mathbb{R}$-vector space, and in fact a $C^\infty(M)$ module.

Definition (Lie bracket). If $X,Y \in \text{Vect}(M)$, then the Lie bracket $[X,Y]$ is (the vector field corresponding to) the derivation $XY - YX \in \text{Vect}(M)$.

Definition (Lie algebra). A Lie algebra is a vector space $V$ with a bracket $[\cdot, \cdot] : V \times V \to V$ such that

(i) $[\cdot, \cdot]$ is bilinear.

(ii) $[\cdot, \cdot]$ is antisymmetric, i.e. $[X,Y] = -[Y,X]$.

(iii) The Jacobi identity holds

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$ 

2.2 Flows

Definition (Integral curve). Let $X \in \text{Vect}(M)$. An integral curve of $X$ is a smooth $\gamma : I \to M$ such that $I$ is an open interval in $\mathbb{R}$ and

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$ 

Definition (Maximal integral curve). Let $p \in M$, and $X \in \text{Vect}(M)$. Let $I_p$ be the union of all $I$ such that there is an integral curve $\gamma : I \to M$ with $\gamma(0) = p$. Then there exists a unique integral curve $\gamma : I_p \to M$, known as the maximal integral curve.

Definition (Complete vector field). A vector field is complete if $I_p = \mathbb{R}$ for all $p \in M$.

2.3 Lie derivative

Notation. Let $F : M \to M$ be a diffeomorphism, and $g \in C^\infty(M)$. We write

$$F^*g = g \circ F \in C^\infty(M).$$ 

Definition (Lie derivative of a function). Let $X$ be a complete vector field, and $\Theta$ be its flow. We define the Lie derivative of $g$ along $X$ by

$$\mathcal{L}_X(g) = \frac{d}{dt} \bigg|_{t=0} \Theta^*_t g.$$ 

Here this is defined pointwise, i.e. for all $p \in M$, we define

$$\mathcal{L}_X(g)(p) = \frac{d}{dt} \bigg|_{t=0} \Theta^*_t(g)(p).$$
**Notation.** Let $Y \in \text{Vect}(M)$, and $F : M \to M$ be a diffeomorphism. Then $DF^{-1}|_{F(p)} : T_{F(p)}M \to T_pM$. So we can write

$$F^*(Y)|_p = DF^{-1}|_{F(p)}(Y_{F(p)}) \in T_pM.$$ 

Then $F^*(Y) \in \text{Vect}(M)$. If $g \in C^\infty(M)$, then

$$F^*(Y)|_p(g) = Y_{F(p)}(g \circ F^{-1}).$$

Alternatively, we have

$$F^*(Y)|_p(g \circ F) = Y_{F(p)}(g).$$

Removing the $p$'s, we have

$$F^*(Y)(g \circ F) = (Y(g)) \circ F.$$

**Definition** (Lie derivative of a vector field). Let $X \in \text{Vect}(M)$ be complete, and $Y \in \text{Vect}(M)$ be a vector field. Then the **Lie derivative** is given pointwise by

$$\mathcal{L}_X(Y) = \left. \frac{d}{dt} \right|_{t=0} \Theta^*_t(Y).$$
3 Lie groups

Definition (Lie group). A Lie group is a manifold $G$ with a group structure such that multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are smooth maps.

Notation. Let $G$ be a Lie group and $g \in G$. We write $L_g : G \to G$ for the diffeomorphism

$$L_g(h) = gh.$$ 

Definition (Left invariant vector field). Let $X \in \text{Vect}(G)$ be a vector field. This is left invariant if

$$\text{DL}_g|_h(X_h) = X_{gh}$$

for all $g, h \in G$.

We write $\text{Vect}^L(G)$ for the collection of all left invariant vector fields.

Definition (Lie algebra of a Lie group). Let $G$ be a Lie group. The Lie algebra $\mathfrak{g}$ of $G$ is the Lie algebra $T_eG$ whose Lie bracket is induced by that of the isomorphism with $\text{Vect}^L(G)$. So

$$[\xi, \eta] = [X_\xi, X_\eta]|_e.$$ 

We also write $\text{Lie}(G)$ for $\mathfrak{g}$.

Definition (Lie group homomorphisms). Let $G, H$ be Lie groups. A Lie group homomorphism is a smooth map that is also a homomorphism.

Definition (Lie algebra homomorphism). Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. Then a Lie algebra homomorphism is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$ such that

$$\beta[\xi, \eta] = [\beta(\xi), \beta(\eta)]$$

for all $\xi, \eta \in \mathfrak{g}$.

Definition (Exponential map). The exponential map of a Lie group $G$ is $\exp : \mathfrak{g} \to G$ given by

$$\exp(\xi) = \gamma_\xi(1),$$

where $\gamma_\xi$ is the integral curve of $X_\xi$ through $e \in G$.

Definition (Lie subgroup). A Lie subgroup of $G$ is a subgroup $H$ with a smooth structure on $H$ making $H$ an immersed submanifold.
4 Vector bundles

4.1 Tensors

**Definition** (Bilinear map). Let $U, V, W$ be vector spaces. We define $\text{Bilin}(V \times W, U)$ to be the functions $V \times W \to U$ that are bilinear, i.e.

\[
\alpha(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \alpha(v_1, w) + \lambda_2 \alpha(v_2, w)
\]

\[
\alpha(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 \alpha(v, w_1) + \lambda_2 \alpha(v, w_2).
\]

**Definition** (Tensor product). A *tensor product* of two vector spaces $V, W$ is a vector space $V \otimes W$ and a bilinear map $\pi : V \times W \to V \otimes W$ such that any bilinear map $\alpha : V \times W \to U$ can be viewed as a linear map $\tilde{\alpha} : V \otimes W \to U$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V \times W & \xrightarrow{\alpha} & U \\
\downarrow{\pi} & & \\
V \otimes W & \xrightarrow{\alpha} & U
\end{array}
\]

So we have

\[
\text{Bilin}(V \times W, U) \cong \text{Hom}(V \otimes W, U).
\]

Given $v \in V$ and $w \in W$, we obtain $\pi(v, w) \in V \otimes W$, called the *tensor product* of $v$ and $w$.

**Definition** (Covariant tensor). A *covariant tensor* of rank $k$ on $V$ is an element of

\[
\alpha \in V^* \otimes \cdots \otimes V^* \text{, k times},
\]

i.e. $\alpha$ is a multilinear map $V \times \cdots \times V \to \mathbb{R}$.

**Definition** (Tensor). A *tensor* of type $(k, \ell)$ is an element in

\[
T^k_\ell(V) = V^* \otimes \cdots \otimes V^* \otimes V \otimes \cdots \otimes V \text{, k times, \ell times}.
\]

**Definition** (Exterior product). Consider

\[
T(V) = \bigoplus_{k \geq 0} V^\otimes k
\]

as an algebra (with multiplication given by the tensor product) (with $V^\otimes 0 = \mathbb{R}$).

We let $I(V)$ be the ideal (as algebras!) generated by $\{ v \otimes v : v \in V \} \subseteq T(V)$.

We define

\[
\Lambda(V) = T(V)/I(V),
\]

with a projection map $\pi : T(V) \to \Lambda(V)$. This is known as the *exterior algebra*.

We let

\[
\Lambda^k(V) = \pi(V^\otimes k),
\]

the $k$-th *exterior product* of $V$.

We write $a \wedge b$ for $\pi(\alpha \otimes \beta)$. 

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4.2 Vector bundles

**Definition** (Vector bundle). A *vector bundle* of rank $r$ on $M$ is a smooth manifold $E$ with a smooth $\pi : E \to M$ such that

(i) For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an $r$-dimensional vector space,

(ii) For all $p \in M$, there is an open $U \subseteq M$ containing $p$ and a diffeomorphism

$$t : E|_U = \pi^{-1}(U) \to U \times \mathbb{R}^r$$

such that

$$E|_U \xrightarrow{t} U \times \mathbb{R}^r$$

commutes, and the induced map $E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism for all $q \in U$.

We call $t$ a trivialization of $E$ over $U$; call $E$ the total space; call $M$ the base space; and call $\pi$ the projection. Also, for each $q \in M$, the vector space $E_q = \pi^{-1}(\{q\})$ is called the fiber over $q$.

Note that the vector space structure on $E_p$ is part of the data of a vector bundle.

**Definition** (Section). A (smooth) section of a vector bundle $E \to M$ over some open $U \subseteq M$ is a smooth $s : U \to E$ such that $s(p) \in E_p$ for all $p \in U$, that is $\pi \circ s = \text{id}$. We write $C^\infty(U, E)$ for the set of smooth sections of $E$ over $U$.

**Definition** (Transition function). Suppose that $t_\alpha : E|_{U_\alpha} \sim U_\alpha \times \mathbb{R}^r$ and $\tilde{t}_\alpha : \tilde{E}|_{U_\alpha} \sim U_\alpha \times \mathbb{R}^r$ are trivializations of $E$ over $U_\alpha$. Then

$$t_\alpha \circ \tilde{t}_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is fiberwise linear, i.e.

$$t_\alpha \circ \tilde{t}_\alpha^{-1}(q, v) = (q, \varphi_{\alpha\beta}(q)v),$$

where $\varphi_{\alpha\beta}(q)$ is in $\text{GL}_r(\mathbb{R})$.

In fact, $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}_r(\mathbb{R})$ is smooth. Then $\varphi_{\alpha\beta}$ is known as the transition function from $\beta$ to $\alpha$.

**Definition** (Direct sum of vector bundles). Let $E$, $\tilde{E}$ be vector bundles on $M$. Suppose $t_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r$ is a trivialization for $E$ over $U_\alpha$, and $\tilde{t}_\alpha : \tilde{E}|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^{\tilde{r}}$ is a trivialization for $\tilde{E}$ over $U_\alpha$.

We let $\varphi_{\alpha\beta}$ be transition functions for $\{t_\alpha\}$ and $\varphi_{\alpha\beta}$ be transition functions for $\{\tilde{t}_\alpha\}$.

Define

$$E \oplus \tilde{E} = \bigcup p E_p \oplus \tilde{E}_p,$$

and define

$$T_\alpha : (E \oplus \tilde{E})|_{U_\alpha} = E|_{U_\alpha} \oplus \tilde{E}|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^r \oplus \mathbb{R}^{\tilde{r}}) = U_\alpha \times \mathbb{R}^{r+\tilde{r}}$$
be the fiberwise direct sum of the two trivializations. Then $T_\alpha$ clearly gives a linear isomorphism $(E \oplus \tilde{E})_p \cong \mathbb{R}^{r + \tilde{r}}$, and the transition function for $T_\alpha$ is

$$T_\alpha \circ T_\beta^{-1} = \varphi_{\alpha \beta} \oplus \tilde{\varphi}_{\alpha \beta},$$

which is clearly smooth. So this makes $E \oplus \tilde{E}$ into a vector bundle.

**Definition** (Tensor product of vector bundles). Given two vector bundles $E, \tilde{E}$ over $M$, we can construct $E \otimes \tilde{E}$ similarly with fibers $(E \otimes \tilde{E})_p = E_p \otimes \tilde{E}_p$.

**Definition** (Dual vector bundle). Given a vector bundle $E \to M$, we define the dual vector bundle by

$$E^* = \bigcup_{p \in M} (E_p)^*.$$

Suppose again that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ is a local trivialization. Taking the dual of this map gives

$$t^*_\alpha : U_\alpha \times (\mathbb{R}^r)^* \to E^*|_{U_\alpha},$$

since taking the dual reverses the direction of the map. We pick an isomorphism $(\mathbb{R}^r)^* \to \mathbb{R}^r$ once and for all, and then reverse the above isomorphism to get a map

$$E^*|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r.$$

This gives a local trivialization.

**Definition** (Cotangent bundle). The cotangent bundle of a manifold $M$ is

$$T^*M = (TM)^*.$$

In local coordinate charts, we have a frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ of $TM$ over $U$. The dual frame is written as $dx_1, \ldots, dx_n$. In other words, we have

$$dx_i|_p \in (T_pM)^*$$

and

$$dx_i|_p \left( \frac{\partial}{\partial x_j}|_p \right) = \delta_{ij}.$$

**Definition** ($p$-form). A $p$-form on a manifold $M$ over $U$ is a smooth section of $\Lambda^p T^*M$, i.e. an element in $C^\infty(U, \Lambda^p T^*M)$.

**Definition** (Tensors on manifolds). Let $M$ be a manifold. We define

$$T^k_\ell M = \bigotimes_{k \text{ times}} T^*M \otimes \bigotimes_{\ell \text{ times}} TM \otimes \cdots \otimes TM.$$

A tensor of type $(k, \ell)$ is an element of

$$C^\infty(M, T^k_\ell M).$$

The convention when $k = \ell = 0$ is to set $T^0_0 M = M \times \mathbb{R}$.

**Definition** (Riemannian metric). A Riemannian metric on $M$ is a $(2, 0)$-tensor $g$ such that for all $p$, the bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric and positive definite, i.e. an inner product.

Given such a $g$ and $v_p \in T_p M$, we write $\|v_p\|$ for $\sqrt{g_p(v_p, v_p)}$. 

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**Definition (Length of curve).** Let \( \gamma : I \to M \) be a curve. The *length* of \( \gamma \) is
\[
\ell(\gamma) = \int_I \|\dot{\gamma}(t)\| \, dt.
\]

**Definition (Vector bundle morphisms).** Let \( E \to M \) and \( E' \to M' \) be vector bundles. A *bundle morphism* from \( E \) to \( E' \) is a pair of smooth maps \((F : E \to E', f : M \to M')\) such that the following diagram commutes:
\[
\begin{array}{ccc}
E & \xrightarrow{F} & E' \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & M'
\end{array}
\]
i.e. such that \( F_p : E_p \to E'_p \) is linear for each \( p \).

**Definition (Bundle morphism over \( M \)).** Given two bundles \( E, E' \) over the same base \( M \), a *bundle morphism over \( M \) is a bundle morphism \( E \to E' \) of the form \((F, \text{id}_M)\).
5 Differential forms and de Rham cohomology

5.1 Differential forms

Definition (Differential form). We write
\[ \Omega^p(M) = C^\infty(M, \Lambda^p T^* M) = \{ p\text{-forms on } M \}. \]
An element of \( \Omega^p(M) \) is known as a differential \( p \)-form.
In particular, we have
\[ \Omega^0(M) = C^\infty(M, \mathbb{R}). \]

Definition (Non-degenerate form). A 2-form \( \omega \in \Omega^2(M) \) is non-degenerate if
\[ \omega(X_p, X_p) = 0 \] implies \( X_p = 0 \).

Definition (Symplectic form). A symplectic form is a non-degenerate 2-form \( \omega \) such that \( d\omega = 0 \).

Definition (Pullback of differential form). Let \( \omega \in \Omega^p(N) \) and \( F \in C^\infty(M, N) \).
We define the pullback of \( \omega \) along \( F \) to be
\[ F^* \omega |_x = \Lambda^p(DF|_x)^*(\omega|_{F(x)}). \]
In other words, for \( v_1, \cdots, v_p \in T_x M \), we have
\[ (F^* \omega |_x)(v_1, \cdots, v_p) = \omega|_{F(x)}(DF|_x(v_1), \cdots, DF|_x(v_p)). \]

5.2 De Rham cohomology

Definition (Closed form). A \( p \)-form \( \omega \in \Omega^p(M) \) is closed if \( d\omega = 0 \).

Definition (Exact form). A \( p \)-form \( \omega \in \Omega^p(M) \) is exact if there is some \( \sigma \in \Omega^{p-1}(M) \) such that \( \omega = d\sigma \).

Definition (de Rham cohomology). The \( p \)th de Rham cohomology is given by the \( \mathbb{R} \)-vector space
\[ H^p_{\text{dR}}(M) = \frac{\ker d: \Omega^p(M) \to \Omega^{p+1}(M)}{\text{im } d: \Omega^{p-1}(M) \to \Omega^p(M)} = \frac{\text{closed forms}}{\text{exact forms}}. \]
In particular, we have
\[ H^0_{\text{dR}}(M) = \ker d: \Omega^0(M) \to \Omega^1(M). \]

Definition (Smooth homotopy). Let \( F_0, F_1 : M \to N \) be smooth maps. A smooth homotopy from \( F_0 \) to \( F_1 \) is a smooth map \( F : [0, 1] \times M \to N \) such that
\[ F_0(x) = F(0, x), \quad F_1(x) = F(1, x). \]
If such a map exists, we say \( F_0 \) and \( F_1 \) are homotopic.

Definition (Smooth homotopy equivalence). We say two manifolds \( M, N \) are smoothly homotopy equivalent if there are smooth maps \( F : M \to N \) and \( G : N \to M \) such that both \( F \circ G \) and \( G \circ F \) are homotopic to the identity.
5 Differential forms and de Rham cohomology

III Differential Geometry (Definitions)

5.3 Homological algebra and Mayer-Vietoris theorem

**Definition (Cochain complex and exact sequence).** A sequence of vector spaces and linear maps

$$\cdots \rightarrow V_{p-1} \xrightarrow{d_{p-1}} V_p \xrightarrow{d_p} V_{p+1} \xrightarrow{d_{p+1}} \cdots$$

is a **cochain complex** if \( d_p \circ d_{p-1} = 0 \) for all \( p \in \mathbb{Z} \). Usually we have \( V^p = 0 \) for \( p < 0 \) and we do not write them out. Keeping these negative degree \( V^p \) rather than throwing them away completely helps us state our theorems more nicely, so that we don’t have to view \( V^0 \) as a special case when we state our theorems.

It is **exact at** \( p \) if \( \ker d_p = \text{im} d_{p-1} \), and **exact** if it is exact at every \( p \).

**Definition (Cohomology).** Let

$$\cdots \rightarrow V_{p-1} \xrightarrow{d_{p-1}} V_p \xrightarrow{d_p} V_{p+1} \xrightarrow{d_{p+1}} \cdots$$

be a cochain complex. The **cohomology** of \( V^\ast \) at \( p \) is given by

$$H^p(V^\ast) = \frac{\ker d_p}{\text{im} d_{p-1}}.$$

**Definition (Cochain map).** Let \( V^\ast \) and \( W^\ast \) be cochain complexes. A **cochain map** \( V^\ast \rightarrow W^\ast \) is a collection of maps \( f^p : V^p \rightarrow W^p \) such that the following diagram commutes for all \( p \):

\[
\begin{array}{ccc}
V^p & \xrightarrow{f^p} & W^p \\
\downarrow{d_p} & & \downarrow{d_p} \\
V^{p+1} & \xrightarrow{f^{p+1}} & W^{p+1}
\end{array}
\]

**Definition (Short exact sequence).** A **short exact sequence** is an exact sequence of the form

$$0 \rightarrow V^1 \xrightarrow{\alpha} V^2 \xrightarrow{\beta} V^3 \rightarrow 0.$$

This implies that \( \alpha \) is injective, \( \beta \) is surjective, and \( \text{im}(\alpha) = \ker(\beta) \). By the rank-nullity theorem, we know

$$\dim V^2 = \text{rank}(\beta) + \text{null}(\beta) = \dim V^3 + \dim V^1.$$
6 Integration

6.1 Orientation

Definition (Orientation of vector space). Let $V$ be a vector space with $\dim V = n$. An orientation is an equivalence class of elements $\omega \in \Lambda^n(V^*)$, where we say $\omega \sim \omega'$ iff $\omega = \lambda \omega'$ for some $\lambda > 0$. A basis $(e_1, \cdots, e_n)$ is oriented if

$$\omega(e_1, \cdots, e_n) > 0.$$ 

By convention, if $V = \{0\}$, an orientation is just a choice of number in $\{\pm 1\}$.

Definition (Orientation of a manifold). An orientation of a manifold $M$ is defined to be an equivalence class of elements $\omega \in \Omega^n(M)$ that are nowhere vanishing, under the equivalence relation $\omega \sim \omega'$ if there is some smooth $f : M \to \mathbb{R}_{>0}$ such that $\omega = f \omega'$.

Definition (Orientable manifold). A manifold is orientable if it has some orientation.

Definition (Oriented manifold). An oriented manifold is a manifold with a choice of orientation.

Definition (Oriented coordinates). Let $M$ be an oriented manifold. We say coordinates $x_1, \cdots, x_n$ on a chart $U$ are oriented coordinates if

$$\frac{\partial}{\partial x_1} \bigg|_p, \cdots, \frac{\partial}{\partial x_n} \bigg|_p$$

is an oriented basis for $T_pM$ for all $p \in U$.

Definition (Orientation-preserving diffeomorphism). Let $M, N$ be oriented manifolds, and $F \in C^\infty(M, N)$ be a diffeomorphism. We say $F$ preserves orientation if $DF|_p : T_pM \to T_{F(p)}N$ takes an oriented basis to an oriented basis.

Alternatively, this says the pullback of the orientation on $N$ is the orientation on $M$ (up to equivalence).

6.2 Integration

Definition (Domain of integration). Let $D \subseteq \mathbb{R}^n$. We say $D$ is a domain of integration if $D$ is bounded and $\partial D$ has measure zero.

Definition (Integration on $\mathbb{R}^n$). Let $D$ be a compact domain of integration, and

$$\omega = f \, dx_1 \wedge \cdots \wedge dx_n$$

be an $n$-form on $D$. Then we define

$$\int_D \omega = \int_D f(x_1, \cdots, x_n) \, dx_1 \cdots dx_n.$$ 

In general, let $U \subseteq \mathbb{R}^n$ and let $\omega \in \Omega^n(\mathbb{R}^n)$ have compact support. We define

$$\int_U \omega = \int_D \omega$$

for some $D \subseteq U$ containing $\text{supp} \omega$. 

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6 Integration

III Differential Geometry (Definitions)

Definition (Smooth function). Let $D \subseteq \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$. We say $f$ is smooth if it is a restriction of some smooth function $\tilde{f} : U \to \mathbb{R}^m$ where $U \supseteq D$.

Definition (Integration on manifolds). Let $M$ be an oriented manifold. Let $\omega \in \Omega^n(M)$. Suppose that $\text{supp}(\omega)$ is a compact subset of some oriented chart $(U, \varphi)$. We set

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$ 

By the previous lemma, this does not depend on the oriented chart $(U, \varphi)$.

If $\omega \in \Omega^n(M)$ is a general form with compact support, we do the following: cover the support by finitely many oriented charts $\{U_\alpha\}_{\alpha = 1, \ldots, m}$. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. We then set

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \chi_\alpha \omega.$$ 

Definition (Parametrization). Let $M$ be either an oriented manifold of dimension $n$, or a domain of integration in $\mathbb{R}^n$. By a parametrization of $M$ we mean a decomposition

$$M = S_1 \cup \cdots \cup S_n,$$

with smooth maps $F_i : D_i \to S_i$, where $D_i$ is a compact domain of integration, such that

(i) $F_i|_{\hat{D}_i} : \hat{D}_i \to \hat{S}_i$ is an orientation-preserving diffeomorphism

(ii) $\partial S_i$ has measure zero (if $M$ is a manifold, this means $\varphi(\partial S_i \cap U)$ for all charts $(U, \varphi)$).

(iii) For $i \neq j$, $S_i$ intersects $S_j$ only in their common boundary.

6.3 Stokes Theorem

Definition (Manifold with boundary). Let

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$ 

A chart-with-boundary on a set $M$ is a bijection $\varphi : U \to \varphi(U)$ for some $U \subseteq M$ such that $\varphi(U) \subseteq \mathbb{H}^n$ is open. Note that this image may or may not hit the boundary of $\mathbb{H}^n$. So a “normal” chart is also a chart with boundary.

An atlas-with-boundary on $M$ is a cover by charts-with-boundary $(U_\alpha, \varphi_\alpha)$ such that the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth (in the usual sense) for all $\alpha, \beta$.

A manifold-with-boundary is a set $M$ with an (equivalence class of) atlas with boundary whose induced topology is Hausdorff and second-countable.

Definition (Boundary point). If $M$ is a manifold with boundary and $p \in M$, then we say $p$ is a boundary point if $\varphi(p) \in \partial \mathbb{H}^n$ for some (hence any) chart-with-boundary $(U, \varphi)$ containing $p$. We let $\partial M$ be the set of boundary points and $\text{Int}(M) = M \setminus \partial M$.  

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**Definition** (Outward/Inward pointing). Let $p \in \partial M$. We then have an inclusion $T_p \partial M \subseteq T_p M$. If $X_p \in T_p M$, then in a chart, we can write

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where $a_i \in \mathbb{R}$ and $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}$ are a basis for $T_p \partial M$. We say $X_p$ is *outward pointing* if $a_n < 0$, and *inward pointing* if $a_n > 0$.

**Definition** (Induced orientation). Let $M$ be an oriented manifold with boundary. We say a basis $e_1, \cdots, e_{n-1}$ is an oriented basis for $T_p \partial M$ if $(X_p, e_1, \cdots, e_{n-1})$ is an oriented basis for $T_p M$, where $X_p$ is any outward pointing element in $T_p M$. This orientation is known as the *induced orientation*. 
7 De Rham’s theorem*

Definition (Singular $p$-complex). Let $M$ be a manifold. Then a singular $p$-simplex is a continuous map\[
s: \Delta_p \to M,\]
where\[
\Delta_p = \left\{ \sum_{i=0}^{p} t_i e_i : \sum t_i = 1 \right\} \subseteq \mathbb{R}^{n+1}.
\]
We define\[
C_p(M) = \{ \text{formal sums } \sum a_i \sigma_i : a_i \in \mathbb{R}, \sigma_i \text{ a singular } p \text{ simplex} \}.
\]
We define\[
C_p^\infty (m) = \{ \text{formal sums } \sum a_i \sigma_i : a_i \in \mathbb{R}, \sigma_i \text{ a smooth singular } p \text{ simplex} \}.
\]
Definition (Boundary map). The boundary map\[
\partial: C_p(M) \to C_{p-1}(M)
\]
is the linear map such that if $\sigma: \Delta_p \to M$ is a $p$ simplex, then\[
\partial \sigma = \sum (-1)^i \sigma \circ F_{i,p},
\]
where $F_{i,p}$ maps $\Delta_{p-1}$ affine linearly to the face of $\Delta_p$ opposite the $i$th vertex. We similarly have\[
\partial: C_p^\infty (M) \to C_{p-1}^\infty (M).
\]
Definition (Singular homology). The singular homology of $M$ is\[
H_p(M, \mathbb{R}) = \frac{\ker \partial: C_p(M) \to C_{p-1}(M)}{\operatorname{im} \partial: C_{p+1}(M) \to C_p(M)}.
\]
The smooth singular homology is the same thing with $C_p(M)$ replaced with $C_p^\infty (M)$.

Definition (Singular cohomology). The singular cohomology of $M$ is defined as\[
H^p(M, \mathbb{R}) = \operatorname{Hom}(H_p(M, \mathbb{R}), \mathbb{R}).
\]
Similarly, the smooth singular cohomology is\[
H^p_\infty (M, \mathbb{R}) = \operatorname{Hom}(H^\infty_p (M, \mathbb{R}), \mathbb{R}).
\]
Definition (de Rham).\begin{enumerate}
\item We say a manifold $M$ is de Rham if $I$ is an isomorphism.
\item We say an open cover $\{U_\alpha\}$ of $M$ is de Rham if $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ is de Rham for all $\alpha_1, \cdots, \alpha_p$.
\item A de Rham basis is a de Rham cover that is a basis for the topology on $M$.
\end{enumerate}
8 Connections

8.1 Basic properties of connections

Notation. Let $E$ be a vector bundle on $M$. Then we write
\[ \Omega^p(E) = \Omega^p(E \otimes \Lambda^p(T^*M)). \]
So an element in $\Omega^p(E)$ takes in $p$ tangent vectors and outputs a vector in $E$.

Definition (Connection). Let $E$ be a vector bundle on $M$. A \textit{connection} on $E$ is a linear map
\[ d_E : \Omega^0(E) \to \Omega^1(E) \]
such that
\[ d_E(fs) = df \otimes s + f d_E s \]
for all $f \in C^\infty(M)$ and $s \in \Omega^0(E)$.

A connection on $TM$ is called a \textit{linear} or \textit{Koszul connection}.

Definition (Induced connection on tensor product). Let $E, F$ be vector bundles with connections $d_E, d_F$ respectively. The \textit{induced connection} is the connection $d_{E \otimes F}$ on $E \otimes F$ given by
\[ d_{E \otimes F}(s \otimes t) = d_E s \otimes t + s \otimes d_F t \]
for $s \in \Omega^0(E)$ and $t \in \Omega^0(F)$, and then extending linearly.

Definition (Induced connection on dual bundle). Let $E$ be a vector bundle with connection $d_E$. Then there is an induced connection $d_{E^*}$ on $E^*$ given by requiring
\[ d\langle s, \xi \rangle = \langle d_E s, \xi \rangle + \langle s, d_{E^*} \xi \rangle, \]
for $s \in \Omega^0(E)$ and $\xi \in \Omega^0(E^*)$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing $\Omega^0(E) \times \Omega^0(E^*) \to C^\infty(M, \mathbb{R})$.

8.2 Geodesics and parallel transport

Definition (Geodesic). Let $M$ be a manifold with a linear connection $\nabla$. We say that $\gamma : I \to M$ is a \textit{geodesic} if
\[ D_t \dot{\gamma}(t) = 0. \]

Definition (Parallel vector field). Let $\nabla$ be a linear connection on $M$, and $\gamma : I \to M$ be a path. We say a vector field $V \in J(\gamma)$ along $\gamma$ is \textit{parallel} if $D_t V(t) \equiv 0$ for all $t \in I$.

Definition (Parallel transport). Let $\gamma : I \to M$ be a curve. For $t_0, t_1$, we define the \textit{parallel transport map}
\[ P_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M \]
given by $\xi \mapsto V_\xi(t_1)$. 

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8.3 Riemannian connections

**Definition** (Metric connection). A linear connection $\nabla$ is compatible with $g$ (or is a metric connection) if for all $X, Y, Z \in \text{Vect}(M)$, 

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Note that the first term is just $X(g(Y, Z))$.

**Definition** (Torsion of linear connection). Let $\nabla$ be a linear connection on $M$. The torsion of $\nabla$ is defined by 

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for $X, Y \in \text{Vect}(M)$.

**Definition** (Symmetric/torsion free connection). A linear connection is symmetric or torsion-free if $\tau(X,Y) = 0$ for all $X, Y$.

**Definition** (Riemannian/Levi-Civita connection). The unique torsion-free metric connection on a Riemannian manifold is called the Riemannian connection or Levi-Civita connection.

8.4 Curvature

**Definition** (Parallel vector field). We say a vector field $V \in \text{Vect}(M)$ is parallel if $V$ is parallel along any curve in $M$.

**Definition** (Curvature). The curvature of a connection $d_E : \Omega^0(E) \to \Omega^1(E)$ is the map 

$$F_E = d_E \circ d_E : \Omega^0(E) \to \Omega^2(E).$$

**Definition** (Flat connection). A connection $d_E$ is flat if $F_E = 0$.

**Definition** (Curvature of metric). Let $(M, g)$ be a Riemannian manifold with metric $g$. The curvature of $g$ is the curvature of the Levi-Civita connection, denoted by 

$$F_g \in \Omega^2(\text{End}(TM)) = \Omega^0(\Lambda^2 T^* M \otimes TM \otimes T^* M).$$

**Definition** (Flat metric). A Riemannian manifold $(M, g)$ is flat if $F_g = 0$.

**Definition** (Isometry). Let $(M, g)$ and $(N, g')$ be Riemannian manifolds. We say $G \in C^\infty(M, N)$ is an isometry if $G$ is a diffeomorphism and $G^* g' = g$, i.e. 

$$DG|_p : T_p M \to T_{G(p)} N$$

is an isometry for all $p \in M$.

**Definition** (Locally isometric). A manifold $M$ is locally isometric to $N$ if for all $p \in M$, there is a neighbourhood $U$ of $p$ and a $V \subseteq N$ and an isometry $G : U \to V$. 

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**Definition (Holonomy).** Consider a piecewise smooth curve \( \gamma : [0, 1] \to M \) with \( \gamma(0) = \gamma(1) = p \). Say we have a linear connection \( \nabla \). Then we have a notion of parallel transport along \( \gamma \).

The *holonomy* of \( \nabla \) around \( \gamma \) is the map

\[
H : T_p M \to T_p M
\]

given by

\[
H(\xi) = V(1),
\]

where \( V \) is the parallel transport of \( \xi \) along \( \gamma \).