Part III — Analysis of Partial Differential Equations
Theorems with proof

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This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to modern functional analytic techniques, relying on a priori estimates, rather than explicit solutions, although the interaction with classical methods (such as the fundamental solution and Fourier representation) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. Some non-linear equations may also be discussed. The course will end with a discussion of major open problems in PDEs.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

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2.1 The Cauchy–Kovalevskaya theorem

Theorem (Picard–Lindelöf theorem)**.** Suppose that there exists $r, K > 0$ such that $B_r(u_0) \subseteq U$, and

$$\|f(x) - f(y)\| \leq K\|x - u_0\|$$

for all $x, y \in B_r(u_0)$. Then there exists an $\varepsilon > 0$ depending on K, r and a unique C^1 function $u : (-\varepsilon, \varepsilon) \to U$ solving the Cauchy problem.

Proof sketch. If u is a solution, then by the fundamental theorem of calculus, we have

$$u(t) = u_0 + \int_0^t f(u(s))\;\mathrm{d}s.$$

Conversely, if u is a C^0 solution to this integral equation, then it solves the ODE. Crucially, this only requires u to be C^0. Indeed, if u is C^0 and satisfies the integral equation, then u is automatically C^1. So we can work in a larger function space when we seek for u.

Thus, we have reformulated our initial problem into an integral equation. In particular, we reformulated it in a way that assumes less about the function. In the case of PDEs, this is what is known as a *weak formulation*.

Returning to the proof, we have reformulated our problem as looking for a fixed point of the map

$$B : w \mapsto u_0 + \int_0^t f(w(s))\;\mathrm{d}s$$

acting on

$$\mathcal{C} = \{w : [-\varepsilon, \varepsilon] \to \overline{B_{r/2}(u_0)} : w \text{ is continuous}\}.$$

This is a complete metric space when we equip it with the supremum norm (in fact, it is a closed ball in a Banach space).

We then show that for ε small enough, this map $B : \mathcal{C} \to \mathcal{C}$ is a contraction map. There are two parts — to show that it actually lands in $\mathcal{C}$, and that it is a contraction. If we managed to show these, then by the contraction mapping theorem, there is a unique fixed point, and we are done. $\square$

Theorem (Cauchy–Kovalevskaya for ODEs)**.** The series

$$u(t) = \sum_{k=0}^{\infty} u_k \frac{t^k}{k!}.$$

converges to the Picard–Lindelöf solution of the Cauchy problem if f is real analytic in a neighbourhood of u_0.

Lemma.

(i) If $g \gg f$ and g converges for $|x| < r$, then f converges for $|x| < r$.

(ii) If $f(x) = \sum_\alpha f_\alpha x^\alpha$ converges for $x < r$ and $0 < s\sqrt{n} < r$, then f has a majorant which converges on $|x| < s$.

Proof.

(i) Given x, define $\tilde{x} = (|x_1|, |x_2|, \ldots, |x_n|)$. We then note that

$$\sum_\alpha |f_\alpha x^\alpha| = \sum_\alpha |f_\alpha| \tilde{x}^\alpha \leq \sum_\alpha g_\alpha \tilde{x}^\alpha = g(\tilde{x}).$$

Since $|\tilde{x}| = |x| < r$, we know g converges at $\tilde{x}$.

(ii) Let $0 < s\sqrt{n} < r$ and set $y = s(1, 1, \ldots, 1)$. Then we have

$$|y| = s\sqrt{n} < r.$$

So by assumption, we know

$$\sum_\alpha f_\alpha y^\alpha$$

converges. A convergent series has bounded terms, so there exists C such that

$$|f_\alpha y^\alpha| \leq C$$

for all α. But $y^\alpha = s^{|\alpha|}$. So we know

$$|f_\alpha| \leq \frac{C}{s^{|\alpha|}} \leq \frac{C}{s^{|\alpha|}} \frac{|\alpha|!}{\alpha!}.$$

But then if we set

$$g(x) = \frac{Cs}{s - (x_1 + \cdots + x_n)} = C \sum_\alpha \frac{|\alpha|!}{s^{|\alpha|} \alpha!} x^\alpha,$$

we are done, since this converges for $|x| < \frac{s}{\sqrt{n}}$. □

Theorem (Cauchy–Kovalevskaya theorem)**.** Given the above assumptions, there exists a real analytic function $\mathbf{u} = \sum_\alpha \mathbf{u}_\alpha x^\alpha$ solving the PDE in a neighbourhood of the origin. Moreover, it is unique among real analytic functions.

Lemma. For $k = 1, \ldots, m$ and α a multi-index in $\mathbb{N}^n$, there exists a polynomial q_α^k in the power series coefficients of B and $\mathbf{c}$ such that any analytic solution to the PDE must be given by

$$\mathbf{u} = \sum_\alpha \mathbf{q}_\alpha(B, \mathbf{c}) x^\alpha,$$

where $\mathbf{q}_\alpha$ is the vector with entries q_α^k.

Moreover, all coefficients of q_α are non-negative.

Proof. We construct the polynomials q_α^k by induction on α_n. If $\alpha_n = 0$, then since $\mathbf{u} = 0$ on $\{x_n = 0\}$, we conclude that we must have

$$u_\alpha = \frac{\mathrm{D}^\alpha \mathbf{u}(0)}{\alpha!} = 0.$$

For $\alpha_n = 1$, we note that whenever $x^n = 0$, we have $\mathbf{u}_{x_j} = 0$ for $j = 1, \ldots, n-1$. So the PDE reads

$$\mathbf{u}_{x_n}(x', 0) = \mathbf{c}(0, x').$$

Differentiating this relation in directions tangent to $x_n = 0$, we find that if $\alpha = (\alpha', 1)$, then

$$\mathrm{D}^\alpha \mathbf{u}(0) = \mathrm{D}^{\alpha'} \mathbf{c}(0,0).$$

So q^k_α is a polynomial in the power series coefficients of $\mathbf{c}$, and has non-negative coefficients.

Now suppose $\alpha_n = 2$, so that $\alpha = (\alpha', 2)$. Then

$$\begin{aligned}
\mathrm{D}^\alpha \mathbf{u} &= \mathrm{D}^{\alpha'} (\mathbf{u}_{x^n})_{x^n}\\
&= \mathrm{D}^{\alpha'} \left(\sum_j B_j \mathbf{u}_{x^j} + \mathbf{c}\right)_{x^n}\\
&= \mathrm{D}^{\alpha'} \left(\sum_j \left(B_j \mathbf{u}_{x^j, x^n} + \sum_p \left(B_{u_p} \mathbf{u}_{x^j}\right) u^p_{x^n}\right) + \sum_p \mathbf{c}_{u_p} u^p_{x^n}\right)
\end{aligned}$$

We don't really care what this looks like. The point is that when we evaluate at 0, and expand all the terms out, we get a polynomial in the derivatives of B_j and $\mathbf{c}$, and also $\mathrm{D}^\beta \mathbf{u}$ with $\beta_n < 2$. The derivatives of B_j and $\mathbf{c}$ are just the coefficients of the power series expansion of B_j and $\mathbf{c}$, and by the induction hypothesis, we can also express the $\mathrm{D}^\beta \mathbf{u}$ in terms of these power series coefficients. Thus, we can use this to construct $\mathbf{q}_\alpha$. By inspecting what the formula looks like, we see that all coefficients in $\mathbf{q}_\alpha$ are non-negative.

We see that we can continue doing the same computations to obtain all $\mathbf{q}_\alpha$. □

Lemma. If $\tilde{B}_j \gg B_j$ and $\tilde{\mathbf{c}} \gg \mathbf{c}$, then

$$q^k_\alpha(\tilde{B}, \tilde{\mathbf{c}}) > q^k_\alpha(B, \mathbf{c}).$$

for all α. In particular, $\tilde{\mathbf{u}} \gg \mathbf{u}$.

Lemma. For any C and r, define

$$h(z, x') = \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)}$$

If B and $\mathbf{c}$ are given by

$$B_j^*(z, x') = h(z, x') \begin{pmatrix} 1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{pmatrix},\quad \mathbf{c}^*(z, x') = h(z, x') \begin{pmatrix} 1\\ \vdots\\ 1\end{pmatrix},$$

then the power series

$$\mathbf{u} = \sum_\alpha \mathbf{q}_\alpha(B, \mathbf{c}) x^\alpha$$

converges in a neighbourhood of the origin.

Proof. We define

$$v(x) = \frac{1}{mn}\left(r - (x^1 + \cdots + x^{n-1}) - \sqrt{(r - (x^1 + \cdots + x^{n-1}))^2 - 2mnCrx^n}\right),$$

which is real analytic around the origin, and vanishes when $x^n = 0$. We then observe that

$$\mathbf{u}(x) = v(x) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

gives a solution to the corresponding PDE, and is real analytic around the origin. Hence it must be given by that power series, and in particular, the power series must converge. □

2.2 Reduction to first-order systems

3 Function spaces

3.1 The Hölder spaces

3.2 Sobolev spaces

Theorem. $L^P(U)$ is a Banach space with the L^p norm. □

Lemma. Suppose $v, \tilde{v} \in L^1_{loc}(U)$ are both αth weak derivatives of $u \in L^1_{loc}(U)$, then $v = \tilde{v}$ almost everywhere.

Proof. For any $\phi \in C_c^\infty(U)$, we have

$$\int_U (v - \tilde{v})\phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_U (u - u)\mathrm{D}^\alpha \phi \, \mathrm{d}x = 0.$$

Therefore $v - \tilde{v} = 0$ almost everywhere. □

Theorem. For each $k = 0, 1, \ldots$ and $1 \le p \le \infty$, the space $W^{k,p}(U)$ is a Banach space.

Proof. Homogeneity and positivity for the Sobolev norm are clear. The triangle inequality follows from the Minkowski inequality.

For completeness, note that

$$\|\mathrm{D}^\alpha u\|_{L^p(U)} \le \|u\|_{W^{k,p}(U)}$$

for $|\alpha| \le k$.

So if $(u_i)_{i=1}^\infty$ is Cauchy in $W^{k,p}(U)$, then $(\mathrm{D}^\alpha u_i)_{i=1}^\infty$ is Cauchy in $L^p(U)$ for $|\alpha| \le k$. So by completeness of $L^p(U)$, we have

$$\mathrm{D}^\alpha u_i \to u^\alpha \in L^p(U)$$

for some u^α. It remains to show that $u^\alpha = \mathrm{D}^\alpha u$, where $u = u^{(0,0,\ldots,0)}$. Let $\phi \in C_c^\infty(U)$. Then we have

$$(-1)^{|\alpha|} \int_U u_j \mathrm{D}^\alpha \phi \, \mathrm{d}x = \int_U \mathrm{D}^\alpha u_j \phi \, \mathrm{d}x$$

for all j. We send $j \to \infty$. Then using $\mathrm{D}^\alpha u_j \to u^\alpha$ in $L^p(U)$, we have

$$(-1)^{|\alpha|} \int_U u \mathrm{D}^\alpha \phi \, \mathrm{d}x = \int_U u^\alpha \phi \, \mathrm{d}x.$$

So $\mathrm{D}^\alpha u = u^\alpha \in L^p(U)$ and we are done. □

3.3 Approximation of functions in Sobolev spaces

Theorem. Let $f \in L^1_{loc}(U)$. Then

(i) $f_\varepsilon \in C^\infty(U_\varepsilon)$.

(ii) $f_\varepsilon \to f$ almost everywhere as $\varepsilon \to 0$.

(iii) If in fact $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$, then $f_\varepsilon \to f$ in $L^p_{loc}(U)$, i.e. we have convergence in L^p on any $V \Subset U$. □

Lemma. Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set

$$u_\varepsilon = \eta_\varepsilon * u \text{ on } U_\varepsilon.$$

Then

(i) $u_\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$

(ii) If $V \Subset U$, then $u_\varepsilon \to u$ in $W^{k,p}(V)$.

Proof.

(i) As above.

(ii) We claim that

$$\mathrm{D}^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u$$

for $|\alpha| \leq k$ in U_ε.

To see this, we have

$$\begin{aligned} \mathrm{D}^\alpha u^\varepsilon(x) &= \mathrm{D}^\alpha \int_U \eta_\varepsilon(x-y)u(y)\;\mathrm{d}y \\ &= \int_U \mathrm{D}^\alpha_x \eta_\varepsilon(x-y)u(y)\;\mathrm{d}y \\ &= \int_U (-1)^{|\alpha|}\mathrm{D}^\alpha_y \eta_\varepsilon(x-y)u(y)\;\mathrm{d}y \end{aligned}$$

For a fixed $x \in U_\varepsilon$, $\eta_\varepsilon(x - \cdot\,) \in C^\infty_c(U)$, so by the definition of a weak derivative, this is equal to

$$\begin{aligned} &= \int_U \eta_\varepsilon(x-y)\mathrm{D}^\alpha u(y)\;\mathrm{d}y \\ &= \eta_\varepsilon * \mathrm{D}^\alpha u. \end{aligned}$$

It is an exercise to verify that we can indeed move the derivative past the integral.

Thus, if we fix $V \Subset U$. Then by the previous parts, we see that $\mathrm{D}^\alpha u_\varepsilon \to \mathrm{D}^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$ for $|\alpha| \leq k$. So

$$\|u_\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha u_\varepsilon - \mathrm{D}^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \to 0$. □

Theorem (Global approximation)**.** Let $1 \leq p < \infty$, and $U \subseteq \mathbb{R}^n$ be open and bounded. Then $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

Proof. For $i \geq 1$, define

$$\begin{aligned} U_i &= \left\{x \in U \mid \mathrm{dist}(x, \partial U) > \tfrac{1}{i}\right\} \\ V_i &= U_{i+3} - \bar{U}_{i+1} \\ W_i &= U_{i+4} - \bar{U}_i. \end{aligned}$$

We clearly have $U = \bigcup_{i=1}^\infty U_i$, and we can choose $V_0 \Subset U$ such that $U = \bigcup_{i=0}^\infty V_i$.

Let $\{\zeta_i\}_{i=0}^\infty$ be a partition of unity subordinate to $\{V_i\}$. Thus, we have $0 \leq \zeta_i \leq 1$, $\zeta_i \in C_c^\infty(V_i)$ and $\sum_{i=0}^\infty \zeta_i = 1$ on U.

Fix $\delta > 0$. Then for each i, we can choose ε_i sufficiently small such that

$$u^i = \eta_{\varepsilon_i} * \zeta_i u$$

satisfies $\operatorname{supp} u_i \subseteq W_i$ and

$$\|u_i - \zeta_i u\|_{W^{k,p}(U)} = \|u_i - \zeta_i u\|_{W^{k,p}(W_i)} \leq \frac{\delta}{2^{i+1}}.$$

Now set

$$v = \sum_{i=0}^\infty u_i \in C^\infty(U).$$

Note that we do not know (yet) that $v \in W^{k,p}(U)$. But it certainly is when we restrict to some $V \Subset U$.

In any such subset, the sum is finite, and since $u = \sum_{i=0}^\infty \zeta_i u$, we have

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{i=0}^\infty \|u^i - \zeta_i u\|_{W^{k,p}(V)} \leq \delta \sum_{i=0}^\infty 2^{-(i+1)} = \delta.$$

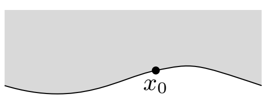
Since the bound δ does not depend on V, by taking the supremum over all V, we have

$$\|v - u\|_{W^{k,p}(U)} \leq \delta.$$

So we are done. □

Theorem (Smooth approximation up to boundary)**.** Let $1 \leq p < \infty$, and $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose ∂U is $C^{0,1}$. Then $C^\infty(\bar{U}) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

Proof. Previously, the reason we didn't get something in $C^\infty(\bar{U})$ was that we had to glue together infinitely many mollifications whose domain collectively exhaust U, and there is no hope that the resulting function is in $C^\infty(\bar{U})$. In the current scenario, we know that U locally looks like



The idea is that given a u defined on U, we can shift it downwards by some ε. It is a known result that translation is continuous, so this only changes u by a tiny bit. We can then mollify with a $\bar{\varepsilon} < \varepsilon$, which would then give a function defined on U (at least locally near x_0).

So fix some $x_0 \in \partial U$. Since ∂U is $C^{0,1}$, there exists $r > 0$ such that $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$ such that

$$U \cap B_r(x_0) = \{(x', x_n) \in B_r(x') \mid x_n > \gamma(x')\}.$$

Set

$$V = U \cap B_{r/2}(x^0).$$

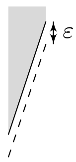
Define the shifted function u_ε to be

$$u_\varepsilon(x) = u(x + \varepsilon e_n).$$

Now pick $\bar{\varepsilon}$ sufficiently small such that

$$v^{\varepsilon, \bar{\varepsilon}} = \eta_{\bar{\varepsilon}} * u_\varepsilon$$

is well-defined. Note that here we need to use the fact that ∂U is $C^{0,1}$. Indeed, we can see that if the slope of ∂U is very steep near a point x:



then we need to choose a $\bar{\varepsilon}$ much smaller than ε. By requiring that γ is 1-Hölder continuous, we can ensure there is a single choice of $\bar{\varepsilon}$ that works throughout V. As long as $\bar{\varepsilon}$ is small enough, we know that $v^{\varepsilon, \bar{\varepsilon}} \in C^\infty(\bar{V})$.

Fix $\delta > 0$. We can now estimate

$$\begin{aligned}
\|v^{\varepsilon, \tilde{\varepsilon}} - u\|_{W^{k.p}(V)} &= \|v^{\varepsilon, \tilde{\varepsilon}} - u_\varepsilon + u_\varepsilon - u\|_{W^{k,p}(V)} \\
&\leq \|v^{\varepsilon, \tilde{\varepsilon}} - u_\varepsilon\|_{W^{k,p}(V)} + \|u_\varepsilon - u\|_{W^{k.p}(V)}.
\end{aligned}$$

Since translation is continuous in the L^p norm for $p < \infty$, we can pick $\varepsilon > 0$ such that $\|u_\varepsilon - u\|_{W^{k.p}(V)} < \frac{\delta}{2}$. Having fixed such an ε, we can pick $\tilde{\varepsilon}$ so small that we also have $\|v^{\varepsilon, \tilde{\varepsilon}} - u_\varepsilon\|_{W^{k.p}(V)} < \frac{\delta}{2}$.

The conclusion of this is that for any $x_0 \in \partial U$, we can find a neighbourhood $V \subseteq U$ of x_0 in U such that for any $u \in W^{k,p}(U)$ and $\delta > 0$, there exists $v \in C^\infty(\bar{V})$ such that $\|u - v\|_{W^{k,p}(V)} \leq \delta$.

It remains to patch all of these together using a partition of unity. By the compactness of ∂U, we can cover ∂U by finitely many of these V, say $V_1, \ldots, V_N$. We further pick a V_0 such that $V_0 \Subset U$ and

$$U = \bigcup_{i=0}^N V_i.$$

We can pick approximations $v_i \in C^\infty(\bar{V}_i)$ for $i = 0, \ldots, N$ (the $i = 0$ case is given by the previous global approximation theorem), satisfying $\|v_i - u\|_{W^{k,p}(V_i)} \leq \delta$. Pick a partition of unity $\{\zeta_i\}_{i=0}^N$ of $\bar{U}$ subordinate to $\{V_i\}$. Define

$$v = \sum_{i=0}^N \zeta_i v_i.$$

Clearly $v \in C^\infty(\bar{U})$, and we can bound

$$\begin{aligned}
\|D^\alpha v - \mathrm{D}^\alpha u\|_{L^p(U)} &= \left\| \mathrm{D}^\alpha \sum_{i=0}^N \zeta_i v_i - \mathrm{D}^\alpha \sum_{i=0}^N \zeta_i u \right\|_{L^p(U)} \\
&\leq C_k \sum_{i=0}^N \|v_i - u\|_{W^{k.p}(V_i)} \\
&\leq C_k (1 + N)\delta,
\end{aligned}$$

where C_k is a constant that solely depends on the derivatives of the partition of unity, which are fixed. So we are done. □

3.4 Extensions and traces

Theorem (Extension of $W^{1,p}$ functions)**.** Suppose U is open, bounded and ∂U is C^1. Pick a bounded V such that $U \Subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

for $1 \le p < \infty$ such that for any $u \in W^{1,p}(U)$,

(i) $Eu = u$ almost everywhere in U

(ii) Eu has support in V

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$, where the constant C depends on U, V, p but not u.

Proof. First note that $C^1(\bar{U})$ is dense in $W^{1,p}(U)$. So it suffices to show that the above theorem holds with $W^{1,p}(U)$ replaced with $C^1(\bar{U})$, and then extend by continuity.

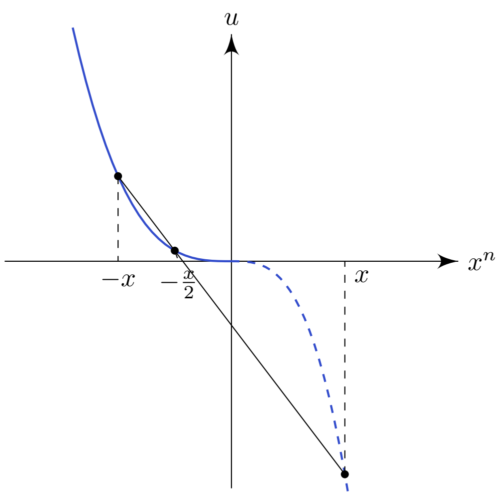
We first show that we can do this locally, and then glue them together using partitions of unity.

Suppose $x^0 \in \partial U$ is such that ∂U near x^0 lies in the plane $\{x_n = 0\}$. In other words, there exists $r > 0$ such that

$$\begin{aligned} B_+ &= B_r(x^0) \cap \{x_n \ge 0\} \subseteq \bar{U} \\ B_- &= B_r(x^0) \cap \{x_n \le 0\} \subseteq \mathbb{R}^n \setminus U. \end{aligned}$$

The idea is that we want to reflect $u|_{B_+}$ across the $x_n = 0$ boundary to get a function on B_-, but the derivative will not be continuous if we do this. So we define a "higher order reflection" by

$$\bar{u}(x) = \begin{cases} u(x) & x \in B^+ \\ -3u(x', -x_n) + 4\left(ux', -\frac{x_n}{2}\right) & x \in B_- \end{cases}$$



We see that this is a continuous function. Moreover, by explicitly computing the partial derivatives, we see that they are continuous across the boundary. So we know $\bar{u} \in C^1(B_r(x^0))$.

We can then easily check that we have

$$\|\bar{u}\|_{W^{1,p}(B_r(x^0))} \leq C\|u\|_{W^{1,p}(B_+)}$$

for some constant C.

If ∂U is not necessarily flat near $x^0 \in \partial U$, then we can use a C^1 diffeomorphism to straighten it out. Indeed, we can pick $r > 0$ and $\gamma \in C^1(\mathbb{R}^{n-1})$ such that

$$U \cap B_r(p) = \{(x', x^n) \in B_r(p) \mid x^n > \gamma(x')\}.$$

We can then use the C^1-diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{aligned} \Phi(x)^i &= x^i & i &= 1, \ldots, n-1 \\ \Phi(x)^n &= x_n - \gamma(x_1, \ldots, x_n) \end{aligned}$$

Then since C^1 diffeomorphisms induce bounded isomorphisms between $W^{1,p}$, this gives a local extension.

Since ∂U is compact, we can take a finite number of points $x_i^0 \in \partial W$, sets W_i and extensions $u_i \in C^1(W_i)$ extending u such that

$$\partial U \subseteq \bigcup_{i=1}^N W_i.$$

Further pick $W_0 \Subset U$ so that $U \subseteq \bigcup_{i=0}^N W_i$. Let $\{\zeta_i\}_{i=0}^N$ be a partition of unity subordinate to $\{W_i\}$. Write

$$\bar{u} = \sum_{i=0}^N \zeta_i \bar{u}_i$$

where $\bar{u}_0 = u$. Then $\bar{u} \in C^1(\mathbb{R}^n)$, $\bar{u} = u$ on U, and we have

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

By multiplying $\bar{u}$ by a cut-off, we may assume $\operatorname{supp} \bar{u} \subseteq V$ for some $V \Supset U$.

Now notice that the whole construction is linear in u. So we have constructed a bounded linear operator from a dense subset of $W^{1,p}(U)$ to $W^{1,p}(V)$, and there is a unique extension to the whole of $W^{1,p}(U)$ by the completeness of $W^{1,p}(V)$. We can see that the desired properties are preserved by this extension. □

Theorem (Trace theorem)**.** Assume U is bounded and has C^1 boundary. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ for $1 \leq p < \infty$ such that $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$.

Proof. It suffices to show that the restriction map defined on C^∞ functions is a bounded linear operator, and then we have a unique extension to $W^{1,p}(U)$. The gist of the argument is that Stokes' theorem allows us to express the integral of a function over the boundary as an integral over the whole of U. In fact, the proof is indeed just the proof of Stokes' theorem.

By a general partition of unity argument, it suffices to show this in the case where $U = \{x_n > 0\}$ and $u \in C^\infty \bar{U}$ with $\operatorname{supp} u \subseteq B_R(0) \cap \bar{U}$. Then

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |u(x', 0)|^p \;\mathrm{d}x' &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\partial}{\partial x_n} |u(x', x_n)|^p \;\mathrm{d}x_n \;\mathrm{d}x'\\ &= \int_U p|u|^{p-1} u_{x_n} \operatorname{sgn} u \;\mathrm{d}x_n \;\mathrm{d}x'.\end{aligned}$$

We estimate this using Young's inequality to get

$$\int_{\mathbb{R}^{n-1}} |u(x', 0)|^p \;\mathrm{d}x' \leq C_p \int_U |u|^p + |u_{x_n}|^p \;\mathrm{d}U \leq C_p \|u\|^p_{W^{1,p}(U)}.$$

So we are done. □

3.5 Sobolev inequalities

Lemma. Let $n \geq 2$ and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For $1 \leq i \leq n$, denote

$$\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

and set

$$f(x) = f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n).$$

Then $f \in L^1(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof. We proceed by induction on n.

If $n = 2$, then this is easy, since

$$f(x_1, x_2) = f_1(x_2) f_2(x_1).$$

So

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x_1, x_2)| \;\mathrm{d}x &= \int |f_1(x_2)| \;\mathrm{d}x_2 \int |f_2(x_1)| \;\mathrm{d}x_1\\ &= \|f_1\|_{L^1(\mathbb{R}^1)} \|f_2\|_{L^1(\mathbb{R}^1)}.\end{aligned}$$

Suppose that the result is true for $n \geq 2$, and consider the $n+1$ case. Write

$$f(x) = f_{n+1}(\tilde{x}_{n+1}) F(x),$$

where $F(x) = f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n)$. Then by Hölder's inequality, we have

$$\int_{x_1, \ldots, x_n} |f(\,\cdot\,, x_{n+1})| \quad \mathrm{d}x \quad \leq \quad \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \|F(\,\cdot\,, x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)}.$$

We now use the induction hypothesis to

$$f_1^{n/(n-1)}(\,\cdot\,, x_{n+1}) f_2^{n/(n-1)}(\,\cdot\,, x_{n+1}) \cdots f_n^{n/(n-1)}(\,\cdot\,, x_{n+1}).$$

So

$$\int_{x_1,\ldots,x_n} |f(\,\cdot\,, x_{n+1})| \;\mathrm{d}x \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \left(\prod_{i=1}^n \|f_i^{\frac{n}{n-1}}(\,\cdot\,, x_n)\|_{L^{n-1}(\mathbb{R}^{n-1})}\right)^{\frac{n-1}{n}}$$

$$= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \|f_i(\,\cdot\,, x_m)\|_{L^n(\mathbb{R}^{n-1})}.$$

Now integrate over x_{n+1}. We get

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^{n+1})} &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \int_{x_{n+1}} \prod_{i=1}^n \|f_i(\,\cdot\,, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})} \;\mathrm{d}x_n.\\ &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^{n+1})} \prod_{i=1}^n \left(\int_{x_{n+1}} \|f_i(\,\cdot\,, x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})} \;\mathrm{d}x_{n+1}\right)^{1/n}\\ &= \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R}^n)}.\end{aligned}$$

□

Theorem (Gagliardo–Nirenberg–Sobolev inequality)**.** Assume $n > p$. Then we have

$$W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*}(\mathbb{R}^n),$$

where

$$p^* = \frac{np}{n-p} > p,$$

and there exists $c > 0$ depending on n, p such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

In other words, $W^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^{p^*}(\mathbb{R}^n)$.

Proof. Assume $u \in C_c^\infty(\mathbb{R}^n)$, and consider $p = 1$. Since the support is compact,

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \;\mathrm{d}y_i.$$

So we know that

$$|u(x)| \leq \int_{-\infty}^\infty |\mathrm{D}u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \;\mathrm{d}y_i \equiv f_i(\tilde{x}_i).$$

Thus, applying this once in each direction, we obtain

$$|u(x)|^{n/(n-1)} \leq \prod_{i=1}^n f_i(\tilde{x}_i)^{1/(n-1)}.$$

If we integrate and then use the lemma, we see that

$$\left(\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)}\right)^{n/(n-1)} \leq C\prod_{i=1}^n \|f_i^{1/(n-1)}\|_{L^{n-1}(\mathbb{R}^{n-1})} = \|\mathrm{D}u\|^{n/(n-1)}_{L^1(\mathbb{R}^n)}.$$

So
$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C\|\mathrm{D}u\|_{L^1(\mathbb{R}^n)}.$$
Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,1}(\mathbb{R}^n)$, the result for $p = 1$ follows.

Now suppose $p > 1$. We apply the $p = 1$ case to
$$v = |u|^\gamma$$
for some $\gamma > 1$, which we choose later. Then we have
$$\mathrm{D}v = \gamma \operatorname{sgn} u \cdot |u|^{\gamma-1}\mathrm{D}u.$$
So
$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\;\mathrm{d}x\right)^{\frac{n-1}{n}} &\leq \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|\mathrm{D}u|\;\mathrm{d}x\\ &\leq \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\;\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\mathrm{D}u|^p\;\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$
We choose γ such that
$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}.$$
So we should pick
$$\gamma = \frac{p(n-1)}{n-p} > 1.$$
Then we have
$$\frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*.$$
So
$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\;\mathrm{d}x\right)^{\frac{n-1}{n}} \leq \frac{p(n-1)}{n-p}\left(\int_{\mathbb{R}^n} |u|^{p^*}\;\mathrm{d}x\right)^{\frac{p-1}{p}} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.$$
So
$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\;\mathrm{d}x\right)^{1/p^*} \leq \frac{p(n-1)}{n-p}\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.$$
This argument is valid for $u \in C_c^\infty(\mathbb{R}^n)$, and by approximation, we can extend to $W^{1,p}(\mathbb{R}^n)$. □

Corollary. Suppose $U \subseteq \mathbb{R}^n$ is open and bounded with C^1-boundary, and $1 \leq p < n$. Then if $p^* = \frac{np}{n-p}$, we have
$$W^{1,p}(U) \subseteq L^{p^*}(U),$$
and there exists $C = C(U, p, n)$ such that
$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Proof. By the extension theorem, we can find $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ with $\bar{u} = u$ almost everywhere on U and
$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$
Then we have
$$\|u\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq \tilde{C}\|u\|_{W^{1,p}(U)}.$$
□

Corollary. Suppose U is open and bounded, and suppose $u \in W_0^{1,p}(U)$. For some $1 \leq p < n$, then we have the estimates

$$\|u\|_{L^q(U)} \leq C\|\mathrm{D}u\|_{L^p(U)}$$

for any $q \in [1, p^*]$. In particular,

$$\|u\|_{L^p(U)} \leq C\|\mathrm{D}u\|_{L^p(U)}.$$

Proof. Since $u \in W_0^{1,p}(U)$, there exists $u_0 \in C_c^\infty(U)$ converging to u in $W^{1,p}(U)$. Extending u_m to vanish on U^c, we have

$$u_m \in C_c^\infty(\mathbb{R}^n).$$

Applying Gagliardo–Nirenberg–Sobolev, we find that

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\mathrm{D}u_m\|_{L^p(\mathbb{R}^n)}.$$

So we know that

$$\|u_m\|_{L^{p^*}(U)} \leq C\|\mathrm{D}u_m\|_{L^p(U)}.$$

Sending $m \to \infty$, we obtain

$$\|u\|_{L^{p^*}(U)} \leq C\|\mathrm{D}u\|_{L^p(U)}.$$

Since U is bounded, by Hölder, we have

$$\left(\int_U |u|^q \;\mathrm{d}x\right)^{1/q} \leq \left(\int_U 1 \;\mathrm{d}x\right)^{1/rq} \left(\int_U |u|^{qs} \;\mathrm{d}s\right)^{1/sq} \leq C\|u\|_{L^{p^*}(U)}$$

provided $q \leq p^*$, where we choose s such that $qs = p^*$, and r such that $\frac{1}{r} + \frac{1}{s} = 1$. □

Theorem (Morrey's inequality)**.** Suppose $n < p < \infty$. Then there exists a constant C depending only on p and n such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$ where $C = C(p, n)$ and $\gamma = 1 - \frac{n}{p} < 1$.

Proof. We first prove the Hölder part of the estimate.

Let Q be an open cube of side length $r > 0$ and containing 0. Define

$$\bar{u} = \frac{1}{|Q|}\int_Q u(x) \;\mathrm{d}x.$$

Then

$$\begin{aligned} |\bar{u} - u(0)| &= \left|\frac{1}{|Q|}\int_Q [u(x) - u(0)] \;\mathrm{d}x\right| \\ &\leq \frac{1}{|Q|}\int_Q |u(x) - u(0)| \;\mathrm{d}x. \end{aligned}$$

Note that

$$u(x) - u(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(tx)\;\mathrm{d}t = \sum_i \int_0^1 x^i \frac{\partial u}{\partial x^i}(tx)\;\mathrm{d}t.$$

So

$$|u(x) - u(0)| \leq r \int_0^1 \sum_i \left| \frac{\partial u}{\partial x^i}(tx)\right| \;\mathrm{d}t.$$

So we have

$$\begin{aligned}
|\bar{u} - u(0)| &\leq \frac{r}{|Q|} \int_Q \int_0^1 \sum_i \left|\frac{\partial u}{\partial x^i}(tx)\right| \;\mathrm{d}t\;\mathrm{d}x\\
&= \frac{r}{|Q|} \int_0^1 t^{-n} \left(\int_{tQ} \sum_i \left|\frac{\partial u}{\partial x^i}(y)\right| \;\mathrm{d}y\right)\;\mathrm{d}t\\
&\leq \frac{r}{|Q|} \int_0^1 t^{-n} \left(\sum_{i=1}^n \left\|\frac{\partial u}{\partial x^i}\right\|_{L^p(tQ)} |tQ|^{1/p'}\right)\;\mathrm{d}t.
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Using that $|Q| = r^n$, we obtain

$$\begin{aligned}
|\bar{u} - u(0)| &\leq c r^{1 - n + \frac{n}{p'}} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \int_0^1 t^{-n + \frac{n}{p'}}\;\mathrm{d}t\\
&\leq \frac{c}{1 - n/p} r^{1 - n/p} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Note that the right hand side is decreasing in r. So when we take r to be very small, we see that $u(0)$ is close to the average value of u around 0.

Indeed, suppose $x, y \in \mathbb{R}^n$ with $|x - y| = \frac{r}{2}$. Pick a box containing x and y of side length r. Applying the above result, shifted so that x, y play the role of 0, we can estimate

$$|u(x) - u(y)| \leq |u(x) - \bar{u}| + |u(y) - \bar{u}| \leq \tilde{C} r^{1 - n/p} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.$$

Since $r < \|x - y\|$, it follows that

$$\frac{|u(x) - u(y)|}{|x - y|^{1 - n/p}} \leq C \cdot 2^{1 - n/p} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.$$

So we conclude that $[u]_{C^{0, \gamma}(\mathbb{R}^n)} \leq C \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$.

Finally, to see that u is bounded, any $x \in \mathbb{R}^n$ belongs to some cube Q of side length 1. So we have

$$|u(x)| \leq |u(x) - \bar{u} + \bar{u}| \leq |\bar{u}| + C \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}.$$

But also

$$|\bar{u}| \leq \int_Q |u(x)|\;\mathrm{d}x \leq \|u\|_{L^p(\mathbb{R}^n)} \|1\|_{L^p(Q)} = \|u\|_{L^p(\mathbb{R}^n)}.$$

So we are done. □

Corollary. Suppose $u \in W^{1, p}(U)$ for U open, bounded with C^1 boundary. Then there exists $u^* \in C^{0, \gamma}(U)$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0, \gamma}(U)} \leq C \|u\|_{W^{1, p}(U)}$.

4 Elliptic boundary value problems

4.1 Existence of weak solutions

Theorem (Lax–Milgram theorem)**.** Let H be a real Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$. Suppose $B : H \times H \to \mathbb{R}$ is a bilinear mapping such that there exists constants $\alpha, \beta > 0$ so that

– $|B[u, v]| \leq \alpha \|u\| \|v\|$ for all $u, v \in H$ (boundedness)

– $\beta \|u\|^2 \leq B[u, u]$ (coercivity)

Then if $f : H \to \mathbb{R}$ is a bounded linear map, then there exists a unique $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Proof. By the Riesz representation theorem, we may assume that there is some w such that

$$\langle f, v \rangle = (u, v).$$

For each fixed $u \in H$, the map

$$v \mapsto B[u, v]$$

is a bounded linear functional on H. So by the Riesz representation theorem, we can find some Au such that

$$B[u, v] = (Au, v).$$

It then suffices to show that A is invertible, for then we can take $u = A^{-1}w$.

– Since B is bilinear, it is immediate that $A : H \to H$ is linear.

– A is bounded, since we have

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|.$$

– A is injective and has closed image. Indeed, by coercivity, we know

$$\beta \|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\| \|u\|.$$

Dividing by $\|u\|$, we see that A is bounded below, hence is injective and has closed image (since H is complete).

(Indeed, injectivity is clear, and if $Au_m \to v$ for some v, then $\|u_m - u_n\| \leq \frac{1}{\beta} \|Au_m - Au_n\| \to 0$ as $m, n \to \infty$. So (u_n) is Cauchy, and hence has a limit u. Then by continuity, $Au = v$, and in particular, $v \in \operatorname{im} A$)

– Since $\operatorname{im} A$ is closed, we know

$$H = \operatorname{im} A \oplus \operatorname{im} A^\perp.$$

Now let $w \in \operatorname{im} A^\perp$. Then we can estimate

$$\beta \|w\|^2 \leq B[w, w] = (Aw, w) = 0.$$

So $w = 0$. Thus, in fact $\operatorname{im} A^\perp = \{0\}$, and so A is surjective. $\square$

Theorem (Energy estimates for B)**.** Suppose $a^{ij} = a^{ji}, b^i, c \in L^\infty(U)$, and there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for almost every $x \in U$ and $\xi \in \mathbb{R}^n$. Then if B is defined by

$$B[u,v] = \int_U \left(\sum_{ij} v_{x_i} a^{ij} u_{x_j} + \sum_i b^i u_{x_i} v + cuv\right)\,\mathrm{d}x,$$

then there exists $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

(i) $|B[u,v]| \leq \alpha\|u\|_{H^1(U)}\|v\|_{H^1(U)}$ for all $u, v \in H_0^1(U)$

(ii) $\beta\|u\|^2_{H^1(U)} \leq B[u,u] + \gamma\|u\|^2_{L^2(U)}$.

Moreover, if $b^i \equiv 0$ and $c \geq 0$, then we can take γ.

Proof.

(i) We estimate

$$\begin{aligned}
|B[u,v]| &\leq \sum_{i,j}\|a^{ij}\|_{L^\infty(U)}\int_U |\mathrm{D}u||\mathrm{D}v|\,\mathrm{d}x\\
&\quad + \sum_i \|b\|_{C^\infty(U)}\int_U |\mathrm{D}u||v|\,\mathrm{d}x\\
&\quad + \|c\|_{L^\infty(U)}\int_U |u||v|\,\mathrm{d}x\\
&\leq c_1\|\mathrm{D}u\|_{L^2(U)}\|\mathrm{D}v\|_{L^2(u)} + c_2\|\mathrm{D}u\|_{L^2(U)}\|v\|_{L^2(U)}\\
&\quad + c_3\|u\|_{L^2(U)}\|v\|_{L^2(u)}\\
&\leq \alpha\|u\|_{H^1(U)}\|v\|_{H^1(U)}
\end{aligned}$$

for some α.

(ii) We start from uniform ellipticity. This implies

$$\begin{aligned}
\theta\int_U |\mathrm{D}u|^2\,\mathrm{d}x &\leq \int_U \sum_{i,j=1}^n a^{ij}(x)u_{x_i}u_{x_j}\,\mathrm{d}x\\
&= B[u,u] - \int_U \sum_{i=1}^n b^i u_{x_i}u + cu^2\,\mathrm{d}x\\
&\leq B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)}\int |\mathrm{D}u||u|\,\mathrm{d}x\\
&\qquad + \|c\|_{L^\infty(U)}\int_U |u|^2\,\mathrm{d}x.
\end{aligned}$$

Now by Young's inequality, we have

$$\int_U |\mathrm{D}u||u|\,\mathrm{d}x \leq \varepsilon\int_U |\mathrm{D}u|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_U |u|^2\,\mathrm{d}x$$

for any $\varepsilon > 0$. We choose ε small enough so that

$$\varepsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \leq \frac{\theta}{2}.$$

So we have

$$\theta \int_U |\mathrm{D}u|^2 \;\mathrm{d}x \leq B[u,u] + \frac{\theta}{2} \int_U |\mathrm{D}u|^2 \;\mathrm{d}x + \gamma \int_U |u|^2 \;\mathrm{d}x$$

for some γ. This implies

$$\frac{\theta}{2} \|\mathrm{D}u\|^2_{L^2(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

We can add $\frac{\theta}{2}\|u\|^2_{L^2(U)}$ on both sides to get the desired bound on $\|u\|_{H^1(U)}$. To get the "moreover" statement, we see that under these conditions, we have

$$\theta \int |\mathrm{D}u|^2 \;\mathrm{d}x \leq B[u,u].$$

Then we apply the *Poincaré's inequality*, which tells us there is some $C > 0$ such that for all $u \in H^1_0(U)$, we have

$$\|u\|_{L^2(U)} \leq C \|\mathrm{D}u\|_{L^2(U)}.$$ □

Theorem. Let U, L be as above. There is a $\gamma \geq 0$ such that for any $\mu \geq \gamma$ and any $f \in L^2(U)$, there exists a unique weak solution to

$$\begin{aligned} Lu + \mu u &= f \text{ on } U \\ u &= 0 \text{ on } \partial U. \end{aligned}$$

Moreover, we have

$$\|u\|_{H^1(U)} \leq C \|f\|_{L^2(U)}$$

for some $C = C(L, U) \geq 0$.

Again, if $b^i \equiv 0$ and $c \geq 0$, then we may take $\gamma = 0$.

Proof. Take γ from the previous theorem when applied to L. Then if $\mu \geq \gamma$ and we set

$$B_\mu[u,v] = B[u,v] + \mu(u,v)_{L^2(U)},$$

This is the bilinear form corresponding to the operator

$$L_\mu = L + \mu.$$

Then by the previous theorem, B_μ satisfies boundedness and coercivity. So if we fix any $f \in L^2$, and think of it as an element of $H^1_0(U)^*$ by

$$\langle f, v\rangle = (f, u)_{L^2(U)} = \int_U fv \;\mathrm{d}x,$$

then we can apply Lax–Milgram to find a unique $u \in H^1_0(U)$ satisfying $B_\mu[u,v] = \langle f, v\rangle = (f,v)_{L^2(U)}$ for all $v \in H^1_0(U)$. This is precisely the condition for u to be a weak solution.

Finally, the Gårding inequality tells us

$$\beta \|u\|^2_{H^1(U)} \leq B_\mu[u,u] = (f,u)_{L^2(U)} \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}.$$

So we know that

$$\beta \|u\|_{H^1(U)} \leq \|f\|_{L^2(U)}.$$ □

4.2 The Fredholm alternative

Theorem (Fredholm alternative)**.** Consider the problem

$$Lu = f, \quad u|_{\partial U} = 0. \tag{$*$}$$

For L a uniformly elliptic operator on an open bounded set U with C^1 boundary, either

(i) For each $f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ to $(*)$; or

(ii) There exists a non-zero weak solution $u \in H_0^1(U)$ to the *homogeneous problem*, i.e. $(*)$ with $f = 0$.

Theorem (Fredholm alternative)**.** Let H be a Hilbert space and $K : H \to H$ be a compact operator. Then

(i) $\ker(I - K)$ is finite-dimensional.

(ii) $\mathrm{im}(I - K)$ is finite-dimensional.

(iii) $\mathrm{im}(I - K) = \ker(I - K^\dagger)^\perp$.

(iv) $\ker(I - K) = \{0\}$ iff $\mathrm{im}(I - K) = H$.

(v) $\dim \ker(I - K) = \dim \ker(I - K^\dagger) = \dim \mathrm{coker}(I - K)$.

Lemma. Weak limits are unique. □

Lemma. Strong convergence implies weak convergence. □

Theorem (Weak compactness)**.** Let H be a separable Hilbert space, and suppose $(u_m)_{m=1}^\infty$ is a bounded sequence in H with $\|u_m\| \leq K$ for all m. Then u_m admits a subsequence $(u_{m_j})_{j=1}^\infty$ such that $u_{m_j} \rightharpoonup u$ for some $u \in H$ with $\|u\| \leq K$.

Proof. Let $(e_i)_{i=1}^\infty$ be an orthonormal basis for H. Consider (e_1, u_m). By Cauchy–Schwarz, we have

$$|(e_1, u_m)| \leq \|e_1\|\|e_m\| \leq K.$$

So by Bolzano–Weierstrass, there exists a subsequence (u_{m_j}) such that (e_1, u_{m_j}) converges.

Doing this iteratively, we can find a subsequence (v_ℓ) such that for each i, there is some c_i such that $(e_i, v_\ell) \to c_i$ as $\ell \to \infty$.

We would expect the weak limit to be $\sum c_i e_i$. To prove this, we need to first show it converges. We have

$$\begin{aligned}
\sum_{j=1}^p |c_j|^2 &= \lim_{k \to \infty} \sum_{j=1}^p |(e_j, v_\ell)|^2 \\
&\leq \sup \sum_{j=1}^p |(e_j, v_\ell)|^2 \\
&\leq \sup \|v_k\|^2 \\
&\leq K^2,
\end{aligned}$$

using Bessel's inequality. So

$$u = \sum_{j=1}^{\infty} c_j e_j$$

converges in H, and $\|u\| \leq K$. We already have

$$(e_j, v_\ell) \to (e_j, u)$$

for all j. Since $\|v_\ell - u\|$ is bounded by $2K$, it follows that the set of all w such that

$$(w, v_\ell) \to (v, u) \tag{$\dagger$}$$

is closed under finite linear combinations and taking limits, hence is all of H. To see that it is closed under limits, suppose $w_k \to w$, and w_k satisfy ($\dagger$). Then

$$|(w, v_\ell) - (w, u)| \leq |(w - w_k, v_\ell - u)| + |(w_k, v_\ell - u)| \leq 2K\|w - w_k\| + |(w_k, v_\ell - u)|$$

So we can first find k large enough such that the first term is small, then pick ℓ such that the second is small. $\square$

Lemma (Poincaré revisited)**.** Suppose $u \in H^1(\mathbb{R}^n)$. Let $Q = [\xi_1, \xi_1 + L] \times \cdots \times [\xi_n, \xi_n + L]$ be a cube of length L. Then we have

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u(x)\;\mathrm{d}x\right)^2 + \frac{nL^2}{2}\|\mathrm{D}u\|^2_{L^2(Q)}.$$

Proof. By approximation, we can assume $u \in C^\infty(\bar{Q})$. For $x, y \in Q$, we write

$$\begin{aligned} u(x) - u(y) = &\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_1, \ldots, x_n)\;\mathrm{d}t \\ &+ \int_{y_2}^{x_2} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, t, x_3, \ldots, x_n)\;\mathrm{d}t \\ &+ \cdots \\ &+ \int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \ldots, y_{n-1}, t)\;\mathrm{d}t. \end{aligned}$$

Squaring, and using $2ab \leq a^2 + b^2$, we have

$$\begin{aligned} u(x)^2 + u(y)^2 - 2u(x)u(y) \leq &\; n\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_1, \ldots, x_n)\;\mathrm{d}t\right)^2 \\ &+ \cdots \\ &+ n\left(\int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t} u(y_1, \ldots, y_{n-1}, t)\;\mathrm{d}t\right)^2. \end{aligned}$$

Now integrate over x and y. On the left, we get

$$\iint_{Q \times Q} \mathrm{d}x\;\mathrm{d}y\;(u(x)^2 + u(y)^2 - 2u(x)u(y)) = 2|Q|\|u\|^2_{L^2(Q)} - 2\left(\int_Q u(x)\;\mathrm{d}x\right)^2.$$

On the right we have

$$\begin{aligned}
I_1 &= \left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_n)\ \mathrm{d}t\right)^2 \\
&\leq \int_{y_1}^{x_1} \mathrm{d}t \int_{y_1}^{x_1} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_n)\right)^2\ \mathrm{d}t \qquad \text{(Cauchy–Schwarz)} \\
&\leq L \int_{\xi_1}^{\xi_1 + L} \left(\frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2, \ldots, x_n)\right)^2\ \mathrm{d}t.
\end{aligned}$$

Integrating over all $x, y \in Q$, we get

$$\iint_{Q \times Q} \mathrm{d}x\ \mathrm{d}y\ I_1 \leq L^2 |Q| \|\mathrm{D}_1 u\|^2_{L^2(Q)}.$$

Similarly estimating the terms on the right-hand side, we find that

$$2|Q| \|u\|_{L^2(Q)} - 2\left(\int_Q u(x)\ \mathrm{d}x\right)^2 \leq n|Q| \sum_{i=1}^n \|\mathrm{D}_i u\|^2_{L^2(Q)} = n|Q|L^2 \|\mathrm{D}u\|^2_{L^2(Q)}.$$

□

Theorem (Rellich–Kondrachov)**.** Let $U \subseteq \mathbb{R}^n$ be open, bounded with C^1 boundary. Then if $(u_m)_{m=1}^\infty$ is a sequence in $H^1(U)$ with $u_m \rightharpoonup u$, then $u_m \to u$ in L^2.

In particular, by weak compactness any sequence in $H^1(U)$ has a subsequence that is convergent in $L^2(U)$.

Proof. By the extension theorem, we may assume $U = Q$ for some large cube Q with $U \Subset Q$.

We subdivide Q into N many cubes of side length δ, such that the cubes only intersect at their faces. Call these $\{Q_a\}_{a=1}^N$.

We apply Poincaré separately to each of these to obtain

$$\begin{aligned}
\|u_j - u\|^2_{L^2(Q)} &= \sum_{a=1}^N \|u_j - u\|^2_{L^2(Q_a)} \\
&\leq \sum_{a=1}^N \left[\frac{1}{|Q_a|}\left(\int_{Q_a} (u_i - u)\ \mathrm{d}x\right)^2 + \frac{n\delta^2}{2} \|\mathrm{D}u_i - \mathrm{D}u\|^2_{L^2(Q_a)}\right] \\
&= \sum_{a=1}^N \frac{1}{|Q_a|}\left(\int_{Q_a} (u_i - u)\ \mathrm{d}x\right)^2 + \frac{n\delta^2}{2} \|\mathrm{D}u_i - \mathrm{D}u\|^2_{L^2(Q)}.
\end{aligned}$$

Now since $\|\mathrm{D}u_i - \mathrm{D}u\|^2_{L^2(Q)}$ is fixed, for δ small enough, the second term is $< \frac{\varepsilon}{2}$. Then since $u_i \rightharpoonup u$, we in particular have

$$\int_{Q_1} (u_i - u)\ \mathrm{d}x \to 0 \text{ as } i \to \infty$$

for all a, since this is just the inner product with the constant function 1. So for i large enough, the first term is also $< \frac{\varepsilon}{2}$. □

Corollary. Suppose $K : L^2(U) \to H^1(U)$ is a bounded linear operator. Then the composition

$$L^2(U) \xrightarrow{K} H^1(U) \hookrightarrow L^2(U)$$

is compact.

Proof. Indeed, if $u_m \in L^2(U)$ is bounded, then Ku_m is also bounded. So by Rellich–Kondrachov, there exists a subsequence $u_{m_j} \to u$ in $L^2(U)$. $\square$

Theorem (Fredholm alternative for elliptic BVP)**.** Let L be a uniformly elliptic operator on an open bounded set U with C^1 boundary. Consider the problem

$$Lu = f, \quad u|_{\partial U} = 0. \tag{$*$}$$

Then exactly one of the following are true:

(i) For each $f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ to $(*)$

(ii) There exists a non-zero weak solution $u \in H_0^1(U)$ to the *homogeneous problem*, i.e. $(*)$ with $f = 0$.

If this holds, then the dimension of $N = \ker L \subseteq H_0^1(U)$ is equal to the dimension of $N^* = \ker L^\dagger \subseteq H_0^1(U)$.

Finally, $(*)$ has a solution if and only if $(f, v)_{L^2(U)} = 0$ for all $v \in N^*$

Proof. We know that there exists $\gamma > 0$ such that for any $f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ to

$$L_\gamma u = Lu + \gamma u = f, \quad u|_{\partial U} = 0.$$

Moreover, we have the bound $\|u\|_{H^1(U)} \leq C\|f\|_{L^2(U)}$ (which gives uniqueness). Thus, we can set $L_\gamma^{-1} f$ to be this u, and then $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$ is a bounded linear map. Composing with the inclusion $L^2(U)$, we get a compact endomorphism of $L^2(U)$.

Now suppose $u \in H_0^1$ is a weak solution to $(*)$. Then

$$B[u, v] = (f, v)_{L^2(U)} \text{ for all } v \in H_0^1(U)$$

is true if and only if

$$B_\gamma[u, v] \equiv B[u, v] + \gamma(u, v) = (f + \gamma u, v) \text{ for all } v \in H_0^1(U).$$

Hence, u is a weak solution of $(*)$ if and only if

$$u = L_\gamma^{-1}(f + \gamma u) = \gamma L_\gamma^{-1} u + L_\gamma^{-1} f.$$

In other words, u solves $(*)$ iff

$$u - Ku = h,$$

for

$$K = \gamma L_\gamma^{-1}, \quad h = L_\gamma^{-1} f.$$

Since we know that $K : L^2(U) \to L^2(U)$ is compact, by the Fredholm alternative for compact operators, either

(i) $u - Ku = h$ admits a solution $u \in L^2(U)$ for all $h \in L^2(U)$; or

(ii) There exists a non-zero $u \in L^2(U)$ such that $u - Ku = 0$. Moreover, $\operatorname{im}(I - K) = \ker(I - K^\dagger)^\perp$ and $\dim\ker(I - K) = \dim\operatorname{im}(I - K)^\perp$.

There is a bit of bookkeeping to show that this corresponds to the two alternatives in the theorem.

(i) We need to show that $u \in H^1_0(U)$. But this is trivial, since we have

$$u = \gamma L_\gamma^{-1} u + L_\gamma^{-1} f,$$

and we know that L_γ^{-1} maps $L^2(U)$ into $H^1_0(U)$.

(ii) As above, we know that the non-zero solution u. There are two things to show. First, we have to show that $v - K^\dagger v = 0$ iff v is a weak solution to

$$L^\dagger v = 0, \quad v|_{\partial U} = 0.$$

Next, we need to show that $h = L_\gamma^{-1} f \in (N^*)^\perp$ iff $f \in (N^*)^\perp$.

For the first part, we want to show that $v \in \ker(I - K^\dagger)$ iff $B^\dagger[v, u] = B[u, v] = 0$ for all $u \in H^1_0(U)$.

We are good at evaluating $B[u, v]$ when u is of the form $L_\gamma^{-1} w$, by definition of a weak solution. Fortunately, $\operatorname{im} L_\gamma^{-1}$ contains $C_c^\infty(U)$, since $L_\gamma^{-1} L_\gamma \phi = \phi$ for all $\phi \in C_c^\infty(U)$. In particular, $\operatorname{im} L_\gamma^{-1}$ is dense in $H^1_0(U)$. So it suffices to show that $v \in \ker(I - K^\dagger)$ iff $B[L_\gamma^{-1} w, v] = 0$ for $w \in L^2(U)$. This is immediate from the computation

$$B[L_\gamma^{-1} w, v] = B_\gamma[L_\gamma^{-1} w, v] - \gamma(L_\gamma^{-1} w, v) = (w, v) - (Kw, v) = (w, v - K^\dagger v).$$

The second is also easy — if $v \in N^* = \ker(I - K^\dagger)$, then

$$(L_\gamma^{-1} f, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^\dagger v) = \frac{1}{\gamma}(f, v). \qquad \square$$

4.3 The spectrum of elliptic operators

Theorem (Spectral theorem of compact operators)**.** Let $\dim H = \infty$, and $K : H \to H$ a compact operator. Then

- $\sigma(K) = \sigma_p(K) \cup \{0\}$. Note that 0 may or may not be in $\sigma_p(K)$.
- $\sigma(K) \setminus \{0\}$ is either finite or is a sequence tending to 0.
- If $\lambda \in \sigma_p(K)$, then $\ker(K - \lambda I)$ is finite-dimensional.
- If K is self-adjoint, i.e. $K = K^\dagger$ and H is separable, then there exists a countable orthonormal basis of eigenvectors.

Theorem (Spectrum of L)**.**

(i) There exists a countable set $\Sigma \subseteq \mathbb{R}$ such that there is a non-trivial solution to $Lu = \lambda u$ iff $\lambda \in \Sigma$.

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$, the values of an increasing sequence with $\lambda_k \to \infty$.

(iii) To each $\lambda \in \Sigma$ there is an associated finite-dimensional space

$$\mathcal{E}(\lambda) = \{u \in H_0^1(U) \mid u \text{ is a weak solution of } (*) \text{ with } f = 0\}.$$

We say $\lambda \in \Sigma$ is an *eigenvalue* and $u \in \mathcal{E}(\lambda)$ is the associated *eigenfunction*.

Proof. Apply the spectral theorem to compact operator $L_\gamma^{-1} : L^2(U) \to L^2(U)$, and observe that

$$L_\gamma^{-1} u = \lambda u \iff u = \lambda(L+\gamma)u \iff Lu = \frac{1-\lambda\gamma}{\lambda} u.$$

Note that L_γ^{-1} does not have a zero eigenvalue. □

Theorem. Suppose L is a formally self-adjoint, positive, uniformly elliptic operator on U, an open bounded set with C^1 boundary. Then we can represent the eigenvalues of L as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

where each eigenvalue appears according to its multiplicity ($\dim \mathcal{E}(\lambda)$), and there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(U)$ with $w_k \in H_0^1(U)$ an eigenfunction of L with eigenvalue λ_k.

Proof. Note that positivity implies $c \geq 0$. So the inverse $L^{-1} : L^2(U) \to L^2(U)$ exists and is a compact operator. We are done if we can show that L^{-1} is self-adjoint. This is trivial, since for any f, g, we have

$$(L^{-1} f, g)_{L^2(U)} = B[v, u] = B[u, v] = (L^{-1} g, f)_{L^2(U)}.$$ □

4.4 Elliptic regularity

Lemma. If $u \in L^2(U)$, then $u \in H^1(V)$ iff

$$\|\Delta^h u\|_{L^2(V)} \leq C$$

for some C and all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$. In this case, we have

$$\frac{1}{\tilde{C}} \|\mathrm{D} u\|_{L^2(V)} \leq \|\Delta^h u\|_{L^2(V)} \leq \tilde{C} \|\mathrm{D} u\|_{L^2(V)}.$$

Proof. See example sheet. □

Lemma. If w, v and compactly supported in U, then

$$\int_U w \Delta_k^{-h} v \;\mathrm{d}x = \int_U (\Delta_k^h w) v \;\mathrm{d}x$$
$$\Delta_k^h(wv) = (\tau_k^h w)\Delta_k^h v + (\Delta_k^h w) v,$$

where $\tau_k^h w(x) = w(x + he_k)$.

Theorem (Interior regularity)**.** Suppose L is uniformly elliptic on an open set $U \subseteq \mathbb{R}^n$, and assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is such that

$$B[u, v] = (f, v)_{L^2(U)} \tag{†}$$

for all $v \in H_0^1(U)$. Then $u \in H^2_{loc}(U)$, and for each $V \Subset U$, we have

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

with C depending on L, V, U, but not f or u.

Proof. We first show that we may in fact assume $b^i = c = 0$. Indeed, if we know the theorem for such L, then given a general L, we write

$$L'u = -\sum (a^{ij} u_{x_j})_{x_i}, \quad Ru = \sum b^i u_{x_i} + cu.$$

Then if u is a weak solution to $Lu = f$, then it is also a weak solution to $L'u = f - Ru$. Noting that $Ru \in L^2(U)$, this tells us $u \in H^2_{loc}(U)$. Moreover, on $V \Subset U$,

- We can control $\|u\|_{H^2(V)}$ by $\|f - Ru\|_{L^2(V)}$ and $\|u\|_{L^2(V)}$ (by theorem).
- We can control $\|f - Ru\|_{L^2(V)}$ by $\|f\|_{L^2(V)}, \|u\|_{L^2(V)}$ and $\|\mathrm{D}u\|_{L^2(V)}$.
- By Gårding's inequality, we can control $\|\mathrm{D}u\|_{L^2(V)}$ by $\|u\|_{L^2(V)}$ and $B[u, u] = (f, u)_{L^2(V)}$.
- By Hölder, we can control $(f, u)_{L^2(V)}$ by $\|f\|_{L^2(V)}$ and $\|u\|_{L^2(V)}$.

So it suffices to consider the case where L only has second derivatives. Fix $V \Subset U$ and choose W such that $V \Subset W \Subset U$. Take $\xi \in C_c^\infty(W)$ such that $\zeta \equiv 1$ on V.

Recall that our example of Laplace's equation, we considered the integral $\int f^2 \;\mathrm{d}x$ and did some integration by parts. Essentially, what we did was to apply the definition of a weak solution to Δu. There we was lucky, and we could obtain the result in one go. In general, we should consider the second derivatives one by one.

For $k \in \{1, \ldots n\}$, we consider the function

$$v = -\Delta_k^{-h}(\zeta^2 \Delta_k^h u).$$

As we shall see, this is the correct way to express $u_{x_k x_k}$ in terms of difference quotients (the $-h$ in the first Δ_k^{-h} comes from the fact that we want to integrate by parts). We shall put this into the definition of a weak solution to say $B[u, v] = (f, v)$. The plan is to isolate a $\|\Delta_k^h \mathrm{D}u\|_2$ term on the left and then bound it.

We first compute

$$\begin{aligned} B[u, v] &= -\sum_{i,j} \int_U a^{ij} u_{x_i} \Delta_k^{-h}(\zeta^2 \Delta_k^h u)_{x_j} \;\mathrm{d}x \\ &= \sum_{i,j} \int_U \Delta_k^h (a^{ij} u_{x_i})(\zeta^2 \Delta_k^h u)_{x_j} \;\mathrm{d}x \\ &= \sum_{i,j} \int_U (\tau_k^h a^{ij} \Delta_k^h u_{x_i} + (\Delta_k^h a^{ij}) u_{x_i})(\zeta^2 \Delta_k^h u_{x_j} + 2\zeta\zeta_{x_j} \Delta_k^h u) \;\mathrm{d}x \\ &\equiv A_1 + A_2, \end{aligned}$$

where

$$A_1 = \sum_{i,j} \int_U \xi^2 (\tau_k^h a^{ij})(\Delta_k^h u_{x_i})(\Delta_k^h u_{x_j})\;\mathrm{d}x$$

$$A_2 = \sum_{i,j} \int_U \left[(\Delta_k^h a^{ij}) u_{x_i} \zeta^2 \Delta_k^h u_{x_j} + 2\zeta\zeta_{x_j} \Delta_k^h u (\tau_k^h a^{ij} \Delta_k^h u_{x_i} + (\Delta_k^h a^{ij}) u_{x_i})\right]\;\mathrm{d}x.$$

By uniform ellipticity, we can bound

$$A_1 \geq \theta \int_U \xi^2 |\Delta_k^h \mathrm{D}u|^2\;\mathrm{d}x.$$

This is what we want to be small.

Note that A_2 looks scary, but every term either only involves "first derivatives" of u, or a product of a second derivative of u with a first derivative. Thus, applying Young's inequality, we can bound $|A_2|$ by a linear combination of $|\Delta_k^h \mathrm{D}u|^2$ and $|\mathrm{D}u|^2$, and we can make the coefficient of $|\Delta_k^h \mathrm{D}u|^2$ as small as possible.

In detail, since $a^{ij} \in C^1(U)$ and ζ is supported in W, we can uniformly bound $a^{ij}, \Delta_k^h a^{ij}, \zeta_{x_j}$, and we have

$$|A_2| \leq C \int_W \left[\zeta |\Delta_k^h \mathrm{D}u||\mathrm{D}u| + \zeta |\mathrm{D}u||\Delta_k^h u| + \zeta |\Delta_k^h \mathrm{D}u||\Delta_k^h u|\right]\;\mathrm{d}x.$$

Now recall that $\|\Delta_k^h u\|$ is bounded by $\|\mathrm{D}u\|$. So applying Young's inequality, we may bound (for a different C)

$$|A_2| \leq \varepsilon \int_W \zeta^2 |\Delta_k^h \mathrm{D}u|^2 + C \int_W |\mathrm{D}u|^2\;\mathrm{d}x.$$

Thus, taking $\varepsilon = \frac{\theta}{2}$, it follows that

$$(f, v) = B[u, v] \geq \frac{\theta}{2} \int_U \zeta^2 |\Delta_k^h \mathrm{D}u|^2\;\mathrm{d}x - C \int_W |\mathrm{D}u|^2\;\mathrm{d}x.$$

This is promising.

It now suffices to bound (f, v) from above. By Young's inequality,

$$\begin{aligned}
|(f, v)| &\leq \int |f||\Delta_k^{-h}(\zeta^2 \Delta_k^h u)|\;\mathrm{d}x\\
&\leq C \int |f||\mathrm{D}(\zeta^2 \Delta_k^h u)|\;\mathrm{d}x\\
&\leq \varepsilon \int |\mathrm{D}(\zeta^2 \Delta_k^h u)|^2\;\mathrm{d}x + C \int |f|^2\;\mathrm{d}x\\
&\leq \varepsilon \int |\zeta^2 \Delta_k^h \mathrm{D}u|^2\;\mathrm{d}x + C(\|f\|_{L^2(U)}^2 + \|\mathrm{D}u\|_{L^2(U)}^2)
\end{aligned}$$

Setting $\varepsilon = \frac{\theta}{4}$, we get

$$\int_U \zeta^2 |\Delta_k^h \mathrm{D}u|^2\;\mathrm{d}x \leq C(\|f\|_{L^2(W)}^2 + \|\mathrm{D}u\|_{L^2(W)}^2),$$

and so, in particular, we get a uniform bound on $\|\Delta_k^h \mathrm{D}u\|_{L^2(V)}$. Now as before, we can use Gårding to get rid of the $\|\mathrm{D}u\|_{L^2(W)}$ dependence on the right. □

Theorem (Elliptic regularity). If a^{ij}, b^i and c are $C^{m+1}(U)$ for some $m \in \mathbb{N}$, and $f \in H^m(U)$, then $u \in H_{loc}^{m+2}(U)$ and for $V \Subset W \Subset U$, we can estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(W)} + \|u\|_{L^2(W)}).$$

In particular, if m is large enough, then $u \in C_{loc}^2(U)$, and if all a^{ij}, b^i, c, f are smooth, then u is also smooth.

Theorem (Boundary H^2 regularity). Assume $a^{ij} \in C^1(\bar{U})$, $b^1, c \in L^\infty(U)$, and $f \in L^2(U)$. Suppose $u \in H_0^1(U)$ is a weak solution of $Lu = f, u|_{\partial U} = 0$. Finally, we assume that ∂U is C^2. Then

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

If u is the *unique* weak solution, we can drop the $\|u\|_{L^2(U)}$ from the right hand side.

Proof. Note that we already know that u is locally in $H_{loc}^2(U)$. So we only have to show that the second-derivative is well-behaved near the boundary.

By a partition of unity and change of coordinates, we may assume we are in the case

$$U = B_1(0) \cap \{x_n > 0\}.$$

Let $V = B^{1/2}(0) \cap \{x_n > 0\}$. Choose a $\zeta \in \mathbb{C}_c^\infty(B_1(0))$ with $\zeta \equiv 1$ on V and $0 \leq \zeta \leq 1$.

Most of the proof in the previous proof goes through, as long as we restrict to

$$v = -\Delta_k^{-h}(\zeta^2 \Delta_k^h u)$$

with $k \neq n$, since all the translations keep us within U, and hence are well-defined.

Thus, we control all second derivatives of the form $\mathrm{D}_k \mathrm{D}_i u$, where $k \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, n\}$. The only remaining second-derivative to control is $\mathrm{D}_n \mathrm{D}_n u$. To understand this, we go back to the PDE and look at the PDE itself. Recall that we know it holds pointwise almost everywhere, so

$$\sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu = f.$$

So we can write $a^{nn} u_{x_n} u_{x_n} = F$ almost everywhere, where F depends on a, b, c, f and all (up to) second derivatives of u that are not $u_{x_n x_n}$. Thus, F is controlled in L^2. But uniform ellipticity implies a^{nn} is bounded away from 0. So we are done. $\square$

5 Hyperbolic equations

Theorem (Uniqueness of weak solution). A weak solution, if exists, is unique.

Proof. It suffices to consider the case $f = \psi = \psi' = 0$, and show any solution must be zero. Let

$$v(x, t) = \int_t^T e^{-\lambda s} u(x, s)\;\mathrm{d}s,$$

where λ is a real number we will pick later. The point of introducing this $e^{-\lambda t}$ is that in general, we do not expect conservation of energy. There could be some exponential growth in the energy, so want to suppress this.

Then this function belongs to $H^1(U_T)$, $v = 0$ on $\Sigma_T \cup \partial^* U_T$, and

$$v_t = -e^{-\lambda t} u.$$

Using the fact that u is a weak solution, we have

$$\int_{U_T} \left(u_t u e^{-\lambda t} - \sum v_{tx_j} v_{x_i} e^{\lambda t} + \sum_i b^i u_{x_i} v + (c-1)uv - v v_t e^{\lambda t} \right)\;\mathrm{d}x\;\mathrm{d}t = 0.$$

Integrating by parts, we can write this as $A = B$, where

$$\begin{aligned} A &= \int_{U_T} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} u^2 e^{-\lambda t} - \sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2} v^2 e^{\lambda t} \right) \right. \\ &\qquad \left. + \frac{\lambda}{2} \left(u^2 e^{-\lambda t} + \sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t} \right) \right)\;\mathrm{d}x\;\mathrm{d}t \\ B &= -\int_{U_T} \left(e^{\lambda t} \sum a^{ij} v_{x_i} v_{x_j} - \sum b^i_{x_i} uv - \sum b^i v_{x_i} u + (c-1)uv \right)\;\mathrm{d}x\;\mathrm{d}t. \end{aligned}$$

Here A is the nice bit, which we can control, and B is the junk bit, which we will show that we can absorb elsewhere.

Integrating the time derivative in A, using $v = 0$ on Σ_T and $u = 0$ on Σ_0, we have

$$\begin{aligned} A = e^{\lambda T} \int_{\Sigma_T} \frac{1}{2} u^2\;\mathrm{d}x &+ \int_{\Sigma_0} \sum \left(a^{ij} v_{x_i} v_{x_j} + v^2 \right)\;\mathrm{d}x \\ &\frac{\lambda}{2} \int_{U_T} \left(u^2 e^{-\lambda t} + \sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t} \right)\;\mathrm{d}x\;\mathrm{d}t. \end{aligned}$$

Using the uniform ellipticity condition (and the observation that the first line is always non-negative), we can bound

$$A \geq \frac{\lambda}{2} \int_{U_T} \left(u^2 e^{-\lambda t} + \theta |\mathrm{D}v|^2 e^{\lambda t} + v^2 e^{\lambda t} \right)\;\mathrm{d}x\;\mathrm{d}t.$$

Doing some integration by parts, we can also bound

$$B \leq \frac{c}{2} \int_{U_T} \left(u^2 e^{-\lambda t} + \theta |\mathrm{D}v|^2 e^{\lambda t} + v^2 e^{\lambda t} \right)\;\mathrm{d}x\;\mathrm{d}t,$$

where the constant c does not depend on λ. Taking this together, we have

$$\frac{\lambda - c}{2} \int_{U_T} \left(u^2 e^{-\lambda t} + \theta |\mathrm{D}v|^2 e^{\lambda t} + v^2 e^{\lambda t} \right)\;\mathrm{d}x\;\mathrm{d}t \leq 0.$$

Taking $\lambda > c$, this tells us the integral must vanish. In particular, the integral of $u^2 e^{\lambda t} = 0$. So $u = 0$. $\square$

Theorem (Existence of weak solution)**.** Given $\psi \in H_0^1(U)$ and $\psi' \in L^2(U)$, $f \in L^2(U_T)$, there exists a (unique) weak solution with

$$\|u\|_{H^1(U_T)} \leq C(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}). \tag{†}$$

Proof. We use *Galerkin's method*. The way we write our equations suggests we should think of our hyperbolic PDE as a second-order ODE taking values in the infinite-dimensional space $H_0^1(U)$. To apply the ODE theorems we know, we project our equation onto a finite-dimensional subspace, and then take the limit.

First note that by density arguments, we may assume $\psi, \psi' \in C_c^\infty(U)$ and $f \in C_c^\infty(U_T)$, as long as we prove the estimate (†). So let us do so.

Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(U)$, with $\varphi_k \in H_0^1(U)$. For example, we can take φ_k to be eigenfunctions of $-\Delta$ with Dirichlet boundary conditions.

We shall consider "solutions" of the form

$$u^N(x,t) = \sum_{k=1}^N u_k(t) \varphi_k(x).$$

We want this to be a solution after projecting to the subspace spanned by $\varphi_1, \ldots, \varphi_N$. Thus, we want $(u_{tt} + Lu - f, \varphi_k)_{L^2(\Sigma_t)} = 0$ for all $k = 1, \ldots, N$. After some integration by parts, we see that we want

$$\left(\ddot{u}^N, \varphi_k\right)_{L^2(U)} + \int_{\Sigma_t} \left(\sum a^{ij} u^N_{x_i} (\varphi_k)_{x_j} + b^i u^N_{x_i} \varphi_k + c u^N \varphi_k\right) \;\mathrm{d}x = (f, \varphi_k)_{L^2(U)}. \tag{$*$}$$

We also require

$$\begin{aligned} u_k(0) &= (\psi, \varphi_k)_{L^2(U)}\\ \dot{u}_k(0) &= (\psi', \varphi_k)_{L^2(U)}. \end{aligned}$$

Notice that if we have a genuine solution u that can be written as a finite sum of the $\varphi_k(x)$, then these must be satisfied.

This is a system of ODEs for the functions $u_k(t)$, and the RHS is uniformly C^1 in t and linear in the u_k's. By Picard–Lindelöf, a solution exists for $t \in [0, T]$.

So for each N, we have an approximate solution that solves the equation when projected onto $\langle \varphi_1, \ldots, \varphi_N\rangle$. What we need to do is to extract from this solution a genuine weak solution. To do so, we need some estimates to show that the functions u^N converge.

We multiply ($*$) by $e^{-\lambda t} \dot{u}_k(t)$, sum over $k = 1, \ldots, N$, and integrate from 0 to $\tau \in (0, T)$, and end up with

$$\begin{aligned} \int_0^\tau \;\mathrm{d}t \int_U \;\mathrm{d}x \left(\ddot{u}^N \dot{u}^N e^{-\lambda t} + \sum a^{ij} u^N_{x_i} \dot{u}^N_{x_j} + \sum b^i u^N_{x_i} \dot{u}^N + c u^N \dot{u}^N\right) e^{-\lambda t}&\\ = \int_0^\tau \;\mathrm{d}t \int_U \;\mathrm{d}u (f \dot{u}_N e^{-\lambda t})&. \end{aligned}$$

As before, we can rearrange this to get $A = B$, where

$$A = \int_{U_\tau} \mathrm{d}t\, \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} (\dot{u}^N)^2 + \frac{1}{2} \sum a^{ij} u^N_{x_i} u^N_{x_j} + \frac{1}{2} (u^N)^2 e^{-\lambda t} \right) \right.$$
$$\left. + \frac{\lambda}{2} \left((\dot{u}^N)^2 + \sum a^{ij} u^N_{x_i} u^N_{x_j} + (u^N)^2 \right) e^{-\lambda t} \right)$$

and

$$B = \int_{U_\tau} \mathrm{d}t\, \mathrm{d}x \left(\frac{1}{2} \sum \dot{a}^{ij} u^N_{x_i} u^N_{x_j} - \sum b^i u^N_{x_i} \dot{u}^N + (1 - c) u^N \dot{u}^N + f \dot{u}^N \right) e^{-\lambda t}.$$

Integrating in time, and estimating as before, for λ sufficiently large, we get

$$\frac{1}{2} \int_{\Sigma_\tau} \left((\dot{u}^N)^2 + |\mathrm{D}u^N|^2 \right) \mathrm{d}x + \int_{U_\tau} \left((\dot{u}^N)^2 + |\mathrm{D}u^N|^2 + (u^N)^2 \right) \mathrm{d}x\, \mathrm{d}t$$
$$\leq C(\|\psi\|^2_{H^1(U)} + \|\psi'\|^2_{L^2(U)} + \|f\|^2_{U_T}).$$

This, in particular, tells us u^N is bounded in $H^1(U_T)$,

Since $u^N(0) = \sum_{n=1}^N (\psi, \varphi_k) \varphi_k$, we know this tends to ψ in $H^1(U)$. So for N large enough, we have

$$\|u^N\|_{H^1(\Sigma_0)} \leq 2 \|\psi\|_{H^1(U)}.$$

Similarly, $\|\dot{u}^N\|_{L^2(\Sigma_0)} \leq 2 \|\psi'\|_{L^2(U)}$.

Thus, we can extract a convergent subsequence $u^{N_m} \rightharpoonup u$ in $H^1(U)$ for some $u \in H^1(U)$ such that

$$\|u\|_{H^1(U_T)} \leq C(\|\psi\|_{H^1(U)} + \|\psi\|_{L^2(U)} + \|f\|_{L^2(U_T)}).$$

For convenience, we may relabel the sequence so that in fact $u^N \rightharpoonup u$.

To check that u is a solution, suppose $v = \sum_{k=1}^M v_k(t) \varphi_k$ for some $v_k \in H^1((0, T))$ with $v_k(T) = 0$. By definition of u^N, we have

$$(\ddot{u}^N, v)_{L^2(U)} + \int_{\Sigma_t} \sum_{i,j} a^{ij} u^N_{x_i} v_{x_j} + \sum_i b^i u^N_{x_i} v + cuv\, \mathrm{d}x = (f, v)_{L^2(U)}.$$

Integrating $\int_0^T\, \mathrm{d}t$ using $v(T) = 0$, we have

$$\int_{U_T} \left(-u^N_t v_t + \sum {x_i}^N v_{x_j} + \sum b^i u^N_{x_i} v + cuv \right) \mathrm{d}x\, \mathrm{d}t - \int_{\Sigma_0} u^N_t v\, \mathrm{d}x$$
$$= \int_{U_T} fv\, \mathrm{d}x\, \mathrm{d}t.$$

Now note that if $N > M$, then $\int_{\Sigma_0} u^N_t v\, \mathrm{d}x = \int_{\Sigma_0} \psi' v\, \mathrm{d}x$. Now, passing to the weak limit, we have

$$\int_{U_T} \left(-u_t v_t + \sum a^{ij} u_{x_i} v_{x_j} + \sum b^i u_{x_i} v + cuv \right) \mathrm{d}x\, \mathrm{d}t - \int_{\Sigma_0} \psi' v\, \mathrm{d}x$$
$$= \int_{U_T} fv\, \mathrm{d}x\, \mathrm{d}t.$$

So u_t satisfies the identity required for u to be a weak solution.

Now for $k = 1, \ldots, M$, the map $w \in H^1(U_T) \mapsto \int_{\Sigma_0} w\varphi_k \, \mathrm{d}x$ is a bounded linear map, since the trace is bounded in L^2. So we conclude that

$$\int_{\Sigma_0} u\varphi_k \, \mathrm{d}x = \lim_{N \to \infty} \int_{\Sigma_0} u^N \varphi_k \, \mathrm{d}x = (\psi, \varphi_k)_{L^2(H)}.$$

Since this is true for all φ_k, it follows that $u|_{\Sigma_0} = \psi$, and v of the form considered are dense in $H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. So we are done. □

Theorem. If $a^{ij}, b^i, c \in C^2(U_T)$ and $\partial U \in C^2$, then for $\psi \in H^2(U)$ and $\psi' \in H_0^1(U)$, and $f, f_t \in L^2(U_T)$, we have

$$\begin{aligned} u &\in H^2(U_T) \cap L^\infty((0,T); H^2(U)) \\ u_t &\in L^\infty((0,T), H_0^1(U)) \\ u_{tt} &\in L^\infty((0,T); L^2(U)) \end{aligned}$$

Proof. We return to the Galerkin approximation. Now by assumption, we have a linear system with C^2 coefficients. So $u_k \in C^3((0,T))$. Differentiating with respect to t (assuming as we can $f, f_t \in C^0(\bar{U}_T)$), we have

$$\begin{aligned} (\partial_t^3 u^N, \varphi_k)_{L^2(U)} &+ \int_{\Sigma_t} \left(\sum a^{ij} \dot{u}^N_{x_i} (\varphi_k)_{x_j} + \sum b^i \dot{u}^N_{x_i} \varphi_k + c\dot{u}^N \varphi_k \right) \mathrm{d}x \\ &= (\dot{f}, \varphi_k)_{L^2(U)} - \int_{\Sigma_t} \left(\sum \dot{a}^{ij} u^N_{x_i} (\varphi_k)_{x_j} + \sum \dot{b}^i u^N_{x_i} \varphi_k + \dot{c} u \varphi_k \right) \mathrm{d}x. \end{aligned}$$

Multiplying by $\ddot{u}_k e^{-\lambda t}$, summing $k = 1, \ldots, N$, integrating $\int_0^\tau \mathrm{d}t$, and recalling we already control $u \in H^1(U_T)$, we get

$$\begin{aligned} \sup_{t \in (0,T)} &(\|u^N_t\|_{H^1(\Sigma_t)} + \|u^N_{tt}\|_{L^2(\Sigma_t)} + \|u^N_t\|_{H^2(U_T)}) \\ &\leq C\Big(\|u^N_t\|_{H^1(\Sigma_0)} + \|u^N_{tt}\|_{L^2(\Sigma_0)} + \|\psi\|_{H^1(\Sigma_0)} \\ &\qquad + \|\psi'\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)} \Big). \end{aligned}$$

We know

$$u^N_t|_{t=0} = \sum_{k=1}^N (\psi', \varphi_k)_{L^2(U)} \varphi_k.$$

Since φ_k are a basis for H^1, we have

$$\|u^N_t\|_{H^1(\Sigma_0)} \leq \|\psi'\|_{H^1(\Sigma_0)}.$$

To control u^N_{tt}, let us assume for convenience that in fact φ_k are the eigenfunctions $-\Delta$. From the fact that

$$(\ddot{u}^N, \varphi_k)_{L^2(U)} + \int_{\Sigma_t} \sum_{i,j} a^{ij} u^N_{x_i} (\varphi_k)_{x_j} + \sum_i b^i u^N_{x_i} \varphi_k + c u^N \varphi_k \, \mathrm{d}x \, \mathrm{d}t = (f, \varphi_k)_{L^2(U)},$$

integrate the first term in the integral by parts, multiply by $\ddot{u}_n$, and sum to get

$$\|u^N_{tt}\|_{\Sigma_0} \leq C(\|u^N\|_{H^2(\Sigma_0)} + \|f\|_{L^2(U_T)} + \|f_t\|_{L^2(U_T)}).$$

We need to control $\|u^N\|_{H^2(\Sigma_0)}$ by $\|\psi\|_{H^2(\Sigma_0)}$. Then, using that $\Delta\varphi_k|_{\partial U} = 0$ and u^N is a finite sum of these φ_k's,

$$(\Delta u^N, \Delta u^N)_{L^2(\Sigma_0)} = (u^N, \Delta^2 u^N)_{L^2(\Sigma_0)} = (\psi, \Delta^2 u^N)_{L^2(\Sigma_0)} = (\Delta\psi, \Delta u^N)_{L^2(\Sigma_0)}.$$

So

$$\|u^N\|_{H^2(\Sigma_0)} \leq \|\Delta u^N\|_{L^2(\Sigma_0)} \leq C\|\psi\|_H^2(U).$$

Passing to the weak limit, we conclude that

$$\begin{aligned} u_t &\in H^1(U_T) \\ u_t &\in L^\infty((0,T), H_0^1(U)) \\ u_{tt} &\in L^\infty((0,T), L^2(U)). \end{aligned}$$

Since $u_{tt} + Lu = f$, by an elliptic estimate on (almost) every constant t, we obtain $u \in L^\infty((0,T), H^2(U))$. □

Theorem. If $a^{ij}, b^i, c \in C^{k+1}(\bar{U}_T)$ and ∂U is C^{k+1}, and

$$\begin{aligned} \partial_t^i u|_{\Sigma_0} &\in H_0^1(U) && i = 0, \ldots, k \\ \partial_t^{k+1} u|_{\Sigma_0} &\in L^2(U) \\ \partial_t^i f &\in L^2((0,T); H^{k-i}(U)) && i = 0, \ldots, k \end{aligned}$$

then $u \in H^{k+1}(U)$ and

$$\partial_t^i u \in L^\infty((0,T); H^{k+1-i}(U))$$

for $i = 0, \ldots, k+1$.

In particular, if everything is smooth, then we get a smooth solution.

Theorem. If u is a weak solution of the usual thing, and S' is spacelike, then $u|_D$ depends only on $\psi|_{S_0}, \psi'|_{S_0}$ and $f|_D$.

Proof. Returning to the definition of a weak solution, we have

$$\int_{U_T} -u_t v_t + \sum_{i,j=1}^n a^{ij} u_{x_j} v_{x_i} + \sum_{i=1}^n b^i u_{x_i} + cuv \;\mathrm{d}x\;\mathrm{d}t - \int_{\Sigma_0} \psi' v \;\mathrm{d}x = \int_{U_T} fv \;\mathrm{d}x\;\mathrm{d}t.$$

By linearity it suffices to show that if $u|_{\Sigma_0} = 0$ if $\psi|_{S_0} = \psi'|_{S_0} = 0$ and $f|_D = 0$. We take as test function

$$v(t,x) = \begin{cases} \int_t^{\tau(x)} e^{-\lambda s} u(s,x)\;\mathrm{d}s & (t,x) \in D \\ 0 & (t,x) \not\in D \end{cases}.$$

One checks that this is in $H^1(U_T)$, and $v = 0$ on $\Sigma_T \cup \partial^* U_T$ with

$$\begin{aligned} v_{x_i} &= \tau_{x_i} e^{-\lambda\tau} u(x,\tau) + \int_t^{\tau(x)} e^{-\lambda s} u_{x_i}(x,s)\;\mathrm{d}s \\ v_t &= -e^{-\lambda t} u(x,t). \end{aligned}$$

Plugging these into the definition of a weak solution, we argue as in the previous uniqueness proof. Then

$$\begin{aligned}\int_D \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}u^2 e^{-\lambda t} - \frac{1}{2}\sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \frac{1}{2}v^2 e^{\lambda t}\right) \\ + \frac{\lambda}{2}\left(u^2 e^{-\lambda t} + \sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t}\right)\;\mathrm{d}x\;\mathrm{d}t \\ = \int_D \left(\frac{1}{2}\sum a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - \sum b^i v_{x_i} v - (c-1)uv\right)\;\mathrm{d}x\;\mathrm{d}t\end{aligned}$$

Noting that $\int_D \;\mathrm{d}x\;\mathrm{d}t = \int_{S_0}\;\mathrm{d}x\int_0^{\tau(x)}\;\mathrm{d}t$, we can perform the t integral of the $\frac{\mathrm{d}}{\mathrm{d}t}$ term, and we get contribution from S' which is given by

$$I_{S'} = \int_{S_0}\left(\frac{1}{2}u^2(\tau(x),x)e^{-\lambda\tau(x)} - \frac{1}{2}\sum_{i,j} a^{ij}\tau_{x_i}\tau_{x_j} u^2 e^{-\lambda\tau}\right)\;\mathrm{d}x$$

We have used $v = 0$ on S' and $v_{x_i} = \tau_{x_i} u e^{-\lambda\tau}$. Using the definition of a spacelike surface, we have $I_{S'} > 0$. The rest of the argument of the uniqueness of solutions goes through to conclude that $u = 0$ on D. $\square$