Part III — Analysis of Partial Differential Equations
Definitions

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This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to modern functional analytic techniques, relying on a priori estimates, rather than explicit solutions, although the interaction with classical methods (such as the fundamental solution and Fourier representation) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. Some non-linear equations may also be discussed. The course will end with a discussion of major open problems in PDEs.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Contents

0 Introduction

1 Basics of PDEs

Definition (Partial differential equation)**.** Suppose $U \subseteq \mathbb{R}^n$ is open. A *partial differential equation* (*PDE*) of *order* k is a relation of the form

$$F(x, u(x), \mathrm{D}u(x), \ldots, \mathrm{D}^k u(x)) = 0, \tag{$*$}$$

where $F : U \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$ is a given function, and $u : U \to \mathbb{R}$ is the "unknown".

Definition (Classical solution)**.** We say $u \in C^k(U)$ is a *classical solution* of a PDE if in fact the PDE is identically satisfied on U when $u, \mathrm{D}u, \ldots, \mathrm{D}^k u$ are substituted in.

Notation (Multi-index/Schwartz notation)**.** We say an element $\alpha \in \mathbb{N}^n$ is a *multi-index*. Writing $\alpha = (\alpha_1, \ldots, \alpha_n)$. We write

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Also, we have

$$\mathrm{D}^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

We also write

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!.$$

Definition (Linear PDE)**.** We say a PDE is *linear* if F is a linear function of u and its derivatives. In this case, we can re-write it as

$$\sum_{|\alpha| \leq k} a_\alpha(x) \mathrm{D}^\alpha u = 0.$$

Definition (Semi-linear PDE)**.** We say a PDE is *semi-linear* if it is of the form

$$\sum_{|\alpha| = k} a_\alpha(x) \mathrm{D}^\alpha u(x) + a_0[x, u, \mathrm{D}u, \ldots, \mathrm{D}^{k-1} u] = 0.$$

In other words, the terms involving the highest order derivatives are linear.

Definition (Quasi-linear PDE)**.** We say a PDE is *quasi-linear* if it is of the form

$$\sum_{|\alpha| = k} a_\alpha[x, u, \mathrm{D}u, \ldots, \mathrm{D}^{k-1} u] \mathrm{D}^\alpha u(x) + a_0[x, u, \ldots, \mathrm{D}^{k-1} u] = 0.$$

Definition (Fully non-linear PDE)**.** A PDE is *fully non-linear* if it is not quasi-linear.

2 The Cauchy–Kovalevskaya theorem

2.1 The Cauchy–Kovalevskaya theorem

Definition (Real analytic)**.** Let $U \subseteq \mathbb{R}^n$ be open, and suppose $f : U \to \mathbb{R}$. We say f is *real analytic* near $x_0 \in U$ if there exists $r > 0$ and constants $f_\alpha \in \mathbb{R}$ for each multi-index α such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha$$

for $|x - x_0| < r$.

Definition (Majorant)**.** Let

$$f = \sum_\alpha f_\alpha x^\alpha, \quad g = \sum_\alpha g_\alpha x^\alpha$$

be formal power series. We say g majorizes f (or g is a majorant of f), written $g \gg f$, if $g_\alpha \geq |f_\alpha|$ for all multi-indices α.

If f and A are vector-valued, then this means $g^i \gg f^i$ for all indices i.

2.2 Reduction to first-order systems

Definition (Real analytic hypersurface)**.** We say that $\Sigma \subseteq \mathbb{R}^n$ is a *real analytic hypersurface* near $x \in \Sigma$ if there exists $\varepsilon > 0$ and a real analytic map $\Phi : B_\varepsilon(x) \to U \subseteq \mathbb{R}^n$, where $U = \Phi(B_\varepsilon(x))$, such that

- Φ is bijective and $\Phi^{-1} : U \to B_\varepsilon(x)$ is real analytic.
- $\Phi(\Sigma \cap B_\varepsilon(x)) = \{x_n = 0\} \cap U$ and $\Phi(x) = 0$.

Definition ((Non-)characteristic surface)**.** A surface Σ is *non-characteristic* at $x\Sigma$ provided

$$\sum_{|\alpha|=k} a_\alpha (\mathrm{D}\Phi^n)^\alpha \neq 0.$$

Equivalently, if

$$\sum_{|\alpha|=k} a_\alpha \nu^\alpha \neq 0,$$

where ν is the normal to the surface. We say a surface is *characteristic* if it is not non-characteristic.

3 Function spaces

3.1 The Hölder spaces

Definition (C^k spaces)**.** Let $U \subseteq \mathbb{R}^n$ be an open set. We define $C^k(U)$ to be vector space of all $u : U \to \mathbb{R}$ such that u is k-times differentiable and the partial derivatives $\mathrm{D}^\alpha u : U \to \mathbb{R}$ are continuous for $|\alpha| \leq k$.

Definition ($C^k(\bar{U})$ spaces)**.** We define $C^k(\bar{U}) \subseteq C^k(U)$ to be the subspace of all u such that $\mathrm{D}^\alpha u$ are all bounded and uniformly continuous. We define a norm on $C^k(\bar{U})$ by

$$\|u\|_{C^k(\bar{U})} = \sum_{|\alpha| \leq k} \sup_{x \in U} \|\mathrm{D}^\alpha u(x)\|.$$

This makes $C^k(\bar{U})$ a Banach space.

Definition (Hölder continuity)**.** We say a function $u : U \to \mathbb{R}$ is *Hölder continuous* with index γ if there exists $C \geq 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\gamma$$

for all $x, y \in U$.

We write $C^{0,\gamma}(\bar{U}) \subseteq C^0(\bar{U})$ for the subspace of all Hölder continuous functions with index γ.

We define the γ-Hölder semi-norm by

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x \neq y \in U} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

We can then define a norm on $C^{0,\gamma}(\bar{U})$ by

$$\|u\|_{C^{(0,\gamma}(\bar{U})} = \|u\|_{C^0(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}.$$

We say $u \in C^{k,\gamma}(\bar{U})$ if $u \in C^k(\bar{U})$ and $\mathrm{D}^\alpha u \in C^{0,\gamma}(\bar{U})$ for all $|\alpha| = k$, and we define

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \|u\|_{C^k(\bar{U})} + \sum_{|\alpha| = k} [\mathrm{D}^\alpha u]_{C^{0,\gamma}(\bar{U})}.$$

This makes $C^{k,\gamma}(\bar{U})$ into a Banach space as well.

3.2 Sobolev spaces

Definition (L^p space)**.** Let $U \subseteq \mathbb{R}^n$ be open, and suppose $1 \leq p \leq \infty$. We define the space $L^p(U)$ by

$$L^p(U) = \{u : U \to \mathbb{R} \text{ measurable} \mid \|u\|_{L^p(U)} < \infty\}/\{\text{equality a.e.}\}.$$

where, if $p < \infty$, we define

$$\|u\|_{L^p(U)} = \left(\int_U |u(x)|^p \,\mathrm{d}x\right)^{1/p},$$

and

$$\|u\|_{L^\infty(U)} = \inf\{C \geq 0 \mid |u(x)| \leq C \text{ almost everywhere}\}.$$

Definition (Weak derivative)**.** Suppose $u, v \in L^1_{loc}(U)$ and α is a multi-index. We say that v is the αth weak derivative of u if

$$\int_U u\mathrm{D}^\alpha \phi \;\mathrm{d}x = (-1)^{|\alpha|} \int_U v\phi \;\mathrm{d}x$$

for all $\phi \in C_c^\infty(U)$, i.e. for all smooth, compactly supported function on U. We write $v = \mathrm{D}^\alpha u$.

Definition (Sobolev space)**.** We say that $u \in L^1_{loc}(U)$ belongs to the *Sobolev space* $W^{k,p}(U)$ if $u \in L^p(U)$ and $\mathrm{D}^\alpha u$ exists and is in $L^p(U)$ for all $|\alpha| \leq k$.

If $p = 2$, we write $H^k(U) = W^{k,2}(U)$, which will be a Hilbert space.

If $p < \infty$, we define the $W^{k,p}(U)$ norm by

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha|\leq k} \int_U |\mathrm{D}^\alpha u|^p \;\mathrm{d}x\right)^{1/p}.$$

If $p = \infty$, we define

$$\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha u\|_{L^\infty(U)}.$$

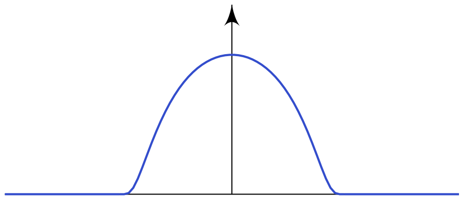
We denote by $W_0^{k,p}(U)$ the completion of $C_c^\infty(U)$ in this norm (and again $H_0^k(U) = W_0^{k,2}(U)$).

3.3 Approximation of functions in Sobolev spaces

Definition (Standard mollifier)**.** Let

$$\eta(x) = \begin{cases} Ce^{1/(|x|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases},$$

where C is chosen so that $\int_{\mathbb{R}^n} \eta(x) \;\mathrm{d}x = 1$.



One checks that this is a smooth function on $\mathbb{R}^n$, peaked at $x = 0$.

For each $\varepsilon > 0$, we set

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Of course, the pre-factor of $\frac{1}{\varepsilon^n}$ is chosen so that η_ε is appropriately normalized.

We call η_ε the *standard mollifier*, and it satisfies $\operatorname{supp}\eta_\varepsilon \subseteq \overline{B_\varepsilon(0)}$.

Definition (Mollification)**.** If $f \in L^1_{loc}(U)$, we define the *mollification* $f_\varepsilon : U_\varepsilon \to \mathbb{R}$ by the convolution

$$f_\varepsilon = \eta_\varepsilon * f.$$

In other words,

$$f_\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y) = \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y) f(y)\,\mathrm{d}y.$$

Definition ($C^{k,\delta}$ boundary)**.** Let $U \subseteq \mathbb{R}^n$ be open and bounded. We say ∂U is $C^{k,\delta}$ if for any point in the boundary $p \in \partial U$, there exists $r > 0$ and a function $\gamma \in C^{k,\delta}(\mathbb{R}^{n-1})$ such that (possibly after relabelling and rotating axes) we have

$$U \cap B_r(p) = \{(x', x_n) \in B_r(p) : x_n > \gamma(x')\}.$$

3.4 Extensions and traces

3.5 Sobolev inequalities

4 Elliptic boundary value problems

4.1 Existence of weak solutions

Definition (Uniform ellipticity)**.** An operator

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_j)_{x_i} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

is *uniformly elliptic* if

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for some $\theta > 0$ and all $x \in U, \xi \in \mathbb{R}^n$.

Definition (Weak solution)**.** We say $u \in H_0^1(U)$ is a *weak solution* of

$$\begin{aligned} Lu &= f \text{ on } U \\ u &= 0 \text{ on } \partial U \end{aligned}$$

for $f \in L^2(U)$ if

$$B[u, v] = (f, v)_{L^2(U)}$$

for all $v \in H_0^1(U)$.

4.2 The Fredholm alternative

Definition (Compact operator)**.** A bounded operator $K : H \to H'$ is *compact* if every bounded sequence $(u_m)_{m=1}^{\infty}$ has a subsequence u_{m_j} such that $(Ku_{m_j})_{j=1}^{\infty}$ converges strongly in H.

Definition (Weak convergence)**.** Suppose $(u_n)_{n=1}^{\infty}$ is a sequence in a Hilbert space H. We say u_n *converges weakly* to $u \in H$ if

$$(u_n, w) \to (u, w)$$

for all $w \in H$. We write $u_n \rightharpoonup u$.

4.3 The spectrum of elliptic operators

Definition (Resolvent set)**.** Let $A : H \to H$ be a bounded linear operator. Then the *resolvent set* is

$$\rho(A) = \{\lambda \in \mathbb{R} : A - \lambda I \text{ is bijective}\}.$$

Definition (Spectrum)**.** The *spectrum* of a bounded linear $A : H \to H$ is

$$\sigma(A) = \mathbb{R} \setminus \rho(A).$$

Definition (Point spectrum)**.** We say $\eta \in \sigma(A)$ belongs to the *point spectrum* of A if

$$\ker(A - \eta I) \neq \{0\}.$$

If $\eta \in \sigma_p(A)$ and w satisfies $Aw = \eta w$, then w is an *associated eigenvector*.

Definition (Formally self-adjoint)**.** An operator L is *formally self-adjoint* if $L = L^\dagger$. Equivalently, if $b^i \equiv 0$.

Definition (Positive operator)**.** We say L is *positive* if there exists $C > 0$ such that

$$\|u\|^2_{H^1_0(U)} \leq CB[u,u] \text{ for all } u \in H^1_0(U).$$

4.4 Elliptic regularity

Definition (Difference quotient)**.** Suppose $U \subseteq \mathbb{R}^n$ is open and $V \Subset U$. For $0 < |h| < \operatorname{dist}(V, \partial U)$, we define

$$\begin{aligned} \Delta_i^h u(x) &= \frac{u(x + he_i) - u(x)}{h} \\ \Delta^k u(x) &= (\Delta_1^h u, \ldots, \Delta_n^h u). \end{aligned}$$

5 Hyperbolic equations

Definition (Hyperbolic PDE)**.** A *second-order linear hyperbolic PDE* is a PDE of the form

$$\sum_{i,j=1}^{n+1} (a^{ij}(y) u_{y_j})_{y_i} + \sum_{i=1}^{n+1} b^i(y) u_{y_i} + c(y) u = f$$

with $y \in \mathbb{R}^{n+1}$, $a^{ij} = a^{ji}, b^i, c \in C^\infty(\mathbb{R}^{n+1})$, such that the *principal symbol*

$$Q(\xi) = \sum_{i,j=1}^{n+1} a^{ij}(y) \xi_i \xi_j$$

has signature $(+, -, -, \ldots)$ for all y. That is to say, after perhaps changing basis, at each point we can write

$$q(\xi) = \lambda_{n+1}^2 \xi_{n+1}^2 - \sum_{i=1}^{n} \lambda_i^2 \xi_i^2$$

with $\lambda_i > 0$.

Definition (Weak solution)**.** Suppose $f \in L^2(U_T)$, $\psi \in H_0^1(\Sigma_0)$ and $\psi' \in L^2(\Sigma_0)$. We say $u \in H^1(U_t)$ is a weak solution to the hyperbolic PDE if

(i) $u|_{\Sigma_0} = \psi$ in the trace sense;

(ii) $u|_{\partial^* U_T} = 0$ in the trace sense; and

(iii) (†) holds for all $v \in H^1(U_T)$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$ in a trace sense.