Part III — Algebraic Topology
Theorems with proof

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Algebraic Topology assigns algebraic invariants to topological spaces; it permeates modern pure mathematics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland’s book. The course will not assume any knowledge of Algebraic Topology, but will go quite fast in order to reach more interesting material, so some previous exposure to simplicial homology or the fundamental group would be helpful. The Part III Differential Geometry course will also contain useful, relevant material.

Hatcher’s book is especially recommended for the course, but there are many other suitable texts.
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1 Homotopy

Proposition. If \( f_0 \simeq f_1 : X \to Y \) and \( g_0 \simeq g_1 : Y \to Z \), then \( g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z \).
2 Singular (co)homology

2.1 Chain complexes

Lemma. If \( f : C \to D \) is a chain map, then \( f_\ast : H_\ast(C) \to H_\ast(D) \) given by \( [x] \mapsto [f_\ast(x)] \) is a well-defined homomorphism, where \( x \in C_\ast \) is any element representing the homology class \([x] \in H_\ast(C)\).

Proof. Before we check if it is well-defined, we first need to check if it is defined at all! In other words, we need to check if \( f_\ast(x) \) is a cycle. Suppose \( x \in C_\ast \) is a cycle, i.e. \( \partial_\ast(x) = 0 \). Then we have

\[
d^n(f_\ast(x)) = f_\ast(\partial^n(x)) = f_\ast(0) = 0.
\]

So \( f_\ast(x) \) is a cycle, and it does represent a homology class.

To check this is well-defined, if \([x] = [y] \in H_\ast(C)\), then \( x - y = \partial_{\ast+1}(z) \) for some \( z \in C_{\ast+1} \). So \( f_\ast(x) - f_\ast(y) = f_\ast(\partial_{\ast+1}(z)) = \partial_{\ast+1}(f_\ast(z)) \) is a boundary. So we have \([f_\ast(x)] = [f_\ast(y)] \in H_\ast(D)\). \( \square \)

2.2 Singular (co)homology

Lemma. If \( i < j \), then \( \delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} : \Delta^{n-2} \to \Delta^n \).

Proof. Both send \((t_0, \cdots, t_{n-2})\) to \((t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{j-2}, 0, t_{j-1}, \cdots, t_{n-2})\). \( \square \)

Corollary. The homomorphism \( d_{n-1} \circ d_n : C_n(X) \to C_{n-2}(X) \) vanishes.

Proof. It suffices to check this on each basis element \( \sigma : \Delta^n \to X \). We have

\[
d_{n-1} \circ d_n(\sigma) = \sum_{i=0}^{n-1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \delta_j \circ \delta_i.
\]

We use the previous lemma to split the sum up into \( i < j \) and \( i \geq j \):

\[
= \sum_{i < j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i
= \sum_{i < j} (-1)^{i+j} \sigma \circ \delta_i \circ \delta_{j-1} + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i
= \sum_{i \leq j} (-1)^{i+j+1} \sigma \circ \delta_i \circ \delta_j + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i
= 0.
\]

\( \square \)

Proposition. If \( f : X \to Y \) is a continuous map of topological spaces, then the maps

\[
f_\ast : C_\ast(X) \to C_\ast(Y)
(\sigma : \Delta^n \to X) \mapsto (f \circ \sigma : \Delta^n \to Y)
\]

give a chain map. This induces a map on the homology (and cohomology).
Proof. To see that the $f_n$ and $d_n$ commute, we just notice that $f_n$ acts by composing on the left, and $d_n$ acts by composing on the right, and these two operations commute by the associativity of functional composition.

**Proposition.** If $f : X \rightarrow Y$ is a homeomorphism, then $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism of abelian groups.

**Proof.** If $g : Y \rightarrow X$ is an inverse to $f$, then $g_*$ is an inverse to $f_*$, as $f_* \circ g_* = (f \circ g)_* = (\text{id})_* = \text{id}$, and similarly the other way round.

**Lemma.** If $X$ is path-connected and non-empty, then $H_0(X) \cong \mathbb{Z}$.

**Proof.** Define a homomorphism $\varepsilon : C_0(X) \rightarrow \mathbb{Z}$ given by 

$$\sum n_{\sigma} \sigma \mapsto \sum n_{\sigma}.$$

Then this is surjective. We claim that the composition

$$C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is zero. Indeed, each simplex has two ends, and a $\sigma : \Delta^1 \rightarrow X$ is mapped to $\sigma \circ \delta_0 - \sigma \circ \delta_1$, which is mapped by $\varepsilon$ to $1 - 1 = 0$.

Thus, we know that $\varepsilon(\sigma) = \varepsilon(\sigma + dr)$ for any $\sigma \in C_0(X)$ and $r \in C_1(X)$. So we obtain a well-defined map $\varepsilon : H_0(X) \rightarrow \mathbb{Z}$ mapping $[x] \mapsto \varepsilon(x)$, and this is surjective as $X$ is non-empty.

So far, this is true for a general space. Now we use the path-connectedness condition to show that this map is indeed injective. Suppose $\sum n_{\sigma} \sigma \in C_0(X)$ lies in ker $\varepsilon$. We choose an $x_0 \in X$. As $X$ is path-connected, for each of $\Delta^0 \rightarrow X$ we can choose a path $\tau_\sigma : \Delta^1 \rightarrow X$ with $\tau_\sigma \circ \delta_0 = \sigma$ and $\tau_\sigma \circ \delta_1 = x_0$.

Given these 1-simplices, we can form a 1-chain $\sum n_{\sigma} \tau_\sigma \in C_1(X)$, and

$$d_1 \left( \sum n_{\sigma} \tau_\sigma \right) = \sum n_{\sigma} (\sigma + x_0) = \sum n_{\sigma} \cdot \sigma - \left( \sum n_{\sigma} \right) x_0.$$

Now we use the fact that $\sum n_{\sigma} = 0$. So $\sum n_{\sigma} \cdot \sigma$ is a boundary. So it is zero in $H_0(X)$.

**Proposition.** For any space $X$, we have $H_0(X)$ is a free abelian group generated by the path components of $X$. 

3 Four major tools of (co)homology

3.1 Homotopy invariance

**Theorem** (Homotopy invariance theorem). Let \( f \simeq g : X \to Y \) be homotopic maps. Then they induce the same maps on (co)homology, i.e.

\[
f_* = g_* : H_*(X) \to H_*(Y)
\]

and

\[
f^* = g^* : H^*(Y) \to H^*(X).
\]

**Corollary.** If \( f : X \to Y \) is a homotopy equivalence, then \( f_* : H_*(X) \to H_*(Y) \) and \( f^* : H^*(Y) \to H^*(X) \) are isomorphisms.

**Proof.** If \( g : Y \to X \) is a homotopy inverse, then

\[
g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}_{H_*(X)}.
\]

Similarly, we have \( f_* \circ g_* = (\text{id}_Y)_* = \text{id}_{H_*(Y)} \). So \( f_* \) is an isomorphism with an inverse \( g_* \).

The case for cohomology is similar. \( \square \)

3.2 Mayer-Vietoris

**Lemma.** In a short exact sequence

\[
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,
\]

the map \( f \) is injective; \( g \) is surjective, and \( C \cong B/A \).

**Theorem** (Mayer-Vietoris theorem). Let \( X = A \cup B \) be the union of two open subsets. We have inclusions

\[
A \cap B \xhookleftarrow{i_A} A \xrightarrow{j_A} X \quad \text{and} \quad B \xhookleftarrow{i_B} B \xrightarrow{j_B} X
\]

Then there are homomorphisms \( \partial_{MV} : H_n(X) \to H_{n-1}(A \cap B) \) such that the following sequence is exact:

\[
\begin{array}{ccccccc}
\partial_{MV} & H_n(A \cap B) & \xrightarrow{i_A_* \oplus i_B_*} & H_n(A) \oplus H_n(B) & \xrightarrow{j_A^* - j_B^*} & H_n(X) \\
& \xrightarrow{\partial_{MV}} & & \xrightarrow{\partial_{MV}} & & \\
& H_{n-1}(A \cap B) & \xrightarrow{i_A^* \oplus i_B^*} & H_{n-1}(A) \oplus H_{n-1}(B) & \xrightarrow{j_A^* - j_B^*} & H_{n-1}(X) & \cdots \\
& & & \xrightarrow{\partial_{MV}} & & \\
& \cdots & & H_0(A) \oplus H_0(B) & \xrightarrow{j_A^* - j_B^*} & H_0(X) & \longrightarrow 0
\end{array}
\]
Furthermore, the Mayer-Vietoris sequence is natural, i.e. if \( f : X = A \cup B \to Y = U \cup V \) satisfies \( f(A) \subseteq U \) and \( f(B) \subseteq V \), then the diagram

\[
\begin{array}{c}
H_{n+1}(X) \xrightarrow{\partial_{MV}} H_n(A \cap B) \xrightarrow{i_A^* \oplus i_B^*} H_n(A) \oplus H_n(B) \xrightarrow{j_A^* - j_B^*} H_n(X) \\
\downarrow f_* & \downarrow f_{|A \cap B*} & \downarrow f_{|A* \oplus f|B*} & \downarrow f_*
\end{array}
\]

\[
H_{n+1}(Y) \xrightarrow{\partial_{MV}} H_n(U \cap V) \xrightarrow{i_U^* \oplus i_V^*} H_n(U) \oplus H_n(V) \xrightarrow{j_U^* - j_V^*} H_n(Y)
\]

commutes.

### 3.3 Relative homology

**Theorem** (Exact sequence for relative homology). There are homomorphisms \( \partial : H_n(X, A) \to H_{n-1}(A) \) given by mapping

\[
\left[ [c] \right] \mapsto [d_n c].
\]

This makes sense because if \( c \in C_n(X) \), then \( [c] \in C_n(X)/C_n(A) \). We know \( [d_n c] = 0 \in C_{n-1}(X)/C_{n-1}(A) \). So \( d_n c \in C_{n-1}(A) \). So this notation makes sense.

Moreover, there is a long exact sequence

\[
\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{q_*} H_{n-1}(X, A) \xrightarrow{\partial} \cdots
\]

\[
\cdots \xrightarrow{} H_0(X) \xrightarrow{q_*} H_0(X, A) \xrightarrow{} 0
\]

where \( i_* \) is induced by \( i : C_*(A) \to C_*(X) \) and \( q_* \) is induced by the quotient \( q : C_*(X) \to C_*(X, A) \).

### 3.4 Excision theorem

**Theorem** (Excision theorem). Let \( (X, A) \) be a pair of spaces, and \( Z \subseteq A \) be such that \( \overline{Z} \subseteq \overline{A} \) (the closure is taken in \( X \)). Then the map

\[
H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)
\]

is an isomorphism.

### 3.5 Applications

**Theorem.** We have

\[
H_i(S^1) = \begin{cases} 
Z & i = 0, 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** We can split \( S^1 \) up as
We want to apply Mayer-Vietoris. We have

\[ A \cong B \cong \mathbb{R} \cong \{p\} \bigcup \{q\}. \]

We obtain

\[
\begin{array}{cccc}
0 & \cdots & H_1(A \cap B) & H_1(A) \oplus H_1(B) \oplus H_1(S^1) \\
\downarrow \cong & & \downarrow \cong & \downarrow \\
H_0(A \cap B) & \oplus & H_0(A) \oplus H_0(B) & H_0(S^1) \rightarrow 0 \\
\cong \mathbb{Z} \oplus \mathbb{Z} & \oplus & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow \mathbb{Z}
\end{array}
\]

Notice that the map into \( H_1(S^1) \) is zero. So the kernel of \( \partial \) is trivial, i.e. \( \partial \) is an injection. So \( H_1(S^1) \) is isomorphic to the image of \( \partial \), which is, by exactness, the kernel of \( i_{A*} \oplus i_{B*} \). So we want to know what this map does.

We know that \( H_0(A \cap B) \cong \mathbb{Z} \oplus \mathbb{Z} \) is generated by \( p \) and \( q \), and the inclusion map sends each of \( p \) and \( q \) to the unique connected components of \( A \) and \( B \). So the homology classes are both sent to \( (1, 1) \in H_0(A) \oplus H_0(B) \cong \mathbb{Z} \oplus \mathbb{Z} \). We then see that the kernel of \( i_{A*} \oplus i_{B*} \) is generated by \( (p - q) \), and is thus isomorphic to \( \mathbb{Z} \). So \( H_1(S^1) \cong \mathbb{Z} \).

By looking higher up the exact sequence, we see that all other homology groups vanish.

\[ \square \]

**Theorem.** For any \( n \geq 1 \), we have

\[
H_i(S^n) = \begin{cases} 
\mathbb{Z} & i = 0, n \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** We again cut up \( S^n \) as

\[
A = S^n \setminus \{N\} \cong \mathbb{R}^n \cong *, \\
B = S^n \setminus \{S\} \cong \mathbb{R}^n \cong *,
\]

where \( N \) and \( S \) are the north and south poles. Moreover, we have

\[
A \cap B \cong \mathbb{R} \times S^{n-1} \cong S^{n-1}
\]

So we can “induct up” using the Mayer-Vietoris sequence:

\[
\begin{array}{cccc}
\cdots & H_i(S^{n-1}) & H_i(*) \oplus H_i(*) & H_i(S^n) \\
\downarrow & \downarrow \cong & \downarrow \cong & \downarrow \\
H_{i-1}(S^{n-1}) & H_{i-1}(*) \oplus H_{i-1}(*) & H_{i-1}(S^n) & \cdots
\end{array}
\]
Now suppose \( n \geq 2 \), as we already did \( S^1 \) already. If \( i > 1 \), then \( H_i(\ast) = 0 = H_{i-1}(\ast) \). So the Mayer-Vietoris map

\[
\begin{array}{c}
H_i(S^n) \\
\end{array} \xrightarrow{\partial} H_{i-1}(S^{n-1})
\]

is an isomorphism.

All that remains is to look at \( i = 0, 1 \). The \( i = 0 \) case is trivial. For \( i = 1 \), we look at

\[
\begin{array}{cccccccccccc}
0 & \rightarrow & H_1(\ast) & \oplus & H_1(\ast) & \rightarrow & H_1(S^n) & \rightarrow & 0 \\
\rightarrow & & f & | & H_0(S^{n-1}) & \rightarrow & H_0(\ast) & \oplus & H_0(\ast) & \rightarrow & H_0(S^n) & \rightarrow & 0 \\
& & & & \downarrow{0} & & \downarrow{1} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} \\
& & & & Z & & Z \oplus Z & & Z & & Z & & Z \\
\end{array}
\]

To conclude that \( H_1(S^n) \) is trivial, it suffices to show that the map \( f \) is injective. By picking the canonical generators, it is given by \( 1 \mapsto (1, 1) \). So we are done. \( \square \)

**Corollary.** If \( n \neq m \), then \( S^{n-1} \not\simeq S^{m-1} \), since they have different homology groups.

**Corollary.** If \( n \neq m \), then \( \mathbb{R}^n \not\simeq \mathbb{R}^m \).

**Proposition.**

(i) \( \deg(id_{S^n}) = 1 \).

(ii) If \( f \) is not surjective, then \( \deg(f) = 0 \).

(iii) We have \( \deg(f \circ g) = (\deg f)(\deg g) \).

(iv) Homotopic maps have equal degrees.

**Proof.**

(i) Obvious.

(ii) If \( f \) is not surjective, then \( f \) can be factored as

\[
S^n \xrightarrow{f} S^n \setminus \{p\} \xrightarrow{\partial} S^n,
\]

where \( p \) is some point not in the image of \( f \). But \( S^n \setminus \{p\} \) is contractible. So \( f_* \) factors as

\[
f_* : H_n(S^n) \rightarrow H_n(\ast) = 0 \rightarrow H_n(S^n).
\]

So \( f_* \) is the zero homomorphism, and is thus multiplication by 0.

(iii) This follows from the functoriality of \( H_n \).

(iv) Obvious as well. \( \square \)
Corollary (Brouwer’s fixed point theorem). Any map \( f : D^n \to D^n \) has a fixed point.

*Proof.* Suppose \( f \) has no fixed point. Define \( r : D^n \to S^{n-1} = \partial D^n \) by taking the intersection of the ray from \( f(x) \) through \( x \) with \( \partial D^n \). This is continuous.

Now if \( x \in \partial D^n \), then \( r(x) = x \). So we have a map

\[
S^{n-1} = \partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n = S^{n-1},
\]

and the composition is the identity. This is a contradiction — contracting \( D^n \) to a point, this gives a homotopy from the identity map \( S^{n-1} \to S^{n-1} \) to the constant map at a point. This is impossible, as the two maps have different degrees.

Proposition. A reflection \( r : S^n \to S^n \) about a hyperplane has degree \(-1\). As before, we cover \( S^n \) by

\[
A = S^n \setminus \{N\} \cong \mathbb{R}^n \simeq *,
\]

\[
B = S^n \setminus \{S\} \cong \mathbb{R}^n \simeq *,
\]

where we suppose the north and south poles lie in the hyperplane of reflection. Then both \( A \) and \( B \) are invariant under the reflection. Consider the diagram

\[
\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\partial_{MV}} & H_{n-1}(A \cap B) \\
| & & | \\
\downarrow{r_*} & & \downarrow{r_*} \\
H_n(S^n) & \xrightarrow{\partial_{MV}} & H_{n-1}(A \cap B) \\
\end{array}
\]

where the \( S^{n-1} \) on the right most column is given by contracting \( A \cap B \) to the equator. Note that \( r \) restricts to a reflection on the equator. By tracing through the isomorphisms, we see that \( \deg(r) = \deg(r|_{\text{equator}}) \). So by induction, we only have to consider the case when \( n = 1 \). Then we have maps

\[
\begin{array}{ccc}
0 & \to & H_1(S^1) \xrightarrow{\partial_{MV}} H_0(A \cap B) \\
\downarrow{r_*} & & \downarrow{r_*} \\
0 & \to & H_1(S^1) \xrightarrow{\partial_{MV}} H_0(A \cap B) \\
\end{array}
\]

Now the middle vertical map sends \( p \mapsto q \) and \( q \mapsto p \). Since \( H_1(S^1) \) is given by the kernel of \( H_0(A \cap B) \to H_0(A) \oplus H_0(B) \), and is generated by \( p - q \), we see that this sends the generator to its negation. So this is given by multiplication by \(-1\). So the degree is \(-1\).
Corollary. The antipodal map $a : S^n \to S^n$ given by
\[ a(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, -x_{n+1}) \]
has degree $(-1)^{n+1}$ because it is a composition of $(n+1)$ reflections.

Corollary (Hairy ball theorem). $S^n$ has a nowhere 0 vector field if $n$ is odd. More precisely, viewing $S^n \subseteq \mathbb{R}^{n+1}$, a vector field on $S^n$ is a map $v : S^n \to \mathbb{R}^{n+1}$ such that $\langle v(x), x \rangle = 0$, i.e. $v(x)$ is perpendicular to $x$.

Proof. If $n$ is odd, say $n = 2k - 1$, then
\[ v(x_1, y_1, x_2, y_2, \cdots, x_k, y_k) = (y_1, -x_1, y_2, -x_2, \cdots, y_k, -x_k) \]
works.

Conversely, if $v : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a vector field, we let $w = \frac{v}{|v|} : S^n \to S^n$. We can construct a homotopy from $w$ to the antipodal map by “linear interpolation”, but in a way so that it stays on the sphere. We let
\[ H : [0, \pi] \times S^n \to S^n \]
\[ (t, x) \mapsto \cos(t)x + \sin(t)w(x) \]
Since $w(x)$ and $x$ are perpendicular, it follows that this always has norm 1, so indeed it stays on the sphere.

Now we have
\[ H(0, x) = x, \quad H(\pi, x) = -x. \]
So this gives a homotopy from id to $a$, which is a contradiction since they have different degrees. \qed

Lemma. Let $M$ be a $d$-dimensional manifold (i.e. a Hausdorff, second-countable space locally homeomorphic to $\mathbb{R}^d$). Then
\[ H_n(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & n = d \\ 0 & \text{otherwise} \end{cases}. \]
This is known as the local homology.

Proof. Let $U$ be an open neighbourhood isomorphic to $\mathbb{R}^d$ that maps $x \mapsto 0$. We let $Z = M \setminus U$. Then
\[ Z = Z \subseteq M \setminus \{x\} = (M \setminus \{x\}). \]
So we can apply excision, and get
\[ H_n(U, U \setminus \{x\}) = H_n(M \setminus Z, (M \setminus \{x\}) \setminus Z) \cong H_n(M, M \setminus \{x\}). \]
So it suffices to do this in the case $M \cong \mathbb{R}^d$ and $x = 0$. The long exact sequence for relative homology gives
\[ H_n(\mathbb{R}^d) \to H_n(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \to H_{n-1}(\mathbb{R}^d \setminus \{0\}) \to H_{n-1}(\mathbb{R}^d). \]
Since $H_n(\mathbb{R}^d) = H_{n-1}(\mathbb{R}^d) = 0$ for $n \geq 2$ large enough, it follows that
\[ H_n(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \cong H_{n-1}(\mathbb{R}^d \setminus \{0\}) \cong H_{n-1}(S^{d-1}), \]
and the result follows from our previous computation of the homology of $S^{d-1}$. We will have to check the lower-degree terms manually, but we shall not. \qed
Theorem. Let \( f : S^d \to S^d \) be a map. Suppose there is a \( y \in S^d \) such that 
\[
f^{-1}(y) = \{x_1, \cdots, x_k\}
\]
is finite. Then 
\[
\deg(f) = \sum_{i=1}^k \deg(f)_{x_i}.
\]

Proof. Note that by excision, instead of computing the local degree at \( x_i \) via 
\[H_d(S^d, S^d \setminus \{x_i\}) \hookrightarrow H_d(S^d, S^d \setminus \{x_i, \cdots, x_k\})\]
we can pick a neighbourhood \( U_i \) of \( x_i \) and a neighbourhood \( V \) of \( x_i \) such that \( f(U_i) \subseteq V \), and then look at the map 
\[f_* : H_d(U_i, U_i \setminus \{x_i\}) \to H_d(V, V \setminus \{y\})\]
instead. Moreover, since \( S^d \) is Hausdorff, we can pick the \( U_i \) such that they are disjoint. Consider the huge commutative diagram:

\[
\begin{array}{ccc}
H_d(S^d) & \xrightarrow{f_*} & H_d(S^d) \\
\downarrow & & \downarrow \\
H_d(S^d, S^d \setminus \{x_1, \cdots, x_k\}) & \xrightarrow{f_*} & H_d(S^d, S^d \setminus \{y\})
\end{array}
\]

\[\xrightarrow{\text{excision}}\]

\[
\begin{array}{ccc}
& & \\
H_d(\coprod U_i, \coprod(U_i \setminus \{x_i\})) & \xrightarrow{\sim} & \coprod_{i=1}^k H_d(U_i, U_i \setminus \{x_i\}) \\
\uparrow & & \uparrow \\
& & \\
\oplus_{i=1}^k H_d(U_i, U_i \setminus \{x_i\}) & \xrightarrow{\oplus f_*} & H_d(V, V \setminus \{y\})
\end{array}
\]

Consider the generator 1 of \( H_d(S^d) \). By definition, its image in \( H_d(S^d) \) is \( \deg(f) \). Also, its image in \( \oplus H_d(U_i, U_i \setminus \{x_i\}) \) is \((1, \cdots, 1)\). The bottom horizontal map sends this to \( \sum \deg(f)_{x_i} \). So by the isomorphisms, it follows that 
\[
\deg(f) = \sum_{i=1}^k \deg(f)_{x_i}. \quad \square
\]

3.6 Repaying the technical debt

Theorem (Snake lemma). Suppose we have a short exact sequence of complexes  
\[
0 \leftrightarrow A \xrightarrow{i_*} B \xrightarrow{q_*} C \leftrightarrow 0.
\]

Then there are maps 
\[\partial : H_n(C) \rightarrow H_{n-1}(A)\]
such that there is a long exact sequence 
\[
\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{q_*} H_n(C) \rightarrow H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{q_*} H_{n-1}(C) \rightarrow \cdots
\]
Proof. The proof of this is in general not hard. It just involves a lot of checking of the details, such as making sure the homomorphisms are well-defined, are actually homomorphisms, are exact at all the places etc. The only important and non-trivial part is just the construction of the map $\partial_*$. 

First we look at the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & A_n & \rightarrow^i & B_n & \rightarrow^q & C_n & \rightarrow & 0 \\
0 & \rightarrow & A_{n-1} & \rightarrow^i & B_{n-1} & \rightarrow^q & C_{n-1} & \rightarrow & 0 \\
\end{array}
\]

To construct $\partial_* : H_n(C) \rightarrow H_{n-1}(A)$, let $[x] \in H_n(C)$ be a class represented by $x \in Z_n(C)$. We need to find a cycle $z \in A_{n-1}$. By exactness, we know the map $q_n : B_n \rightarrow C_n$ is surjective. So there is a $y \in B_n$ such that $q_n(y) = x$. Since our target is $A_{n-1}$, we want to move down to the next level. So consider $d_{n-1}(y) \in B_{n-1}$. We would be done if $d_{n-1}(y)$ is in the image of $i_{n-1}$. By exactness, this is equivalent saying $d_{n-1}(y)$ is in the kernel of $q_{n-1}$. Since the diagram is commutative, we have

\[
d_{n-1} \circ d_{n-1}(y) = 0.
\]

We want to check that $d_{n-1}(z) = 0$. We will use the commutativity of the diagram. In particular, we know

\[
i_{n-2} \circ d_{n-1}(z) = d_{n-1} \circ i_{n-1}(z) = d_{n-1} \circ d_{n}(y) = 0.
\]
By exactness at $A_{n-2}$, we know $i_{n-2}$ is injective. So we must have $d_{n-1}(z) = 0$.

(iii) (a) First, in the proof, suppose we picked a different $y'$ such that $q_n(y') = q_n(y) = x$. Then $q_n(y' - y) = 0$. So $y' - y \in \ker q_n = \text{im} i_n$. Let $a \in A_n$ be such that $i_n(a) = y' - y$. Then

$$d_n(y') = d_n(y' - y) + d_n(y) = d_n \circ i_n(a) + d_n(y) = i_{n-1} \circ d_n(a) + d_n(y).$$

Hence when we pull back $d_n(y')$ and $d_n(y)$ to $A_{n-1}$, the results differ by the boundary $d_n(a)$, and hence produce the same homology class.

(b) Suppose $[x'] = [x]$. We want to show that $\partial_n[x] = \partial_n[x']$. This time, we add a layer above.

$$
\begin{array}{cccccccc}
0 & \rightarrow & A_{n+1} & \overset{i_{n+1}}{\rightarrow} & B_{n+1} & \overset{q_{n+1}}{\rightarrow} & C_{n+1} & \rightarrow & 0 \\
 & d_{n+1} & \downarrow & d_{n+1} & \downarrow & d_{n+1} & \downarrow & & \\
0 & \rightarrow & A_n & \overset{i_n}{\rightarrow} & B_n & \overset{q_n}{\rightarrow} & C_n & \rightarrow & 0 \\
 & d_n & \downarrow & d_n & \downarrow & d_n & \downarrow & & \\
0 & \rightarrow & A_{n-1} & \overset{i_{n-1}}{\rightarrow} & B_{n-1} & \overset{q_{n-1}}{\rightarrow} & C_{n-1} & \rightarrow & 0
\end{array}
$$

By definition, since $[x'] = [x]$, there is some $c \in C_{n+1}$ such that

$$x' = x + d_{n+1}(c).$$

By surjectivity of $q_{n+1}$, we can write $c = q_{n+1}(b)$ for some $b \in B_{n+1}$. By commutativity of the squares, we know

$$x' = x + q_n \circ d_{n+1}(b).$$

The next step of the proof is to find some $y$ such that $q_n(y) = x$. Then

$$q_n(y + d_{n+1}(b)) = x'.$$

So the corresponding $y'$ is $y' = y + d_{n+1}(b)$. So $d_n(y) = d_n(y')$, and hence $\partial_n[x] = \partial_n[x']$.

(iv) This is yet another standard diagram chasing argument. When reading this, it is helpful to look at a diagram and see how the elements are chased along. It is even more beneficial to attempt to prove this yourself.

(a) $\text{im} i_n \subseteq \ker q_n$: This follows from the assumption that $i_n \circ q_n = 0$.

(b) $\ker q_n \subseteq \text{im} i_n$: Let $[b] \in H_n(B)$. Suppose $q_n([b]) = 0$. Then there is some $c \in C_{n+1}$ such that $q_n(b) = d_{n+1}(c)$. By surjectivity of $q_{n+1}$, there is some $b' \in B_{n+1}$ such that $q_{n+1}(b') = c$. By commutativity, we know $q_n(b) = q_n \circ d_{n+1}(b')$, i.e.

$$q_n(b - d_{n+1}(b')) = 0.$$
By exactness of the sequence, we know there is some \( a \in A_n \) such that

\[ i_n(a) = b - d_{n+1}(b'). \]

Moreover,

\[ i_{n-1} \circ d_n(a) = d_n \circ i_n(a) = d_n(b - d_{n+1}(b')) = 0, \]

using the fact that \( b \) is a cycle. Since \( i_{n-1} \) is injective, it follows that \( d_n(a) = 0 \). So \([a] \in H_n(A)\). Then

\[ i_*([a]) = [b] - [d_{n+1}(b')] = [b]. \]

So \([b] \in \text{im } i_*\).

(c) \( \text{im } q_* \subseteq \ker \partial_* \). Let \([b] \in H_n(B)\). To compute \( \partial_*(q_*([b])) \), we first pull back \( q_*(b) \) to \( b \in B_n \). Then we compute \( d_n(b) \) and then pull it back to \( A_{n+1} \). However, we know \( d_n(b) = 0 \) since \( b \) is a cycle. So \( \partial_*(q_*([b])) = 0 \), i.e. \( \partial_* \circ q_* = 0 \).

(d) \( \ker \partial_* \subseteq \text{im } q_* \). Let \([c] \in H_n(C)\) and suppose \( \partial_*(c) = 0 \). Let \( b \in B_n \) be such that \( q_n(b) = c \), and \( a \in A_{n-1} \) such that \( i_{n-1}(a) = d_n(b) \). By assumption, \( \partial_*(c) = [a] = 0 \). So we know \( a \) is a boundary, say \( a = d_n(a') \) for some \( a' \in A_n \). Then by commutativity we know \( d_n(b) = d_n \circ i_n(a') \). In other words,

\[ d_n(b - i_n(a')) = 0. \]

So \([b - i_n(a')] \in H_n(B)\). Moreover,

\[ q_*(b - i_n(a')) = [q_n(b) - q_n \circ i_n(a')] = [c]. \]

So \([c] \in \text{im } q_*\).

(e) \( \text{im } \partial_* \subseteq \ker i_* \). Let \([c] \in H_n(C)\). Let \( b \in B_n \) be such that \( q_n(b) = c \), and \( a \in A_{n-1} \) be such that \( i_n(a) = d_n(b) \). Then \( \partial_*(c) = [a] \). Then

\[ i_*([a]) = [i_n(a)] = [d_n(b)] = 0. \]

So \( i_* \circ \partial_* = 0 \).

(f) \( \ker i_* \subseteq \text{im } \partial_* \). Let \([a] \in H_n(A)\) and suppose \( i_*([a]) = 0 \). So we can find some \( b \in B_{n+1} \) such that \( i_n(a) = d_{n+1}(b) \). Let \( c = q_{n+1}(b) \). Then

\[ d_{n+1}(c) = d_{n+1} \circ q_{n+1}(b) = q_n \circ d_{n+1}(b) = q_n \circ i_n(a) = 0. \]

So \([c] \in H_n(C)\). Then \([a] = \partial_*(c)\) by definition of \( \partial_* \). So \([a] \in \text{im } \partial_*\). \( \square \)

**Lemma** (Five lemma). Consider the following commutative diagram:

\[
\begin{array}{ccccccc}
A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & C & \overset{h}{\longrightarrow} & D & \overset{j}{\longrightarrow} & E \\
\downarrow{j} & & \downarrow{m} & & \downarrow{n} & & \downarrow{p} & & \downarrow{q} \\
A' & \overset{r}{\longrightarrow} & B' & \overset{s}{\longrightarrow} & C' & \overset{t}{\longrightarrow} & D' & \overset{u}{\longrightarrow} & E'
\end{array}
\]

If the two rows are exact, \( m \) and \( p \) are isomorphisms, \( q \) is injective and \( \ell \) is surjective, then \( u \) is also an isomorphism.
Proof. The philosophy is exactly the same as last time.
We first show that \( n \) is surjective. Let \( c' \in C' \). Then we obtain \( d' = t(c') \in D' \).
Since \( p \) is an isomorphism, we can find \( d \in D \) such that \( p(d) = d' \). Then we have
\[
q(j(d)) = u(p(d)) = u(f(c')) = 0.
\]
Since \( q \) is injective, we know \( j(d) = 0 \). Since the sequence is exact, there is some \( c \in C \) such that \( h(c) = d \).
We are not yet done. We do not know that \( n(c) = c' \). All we know is that \( d(n(c)) = d(c') \). So \( d(c' - n(c)) = 0 \). By exactness at \( C' \), we can find some \( b' \) such that \( s(b') = n(c) - c' \). Since \( m \) was surjective, we can find \( b \in B \) such that \( m(b) = b' \). Then we have
\[
n(g(b)) = n(c) - c'.
\]
So we have
\[
n(c - g(b)) = c'.
\]
So \( n \) is surjective.

Showing that \( n \) is injective is similar. \( \square \)

Corollary. Let \( f : (X, A) \to (Y, B) \) be a map of pairs, and that any two of \( f_* : H_\ast(X, A) \to H_\ast(Y, B), H_\ast(X) \to H_\ast(Y) \) and \( H_\ast(A) \to H_\ast(B) \) are isomorphisms. Then the third is also an isomorphism.

Proof. Follows from the long exact sequence and the five lemma. \( \square \)

Lemma. If \( f_* \) and \( g_* \) are chain homotopic, then \( f_* = g_* : H_\ast(C.) \to H_\ast(D.) \).

Proof. Let \( [c] \in H_\ast(C.) \). Then we have
\[
g_\ast(c) - f_\ast(c) = d^D_{n+1}F_n(c) + F_{n-1}(d^C_n(c)) = d^D_{n+1}F_n(c),
\]
where the second term dies because \( c \) is a cycle. So we have \( [g_\ast(c)] = [f_\ast(c)] \). \( \square \)

Theorem (Small simplices theorem). The natural map \( H^D_\ast(X) \to H_\ast(X) \) is an isomorphism.

Proof of Mayer-Vietoris. Let \( X = A \cup B \), with \( A, B \) open in \( X \). We let \( \mathcal{U} = \{A, B\} \), and write \( C_\ast((A + B) = C^D_\ast(X) \). Then we have a natural chain map
\[
C_\ast(A) \oplus C_\ast(B) \xrightarrow{j_{A-B}} C_\ast(A + B)
\]
that is surjective. The kernel consists of \( (x, y) \) such that \( j_A(x) - j_B(y) = 0 \), i.e. \( j_A(x) = j_B(y) \). But \( j \) doesn’t really do anything. It just forgets that the simplices lie in \( A \) or \( B \). So this means \( y = x \) is a chain in \( A \cap B \). We thus deduce that we have a short exact sequence of chain complexes
\[
C_\ast(A \cap B) \xrightarrow{(i_{A-B})} C_\ast(A) \oplus C_\ast(B) \xrightarrow{j_{A-B}} C_\ast(A + B).
\]
Then the snake lemma shows that we obtain a long exact sequence of homology groups. So we get a long exact sequence of homology groups
\[
\cdots \xrightarrow{} H_n(A \cap B) \xrightarrow{(i_{A-B})} H_n(A) \oplus H_n(B) \xrightarrow{j_{A-B}} H^n(X) \xrightarrow{} \cdots.
\]
By the small simplices theorem, we can replace \( H^n(X) \) with \( H_n(X) \). So we obtain Mayer-Vietoris. \( \square \)
Proof of excision. Let $X \supseteq A \supseteq Z$ be such that $\overline{Z} \supseteq \bar{A}$. Let $B = X \setminus Z$. Then again take

$$U = \{A,B\}.$$  

By assumption, their interiors cover $X$. We consider the short exact sequences

$$0 \longrightarrow C_\cdot(A) \longrightarrow C_\cdot(A + B) \longrightarrow C_\cdot(A + B)/C_\cdot(A) \longrightarrow 0$$

$$0 \longrightarrow C_\cdot(A) \longrightarrow C_\cdot(X) \longrightarrow C_\cdot(X,A) \longrightarrow 0$$

Looking at the induced map between long exact sequences on homology, the middle and left terms induce isomorphisms, so the right term does too by the 5-lemma.

On the other hand, the map

$$C_\cdot(B)/C_\cdot(A \cap B) \longrightarrow C_\cdot(A + B)/C_\cdot(A)$$

is an isomorphism of chain complexes. Since their homologies are $H_\cdot(B, A \cap B)$ and $H_\cdot(X, A)$, we infer they the two are isomorphic. Recalling that $B = X \setminus \overline{Z}$, we have shown that

$$H_\cdot(X \setminus Z, A \setminus Z) \cong H_\cdot(X, A).$$

**Lemma.** $\rho^n_X$ is a natural chain map.

**Lemma.** $\rho^n_X$ is chain homotopic to the identity.

**Proof.** No one cares.

**Lemma.** The diameter of each subdivided simplex in $(\rho^n_\Delta)^k(\iota_n)$ is bounded by

$$\left(\frac{n}{n+1}\right)^k \text{diam}(\Delta^n).$$

**Proof.** Basic geometry.

**Proposition.** If $c \in C^M_n(X)$, then $p^X(c) \in C^M_n(X)$.

Moreover, if $c \in C_n(X)$, then there is some $k$ such that $(\rho^n_\Delta)^k(c) \in C^M_n(X)$.

**Proof.** The first part is clear. For the second part, note that every chain is a finite sum of simplices. So we only have to check it for single simplices. We let $\sigma$ be a simplex, and let

$$V = \{\sigma^{-1}U_\alpha\}$$

be an open cover of $\Delta^n$. By the Lebesgue number lemma, there is some $\varepsilon > 0$ such that any set of diameter $< \varepsilon$ is contained in some $\sigma^{-1}U_\alpha$. So we can choose $k > 0$ such that $(\rho^n_\Delta)^k(\iota_n)$ is a sum of simplices which each has diameter $< \varepsilon$. So each lies in some $\sigma^{-1}U_\alpha$. So

$$(\rho^n_\Delta)^k(\iota_n) = C^M_n(\Delta^n).$$

So applying $\sigma$ tells us

$$(\rho^n_\Delta)^k(\sigma) \in C^M_n(X).$$

**Theorem** (Small simplices theorem). The natural map $U : H^M_n(X) \to H_\cdot(X)$ is an isomorphism.
Proof. Let \([c] \in H_n(X)\). By the proposition, there is some \(k > 0\) such that \((\rho_X^k)^*(c) \in C^k(X)\). We know that \(\rho_X^k\) is chain homotopic to the identity. Thus so is \((\rho_X^k)^*\). So \([([\rho_X^k]^*(c)] = [c]\). So the map \(H^U_n(X) \to H_n(X)\) is surjective.

To show that it is injective, we suppose \(U([c]) = 0\). Then we can find some \(z \in H_{n+1}(X)\) such that \(dz = c\). We can then similarly subdivide \(z\) enough such that it lies in \(C^{n+1}_n(X)\). So this shows that \([c] = 0 \in H^U_n(X)\). \(\square\)
4 Reduced homology

**Theorem.** If \((X, A)\) is good, then the natural map

\[
H_*(X, A) \longrightarrow H_*(X/A, A/A) = \tilde{H}_*(X/A)
\]

is an isomorphism.

**Proof.** As \(i: A \hookrightarrow U\) is in particular a homotopy equivalence, the map

\[
H_*(A) \longrightarrow H_*(U)
\]

is an isomorphism. So by the five lemma, the map on relative homology

\[
H_*(X, A) \longrightarrow H_*(X, U)
\]

is an isomorphism as well.

As \(i: A \hookrightarrow U\) is a deformation retraction with homotopy \(H\), the inclusion

\[
\{\ast\} = A/A \hookrightarrow U/A
\]

is also a deformation retraction. So again by the five lemma, the map

\[
H_*(X/A, A/A) \longrightarrow H_*(X/A, U/A)
\]

is also an isomorphism. Now we have

\[
\begin{array}{ccc}
H_n(X, A) & \xrightarrow{\sim} & H_n(X, U) \\
\downarrow & & \downarrow \\
H_n(X/A, A/A) & \xrightarrow{\text{excise } A/A} & H_n(X/A, U/A)
\end{array}
\]

We now notice that \(X \setminus A = X \setminus \frac{\overline{A}}{A}\) and \(U \setminus A = U \setminus \frac{\overline{A}}{A}\). So the right-hand vertical map is actually an isomorphism. So the result follows. \(\square\)
5 Cell complexes

Lemma. If \( A \subseteq X \) is a subcomplex, then the pair \((X, A)\) is good.

Proof. See Hatcher 0.16.

Corollary. If \( A \subseteq X \) is a subcomplex, then

\[
H_n(X, A) \rightarrow \tilde{H}_n(X/A)
\]

is an isomorphism.

Lemma. Let \( X \) be a cell complex. Then

(i) \[
H_i(X^n, X^{n-1}) = \begin{cases} 
0 & i \neq n, \\
\bigoplus_{i \in I_n} \mathbb{Z} & i = n.
\end{cases}
\]

(ii) \( H_i(X^n) = 0 \) for all \( i > n \).

(iii) \( H_i(X^n) \rightarrow H_i(X) \) is an isomorphism for \( i < n \).

Proof.

(i) As \((X^n, X^{n-1})\) is good, we have an isomorphism

\[
H_i(X^n, X^{n-1}) \rightarrow \tilde{H}_i(X^n/X^{n-1}).
\]

But we have

\[
X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n_{\alpha},
\]

the space obtained from \( Y = \coprod_{\alpha \in I_n} S^n_{\alpha} \) by collapsing down the subspace \( Z = \{x_\alpha : \alpha \in I_n\} \), where each \( x_\alpha \) is the south pole of the sphere. To compute the homology of the wedge \( X^n/X^{n-1} \), we then note that \((Y, Z)\) is good, and so we have a long exact sequence

\[
H_i(Z) \rightarrow H_i(Y) \rightarrow \tilde{H}_i(Y/Z) \rightarrow H_{i-1}(Z) \rightarrow \tilde{H}_{i-1}(Y) .
\]

Since \( H_i(Z) \) vanishes for \( i \geq 1 \), the result then follows from the homology of the spheres plus the fact that \( H_i(\coprod X_{\alpha}) = \bigoplus H_i(X_{\alpha}) \).

(ii) This follows by induction on \( n \). We have (part of) a long exact sequence

\[
H_i(X^n-1) \rightarrow H_i(X^n) \rightarrow H_i(X^n, X^{n-1})
\]

We know the first term vanishes by induction, and the third term vanishes for \( i > n \). So it follows that \( H_i(X^n) \) vanishes.

(iii) To avoid doing too much point-set topology, we suppose \( X \) is finite-dimensional, so \( X = X^m \) for some \( m \). Then we have a long exact sequence

\[
H_{i+1}(X^{n+1}, X^n) \rightarrow H_i(X^n) \rightarrow H_i(X^{n+1}) \rightarrow H_i(X^{n+1}, X^n)
\]
Now if \( i < n \), we know the first and last groups vanish. So we have 
\[
H_i(X^n) \cong H_i(X^{n+1}) \cong H_i(X^{n+2}) \cong \cdots \cong H_i(X^m) = H_i(X).
\]

To prove this for the general case, we need to use the fact that any map from a compact space to a cell complex hits only finitely many cells, and then the result would follow from this special case.

**Theorem.**

\[
H^\text{cell}_n(X) \cong H_n(X).
\]

**Proof.** We have

\[
H_n(X) \cong H_n(X^{n+1})
= H_n(X^n) / \text{im}(\partial : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n))
\]

Since \( q_n \) is injective, we apply it to and bottom to get

\[
= q_n(H_n(X^n)) / \text{im}((d^\text{cell})_{n+1} : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}))
\]

By exactness, the image of \( q_n \) is the kernel of \( \partial \). So we have

\[
= \ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})) / \text{im}(d^\text{cell}_{n+1})
= \ker((d^\text{cell})_{n}) / \text{im}(d^\text{cell}_{n+1})
= H^\text{cell}_n(X).
\]

**Corollary.** If \( X \) is a finite cell complex, then \( H_n(X) \) is a finitely-generated abelian group for all \( n \), generated by at most \( |I_n| \) elements. In particular, if there are no \( n \)-cells, then \( H_n(X) \) vanishes.

If \( X \) has a cell-structure with cells in even-dimensional cells only, then \( H_*(X) \) are all free.

**Lemma.** The coefficients \( d_{\alpha \beta} \) are given by the degree of the map

\[
S^{n-1}_\alpha = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1}/X^{n-2} = \bigvee_{\gamma \in I_{n-1}} S^{n-1}_\gamma \xrightarrow{f_{\alpha \beta}} S^{n-1}_\beta,
\]

where the final map is obtained by collapsing the other spheres in the wedge.

In the case of cohomology, the maps are given by the transposes of these.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
H_n(D^n_\alpha, \partial D^n_\alpha) & \xrightarrow{\partial} & H_{n-1}(\partial D^n_\alpha) \\
\downarrow_{\Phi_\alpha} & & \downarrow_{(\varphi_\alpha)_*} \\
H_n(X^n, X^{n-1}) & \xrightarrow{\partial} & H_{n-1}(X^{n-1}) \xrightarrow{q} H_{n-1}(\bigvee S^{n-1}_\gamma) \xrightarrow{\text{excision}} \tilde{H}_{n-1}(X^{n-1}/X^{n-2})
\end{array}
\]

\[
\begin{array}{ccc}
\sim & \text{collaps} & \sim \\
& & \\
\sim & \text{excision} & \sim
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{H}_{n-1}(S^{n-1}_\beta) & \xrightarrow{\text{collapse}} & \tilde{H}_{n-1}(S^{n-1}_\beta)
\end{array}
\]

\[
\begin{array}{ccc}
\text{excision} & \sim & \text{excision}
\end{array}
\]

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By the long exact sequence, the top left horizontal map is an isomorphism.
Now let’s try to trace through the diagram. We can find

\[ \text{1 isomorphism} \quad \xrightarrow{1} \quad \xrightarrow{f_{\alpha\beta}} \quad \xrightarrow{d_{\alpha\beta}} \quad \sum d_{\alpha\gamma} e_{\gamma} \xrightarrow{\sum} \sum d_{\alpha\gamma} e_{\gamma} \]

So the degree of \( f_{\alpha\beta} \) is indeed \( d_{\alpha\beta} \). \( \square \)
6 (Co)homology with coefficients

Algebraic Topology (Theorems with proof)

6 (Co)homology with coefficients
7 Euler characteristic

**Theorem.** We have

\[ \chi = \chi_Z = \chi_F. \]

**Proof.** First note that the number of \( n \) cells of \( X \) is the rank of \( C_n^{cel}(X) \), which we will just write as \( C_n \). Let

\[
\begin{align*}
Z_n &= \ker(d_n : C_n \to C_{n-1}) \\
B_n &= \text{im}(d_{n+1} : C_{n+1} \to C_n).
\end{align*}
\]

We are now going to write down two short exact sequences. By definition of homology, we have

\[
0 \to B_n \to Z_n \to H_n(X; \mathbb{Z}) \to 0.
\]

Also, the definition of \( Z_n \) and \( B_n \) give us

\[
0 \to Z_n \to C_n \to B_{n-1} \to 0.
\]

We will now use the first isomorphism theorem to know that the rank of the middle term is the sum of ranks of the outer terms. So we have

\[
\chi_Z(X) = \sum (-1)^n \text{rank } H_n(X) = \sum (-1)^n (\text{rank } Z_n - \text{rank } B_n).
\]

We also have

\[
\text{rank } B_n = \text{rank } C_{n+1} - \text{rank } Z_{n+1}.
\]

So we have

\[
\chi_Z(X) = \sum (-1)^n (\text{rank } Z_n - \text{rank } C_{n+1} + \text{rank } Z_{n+1})
\]

\[
= \sum (-1)^{n+1} \text{rank } C_{n+1}
\]

\[
= \chi(X).
\]

For \( \chi_F \), we use the fact that

\[
\text{rank } C_n = \dim_F C_n \otimes F.
\]

\[\square\]
8 Cup product

Lemma. If \( \phi \in C^k(X; R) \) and \( \psi \in C^\ell(X; R) \), then

\[
d(\phi \smile \psi) = (d\phi) - \psi + (-1)^k \phi \smile (d\psi).
\]

Proof. This is a straightforward computation.

Let \( \sigma: \Delta^{k+\ell+1} \rightarrow X \) be a simplex. Then we have

\[
((d\phi) \smile \psi)(\sigma) = (d\phi)(\sigma|_{v_0, \ldots, v_{k+1}}) \cdot \psi(\sigma|_{v_{k+1}, \ldots, v_{k+\ell+1}})
\]

\[
= \phi \left( \sum_{i=0}^{k+1} (-1)^i \sigma|_{v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}} \right) \cdot \psi(\sigma|_{v_{k+1}, \ldots, v_{k+\ell+1}})
\]

\[
(\phi \smile (d\psi))(\sigma) = \phi(\sigma|_{v_0, \ldots, v_k}) \cdot (d\psi)(\sigma|_{v_k, \ldots, v_{k+\ell+1}})
\]

\[
= \phi(\sigma|_{v_0, \ldots, v_k}) \cdot \psi \left( \sum_{i=k}^{k+\ell+1} (-1)^{i-k} \sigma|_{v_0, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}} \right)
\]

\[
= (-1)^k \phi(\sigma|_{v_0, \ldots, v_k}) \cdot \psi \left( \sum_{i=k}^{k+\ell+1} (-1)^i \sigma|_{v_k, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}} \right).
\]

We notice that the last term of the first expression, and the first term of the second expression are exactly the same, except the signs differ by \(-1\). Then the remaining terms overlap in exactly 1 vertex, so we have

\[
((d\phi) \smile \psi)(\sigma) + (-1)^k \phi \smile (d\psi)(\sigma) = (\phi \smile \psi)(d\sigma) = (d(\phi \smile \psi))(\sigma)
\]

as required.

\[\square\]

Corollary. The cup product induces a well-defined map

\[
\smile: H^k(X; R) \times H^\ell(X; R) \rightarrow H^{k+\ell}(X; R)
\]

\[
([\phi], [\psi]) \mapsto [\phi \smile \psi]
\]

Proof. To see this is defined at all, as \( d\phi = 0 = d\psi \), we have

\[
d(\phi \smile \psi) = (d\phi) \smile \psi + \phi \smile (d\psi) = 0.
\]

So \( \phi \smile \psi \) is a cocycle, and represents the cohomology class. To see this is well-defined, if \( \phi' = \phi + d\tau \), then

\[
\phi' \smile \psi = \phi \smile \psi + d\tau \smile \psi = \phi \smile \psi + d(\tau \smile \psi) = \phi \smile \psi + (d\tau \smile \psi).
\]

Using the fact that \( d\psi = 0 \), we know that \( \phi' \smile \psi \) and \( \phi \smile \psi \) differ by a boundary, so \( [\phi' \smile \psi] = [\phi \smile \psi] \). The case where we change \( \psi \) is similar. \[\square\]

Proposition. \((H^*(X; R), \smile, [1])\) is a unital ring.

Proposition. Let \( R \) be a commutative ring. If \( \alpha \in H^k(X; R) \) and \( \beta \in H^\ell(X; R) \), then we have

\[
\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha
\]
Proposition. The cup product is natural, i.e. if \( f : X \to Y \) is a map, and \( \alpha, \beta \in H^*(Y; R) \), then
\[
f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).
\]
So \( f^* \) is a homomorphism of unital rings.

Proof of previous proposition. Let \( \rho_n : C_n(X) \to C_n(x) \) be given by
\[
\sigma \mapsto (-1)^{n(n+1)/2} \sigma|_{[v_n, v_{n-1}, \ldots, v_0]}
\]
The \( \sigma|_{[v_n, v_{n-1}, \ldots, v_0]} \) tells us that we reverse the order of the vertices, and the factor of \( (-1)^{n(n+1)/2} \) is the sign of the permutation that reverses \( 0, \ldots, n \). For convenience, we write \( \varepsilon_n = (-1)^n(n+1)/2 \).

Claim. We claim that \( \rho_n \cdot \) is a chain map, and is chain homotopic to the identity. We will prove this later.

Suppose the claim holds. We let \( \phi \in C^k(X; R) \) represent \( \alpha \) and \( \psi \in C^\ell(X; R) \) represent \( \beta \). Then we have
\[
(\rho^* \phi \smile \rho^* \psi)(\sigma) = (\rho^* \phi)(\sigma|_{[v_0, \ldots, v_k]})(\rho^* \psi)(\sigma|_{[v_k, \ldots, v_{k+\ell}]}),
\]
Thus, we can compute
\[
\rho^*(\psi \smile \phi)(\sigma) = (\psi \smile \phi)(\varepsilon_k \cdot \sigma|_{[v_k, \ldots, v_0]}) \psi(\varepsilon_\ell \sigma|_{[v_k+\ell, \ldots, v_0]}).
\]
By checking it directly, we can see that \( \varepsilon_n + \varepsilon_{n+i} \varepsilon_k \varepsilon_\ell = (-1)^{k\ell} \). So we have
\[
\alpha \smile \beta = [\phi \smile \psi] = [\rho^* \phi \smile \rho^* \psi] = (-1)^{k\ell} [\rho^*(\psi \smile \phi)] = (-1)^{k\ell} [\psi \smile \phi] = (-1)^{k\ell} \beta \smile \alpha.
\]

Now it remains to prove the claim. We have
\[
d\rho(\sigma) = \varepsilon_n \sum_{i=0}^{n} (-1)^i \sigma|_{[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]}.
\]
\[
\rho(d\sigma) = \rho \left( \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} \right) = \varepsilon_{n-1} \sum_{j=0}^{n} (-1)^j \sigma|_{[v_n, \ldots, \hat{v}_j, v_0]}.
\]
We now notice that \( \varepsilon_{n-1} (-1)^{n-i} = \varepsilon_n (-1)^i \). So this is a chain map!
We now define a chain homotopy. This time, we need a “twisted prism”. We let
\[ P_n = \sum_i (-1)^i \varepsilon_{n-i}[v_0, \ldots, v_i, w_n, \ldots, w_1] \in C_{n+1}([0, 1] \times \Delta^n), \]
where \( v_0, \ldots, v_n \) are the vertices of \( \{0\} \times \Delta^n \) and \( w_0, \ldots, w_n \) are the vertices of \( \{1\} \times \Delta^n \).

We let \( \pi : [0, 1] \times \Delta^n \to \Delta^n \) be the projection, and let \( P^X_n : C_n(X) \to C_{n+1}(X) \) be given by
\[ \sigma \mapsto (\sigma \circ \pi)(P_n). \]

We calculate
\[ dP^X_n(\sigma) = (\sigma \circ \pi)(dP_n) \]
\[ = (\sigma \circ \pi)( \sum_i \left( \sum_{j \leq i} (-1)^j (-1)^i \varepsilon_{n-i}[v_0, \ldots, \hat{v}_j, \ldots, v_i, w_n, \ldots, w_1] \right) \]
\[ + \sum_{j \geq i} (-1)^{n+i+1-j} (-1)^i \varepsilon_{n-i}[v_0, \ldots, v_i, w_n, \ldots, \hat{w}_j, \ldots, v_1] \) \right). \]

The terms with \( j = i \) give
\[ (\sigma \circ \pi)( \sum_i \varepsilon_{n-i}[v_0, \ldots, v_i-1, w_n, \ldots, w_1] \]
\[ + \sum_i (-1)^{n+i+1} (-1)^i \varepsilon_{n-i}[v_0, \ldots, v_i, w_n, \ldots, w_i+1] ) \right) \]
\[ = (\sigma \circ \pi)(\varepsilon_n[w_n, \ldots, w_0] - [v_0, \ldots, v_n]) \]
\[ = \rho(\sigma) - \sigma \]

The terms with \( j \neq i \) are precisely \(-P^X_{n-1}(d\sigma)\) as required. It is easy to see that the terms are indeed the right terms, and we just have to check that the signs are right. I'm not doing that. \( \square \)
9 K"unneth theorem and universal coefficients theorem

**Theorem** (K"unneth’s theorem). Let $R$ be a commutative ring, and suppose that $H^n(Y; R)$ is a free $R$-module for each $n$. Then the cross product map

$$
\bigoplus_{k+\ell=n} H^k(X; R) \otimes H^\ell(Y; R) \xrightarrow{\times} H^n(X \times Y; R)
$$

is an isomorphism for every $n$, for every finite cell complex $X$.

It follows from the five lemma that the same holds if we have a relative complex $(Y, A)$ instead of just $Y$.

**Proof.** Let $F^n(\_)=\bigoplus_{k+\ell=n} H^k(\_;R) \otimes H^\ell(Y; R)$. We similarly define $G^n(\_)=H^n(\_ \times Y; R)$.

We observe that for each $X$, the cross product gives a map $\times_\_ : F^n(X) \to G^n(X)$, and, crucially, we know $\times_\_ : F^n(*) \to G^n(*)$ is an isomorphism, since $F^n(*) \cong G^n(*) \cong H^n(Y; R)$.

The strategy is to show that $F^n(\_)$ and $G^n(\_)$ have the same formal structure as cohomology and agree on a point, and so must agree on all finite cell complexes.

It is clear that both $F^n$ and $G^n$ are homotopy invariant, because they are built out of homotopy invariant things.

We now want to define the cohomology of pairs. This is easy. We define

$$
F^n(X, A) = \bigoplus_{i+j=n} H^i(X, A; R) \otimes H^j(Y; R)
$$

$$
G^n(X, A) = H^n(X \times Y, A \times Y; R).
$$

Again, the relative cup product gives us a relative cross product, which gives us a map $F^n(X, A) \to G^n(X, A)$.

It is immediate $G^n$ has a long exact sequence associated to $(X, A)$ given by the usual long exact sequence of $(X \times Y, A \times Y)$. We would like to say $F$ has a long exact sequence as well, and this is where our hypothesis comes in.

If $H^*(Y; R)$ is a free $R$-module, then we can take the long exact sequence of $(X, A)$

$$
\cdots \to H^n(A; R) \xrightarrow{\partial} H^n(X, A; R) \to H^n(X; R) \to H^n(A; R) \to \cdots
$$

and then tensor with $H^j(Y; R)$. This preserves exactness, since $H^j(Y; R) \cong R^k$ for some $k$, so tensoring with $H^j(Y; R)$ just takes $k$ copies of this long exact sequence. By adding the different long exact sequences for different $j$ (with appropriate translations), we get a long exact sequence for $F$. 

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We now want to prove Künneth by induction on the number of cells and the dimension at the same time. We are going to prove that if $X = X' \cup_f D^n$ for some $S^{n-1} \to X'$, and $\times : F(X') \to G(X')$ is an isomorphism, then $\times : F(X) \to G(X)$ is also an isomorphism. In doing so, we will assume that the result is true for attaching any cells of dimension less than $n$.

Suppose $X = X' \cup_f D^n$ for some $f : S^{n-1} \to X'$. We get long exact sequences

$$
\begin{array}{cccccc}
F^{*+1}(X') & \to & F^*(X, X') & \to & F^*(X) & \to & F^{*+1}(X, X') \\
\times & & \times & & \times & & \times \\
G^{*+1}(X') & \to & G^*(X, X') & \to & G^*(X) & \to & G^{*+1}(X, X')
\end{array}
$$

Note that we need to manually check that the boundary maps $\partial$ commute with the cross product, since this is not induced by maps of spaces, but we will not do it here.

Now by the five lemma, it suffices to show that the maps on the relative cohomology $\times : F^*(X, X') \to G^*(X, X')$ have an isomorphism.

We now notice that $F^*(-)$ and $G^*(-)$ have excision. Since $(X, X')$ is a good pair, we have a commutative square

$$
\begin{array}{cccc}
F^*(D^n, \partial D^n) & \to & F^*(X, X') \\
\downarrow \times & & \downarrow \times \\
G^*(D^n, \partial D^n) & \to & G^*(X, X')
\end{array}
$$

So we now only need the left-hand map to be an isomorphism. We look at the long exact sequence for $(D^n, \partial D^n)$!

$$
\begin{array}{cccccc}
F^{*+1}(\partial D^n) & \to & F^*(D^n, \partial D^n) & \to & F^*(D^n) & \to & F^*(\partial D^n) & \to & F^{*+1}(D^n, \partial D^n) \\
\times & & \times & & \times & & \times & & \times \\
G^{*+1}(\partial D^n) & \to & G^*(D^n, \partial D^n) & \to & G^*(D^n) & \to & G^*(\partial D^n) & \to & G^{*+1}(D^n, \partial D^n)
\end{array}
$$

But now we know the vertical maps for $D^n$ and $\partial D^n$ are isomorphisms — the ones for $D^n$ are because they are contractible, and we have seen the result of $\ast$ already; whereas the result for $\partial D^n$ follows by induction.

So we are done.

**Theorem** (Universal coefficients theorem for (co)homology). Let $R$ be a PID and $M$ an $R$-module. Then there is a natural map

$$
H_*(X; R) \otimes M \to H_*(X; M).
$$

If $H_*(X; R)$ is a free module for each $n$, then this is an isomorphism. Similarly, there is a natural map

$$
H^*(X; M) \to \text{Hom}_R(H_*(X; R), M),
$$

which is an isomorphism again if $H^*(X; R)$ is free.
Proof. Let $C_n$ be $C_n(X; R)$ and $Z_n \subseteq C_n$ be the cycles and $B_n \subseteq Z_n$ the boundaries. Then there is a short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{g} B_{n-1} \longrightarrow 0,$$

and $B_{n-1} \leq C_{n-1}$ is a submodule of a free $R$-module, and is free, since $R$ is a PID. So by picking a basis, we can find a map $s : B_{n-1} \rightarrow C_n$ such that $g \circ s = \text{id}_{B_{n-1}}$. This induces an isomorphism

$$i \oplus s : Z_n \oplus B_{n-1} \longrightarrow C_n.$$

Now tensoring with $M$, we obtain

$$0 \longrightarrow Z_n \otimes M \longrightarrow C_n \otimes M \longrightarrow B_{n-1} \otimes M \longrightarrow 0,$$

which is exact because we have

$$C_n \otimes M \cong (Z_n \oplus B_{n-1}) \otimes M \cong (Z_n \otimes M) \oplus (B_{n-1} \otimes M).$$

So we obtain a short exact sequence of chain complexes

$$0 \longrightarrow (Z_n \otimes M, 0) \longrightarrow (C_n \otimes M, d \otimes \text{id}) \longrightarrow (B_{n-1} \otimes M, 0) \longrightarrow 0,$$

which gives a long exact sequence in homology:

$$\cdots \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X; M) \longrightarrow B_{n-1} \otimes M \longrightarrow \cdots$$

We’ll leave this for a while, and look at another short exact sequence. By definition of homology, we have a long exact sequence

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(X; R) \longrightarrow 0.$$

As $H_n(X; R)$ is free, we have a splitting $t : H_n(X; R) \rightarrow Z_n$, so as above, tensoring with $M$ preserves exactness, so we have

$$0 \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X; R) \otimes M \longrightarrow 0.$$

Hence we know that $B_n \otimes M \rightarrow Z_n \otimes M$ is injective. So our previous long exact sequence breaks up to

$$0 \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X; M) \longrightarrow 0.$$

Since we have two short exact sequence with first two terms equal, the last terms have to be equal as well.

The cohomology version is similar. □
10 Vector bundles

10.1 Vector bundles

**Theorem** (Tubular neighbourhood theorem). Let $M \subseteq N$ be a smooth submanifold. Then there is an open neighbourhood $U$ of $M$ and a homeomorphism $\nu_{M\subseteq N} \to U$, and moreover, this homeomorphism is the identity on $M$ (where we view $M$ as a submanifold of $\nu_{M\subseteq N}$ by the image of the zero section).

**Proposition.** Partitions of unity exist for any open cover.

**Lemma.** Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is a continuous family of inner products on $E$. In other words, there is a map $E \otimes E \to \mathbb{R}$ which restricts to an inner product on each $E_x$.

**Proof.** We notice that every trivial bundle has an inner product, and since every bundle is locally trivial, we can patch these up using partitions of unity. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $X$ with local trivializations $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^d$. The inner product on $\mathbb{R}^d$ then gives us an inner product on $E|_{U_\alpha}$, say $\langle \cdot, \cdot \rangle_\alpha$. We let $\lambda_\alpha$ be a partition of unity associated to $\{U_\alpha\}$. Then for $u \otimes v \in E \otimes E$, we define

$$\langle u, v \rangle = \sum_{\alpha \in I} \lambda_\alpha(\pi(u)) \langle u, v \rangle_\alpha.$$ 

Now if $\pi(u) = \pi(v)$ is not in $U_\alpha$, then we don’t know what we mean by $\langle u, v \rangle_\alpha$, but it doesn’t matter, because $\lambda_\alpha(\pi(u)) = 0$. Also, since the partition of unity is locally finite, we know this is a finite sum.

It is then straightforward to see that this is indeed an inner product, since a positive linear combination of inner products is an inner product.

**Lemma.** Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some $N$ such that $E$ is a vector subbundle of $X \times \mathbb{R}^N$.

**Proof.** Let $\{U_\alpha\}$ be a trivializing cover of $X$. Since $X$ is compact, we may wlog assume the cover is finite. Call them $U_1, \cdots, U_n$. We let $\varphi_i : E|_{U_i} \to U_i \times \mathbb{R}^d$.

We note that on each patch, $E|_{U_i}$ embeds into a trivial bundle, because it is a trivial bundle. So we can add all of these together. The trick is to use a partition of unity, again.

We define $f_i$ to be the composition

$$E|_{U_i} \xrightarrow{\varphi_i} U_i \times \mathbb{R}^d \xrightarrow{\pi^2} \mathbb{R}^d.$$ 

Then given a partition of unity $\lambda_i$, we define

$$f : E \to X \times (\mathbb{R}^d)^n, \quad v \mapsto (\pi(v), \lambda_1(\pi(v)) f_1(v), \lambda_2(\pi(v)) f_2(v), \cdots, \lambda_n(\pi(v)) f_n(v)).$$ 

We see that this is injective. If $v, w$ belong to different fibers, then the first coordinate distinguishes them. If they are in the same fiber, then there is some $U_i$ with $\lambda_i(\pi(u)) \neq 0$. Then looking at the $i$th coordinate gives us distinguishes them. This then exhibits $E$ as a subbundle of $X \times \mathbb{R}^n$. 

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Corollary. Let $\pi : E \rightarrow X$ be a vector bundle over a compact Hausdorff space. Then there is some $p : F \rightarrow X$ such that $E \oplus F \cong X \times \mathbb{R}^n$. In particular, $E$ embeds as a subbundle of a trivial bundle.

Proof. By above, we can assume $E$ is a subbundle of a trivial bundle. We can then take the orthogonal complement of $E$.

Theorem. There is a correspondence

\[
\begin{align*}
\{ \text{homotopy classes of maps} \} & \leftrightarrow \{ \text{d-dimensional vector bundles} \} \\
\{ f : X \rightarrow \text{Gr}_d(\mathbb{R}^\infty) \} & \leftrightarrow \{ \pi : E \rightarrow X \} \\
[f] & \mapsto f^* \gamma_{\mathbb{R}_d}^X \\
[f_x] & \mapsto \pi
\end{align*}
\]

10.2 Vector bundle orientations

Lemma. Every vector bundle is $\mathbb{F}_2$-orientable.

Proof. There is only one possible choice of generator. \qed

Lemma. If $\{ U_\alpha \}_{\alpha \in I}$ is a family of covers such that for each $\alpha, \beta \in I$, the homeomorphism

\[
(U_\alpha \cap U_\beta) \times \mathbb{R}^d \overset{\varphi_\alpha}{\leftrightarrow} E|_{U_\alpha \cap U_\beta} \overset{\pi}{\rightarrow} (U_\alpha \cap U_\beta) \times \mathbb{R}^d
\]

gives an orientation preserving map from $(U_\alpha \cap U_\beta) \times \mathbb{R}^d$ to itself, i.e. has a positive determinant on each fiber, then $E$ is orientable for any $R$.

Proof. Choose a generator $u \in H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$. Then for $x \in U_\alpha$, we define $\varepsilon_x$ by pulling back $u$ along

\[
E_x \leftarrow E|_{U_\alpha} \overset{\varphi_\alpha}{\leftrightarrow} U_\alpha \times \mathbb{R}^d \overset{\pi_x}{\rightarrow} \mathbb{R}^d.
\]

If $x \in U_\beta$ as well, then the analogous linear isomorphism $\dagger_\alpha$ differs from $\dagger_\beta$ by post-composition with a linear map $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of positive determinant. We now use the fact that any linear map of positive determinant is homotopic to the identity. Indeed, both $L$ and id lies in $\text{GL}_d^+ (\mathbb{R})$, a connected group, and a path between them will give a homotopy between the maps they represent. So we know $(\dagger_\alpha)$ is homotopic to $(\dagger_\beta)$. So they induce the same maps on cohomology classes. \qed

10.3 The Thom isomorphism theorem

Theorem (Thom isomorphism theorem). Let $\pi : E \rightarrow X$ be a $d$-dimensional vector bundle, and $\{ \varepsilon_x \}_{x \in X}$ be an $R$-orientation of $E$. Then

(i) $H^i(E, E^\#; R) = 0$ for $i < d$.

(ii) There is a unique class $u_E \in H^d(E, E^\#; R)$ which restricts to $\varepsilon_x$ on each fiber. This is known as the Thom class.
(iii) The map $\Phi$ given by the composition

\[
H^i(X; R) \xrightarrow{\pi^*} H^i(E; R) \xrightarrow{-u_E} H^{i+d}(E, E^#; R)
\]

is an isomorphism.

Note that (i) follows from (iii), since $H^i(X; R) = 0$ for $i < 0$.

**Theorem.** If there is a section $s : X \to E$ which is nowhere zero, then $e(E) = 0 \in H^d(X; R)$.

**Proof.** Notice that any two sections of $E \to X$ are homotopic. So we have $e \equiv s_0^* u_E = s^* u_E$. But since $u_E \in H^d(E, E^#; R)$, and $s$ maps into $E^#$, we have $s^* u_E$.

Perhaps more precisely, we look at the long exact sequence for the pair $(E, E^#)$, giving the diagram

\[
\begin{array}{ccc}
H^d(E, E^#; R) & \longrightarrow & H^d(E; R) \\
& & \downarrow s^* \\
& & H^d(X; R)
\end{array}
\]

Since $s$ and $s_0$ are homotopic, the diagram commutes. Also, the top row is exact. So $u_E \in H^d(E; R)$ gets sent along the top row to $0 \in H^d(X; R)$, and thus $s^*$ sends it to $0 \in H^d(X; R)$. But the image in $H^d(X; R)$ is exactly the Euler class. So the Euler class vanishes. \qed

**Theorem.** We have

\[
u^* u_E^{-1} = \Phi(e(E)) = \pi^*(e(E))^{-1} \in H^*(E, E^#; R).
\]

**Proof.** By construction, we know the following maps commute:

\[
\begin{array}{ccc}
H^d(E, E^#; R) \otimes H^d(E, E^#; R) & \longrightarrow & H^{2d}(E, E^#; R) \\
\downarrow q^* \otimes \text{id} & & \\
H^d(E; R) \otimes H^d(E, E^#; R) & \longrightarrow & H^d(E, E^#; R)
\end{array}
\]

We claim that the Thom class $u_E \otimes u_E \in H^d(E, E^#; R) \otimes H^d(E, E^#; R)$ is sent to $\pi^*(e(E)) \otimes u_E \in H^d(E; R) \otimes H^d(E, E^#; R)$.

By definition, this means we need

\[
q^* u_E = \pi^*(e(E)),
\]

and this is true because $\pi^*$ is homotopy inverse to $s_0^*$ and $e(E) = s_0^* q^* u_E$. \qed

**Lemma.** If $\pi : E \to X$ is a $d$-dimensional $R$-module vector bundle with $d$ odd, then $2e(E) = 0 \in H^d(X; R)$.

**Proof.** Consider the map $\alpha : E \to E$ given by negation on each fiber. This then gives an isomorphism

\[
a^* : H^d(E, E^#; R) \xrightarrow{\cong} H^d(E, E^#; R).
\]
This acts by negation on the Thom class, i.e.
\[ a^* (u_E) = -u_E, \]
as on the fiber \( E_x \), we know \( a \) is given by an odd number of reflections, each of which acts on \( H^d(E_x, E_x^#; R) \) by \(-1\) (by the analogous result on \( S^n \)). So we change \( \varepsilon_x \) by a sign. We then lift this to a statement about \( u_E \) by the fact that \( u_E \) is the unique thing that restricts to \( \varepsilon_x \) for each \( x \).

But we also know \( a \circ s_0 = s_0 \), which implies
\[ s_0^* (a^* (u_E)) = s_0^* (u_E). \]
Combining this with the result that \( a^* (u_E) = -u_E \), we get that
\[ 2e(E) = 2s_0^* (u_E) = 0. \]

Proof of Thom isomorphism theorem. We will drop the “\( R \)” in all our diagrams for readability (and also so that it fits in the page).

We first consider the case where the bundle is trivial, so \( E = X \times \mathbb{R}^d \). Then we note that
\[ H^* (\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) = \begin{cases} R & * = d \\ 0 & * \neq d. \end{cases} \]

In particular, the modules are free, and (a relative version of) Künneth’s theorem tells us the map
\[ \times : H^* (X) \otimes H^* (\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \cong H^* (X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\})) \]
is an isomorphism. Then the claims of the Thom isomorphism theorem follow immediately.

(i) For \( i < d \), all the summands corresponding to \( H^i (X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\})) \) vanish since the \( H^* (\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \) term vanishes.

(ii) The only non-vanishing summand for \( H^d (X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\})) \) is
\[ H^0 (X) \otimes H^d (\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}). \]

Then the Thom class must be \( 1 \otimes u \), where \( u \) is the object corresponding to \( \varepsilon_x \in H^d (E_x, E_x^#) = H^d (\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \), and this is unique.

(iii) We notice that \( \Phi \) is just given by
\[ \Phi (x) = \pi^* (x) \sim u_E = x \times u_E, \]
which is an isomorphism by Künneth.

We now patch the result up for a general bundle. Suppose \( \pi : E \to X \) is a bundle. Then it has an open cover of trivializations, and moreover if we assume our \( X \) is compact, there are finitely many of them. So it suffices to show that
if \( U, V \subseteq X \) are open sets such that the Thom isomorphism holds for \( E \) restricted to \( U, V, U \cap V \), then it also holds on \( U \cup V \).

The relative Mayer-Vietoris sequence gives us

\[
\begin{array}{c}
H^{d-1}(E|_{U \cap V}, E^#|_{U \cap V}) \xrightarrow{\partial^{MV}} H^d(E|_{U \cup V}, E^#|_{U \cup V}) \\
\end{array}
\]

\[
\xrightarrow{\oplus} H^d(E|_U, E^#|_U) \oplus H^d(E|_V, E^#|_V) \xrightarrow{\partial^{MV}} H^d(E|_{U \cap V}, E^#|_{U \cap V}).
\]

We first construct the Thom class. We have

\[
u_{E|_V} \in H^d(E|_V, E^#), \quad u_{E|_U} \in H^d(E|_U, E^#).
\]

We claim that \((u_{E|_V}, u_{E|_U}) \in H^d(E|_U, E^#|_U) \oplus H^d(E|_V, E^#|_V)\) gets sent to 0 by \(i^*_U - i^*_V\). Indeed, both the restriction of \(u_{E|_V}\) and \(u_{E|_U}\) to \(U \cap V\) are Thom classes, so they are equal by uniqueness, so the difference vanishes.

Then by exactness, there must be some \(u_{E|_{U \cup V}} \in H^d(E|_{U \cup V}, E^#|_{U \cup V})\) that restricts to \(u_{E|_U}\) and \(u_{E|_V}\) in \(U\) and \(V\) respectively. Then this must be a Thom class, since the property of being a Thom class is checked on each fiber. Moreover, we get uniqueness because \(H^{d-1}(E|_{U \cap V}, E^#|_{U \cap V}) = 0\), so \(u_{E|_U}\) and \(u_{E|_V}\) must be the restriction of a unique thing.

The last part in the Thom isomorphism theorem comes from a routine application of the five lemma, and the first part follows from the last as previously mentioned.

10.4 Gysin sequence
11 Manifolds and Poincaré duality

11.1 Compactly supported cohomology

**Theorem.** For any space $X$, we let

$$\mathcal{K}(X) = \{K \subseteq X : K \text{ is compact}\}.$$  

This is a directed set under inclusion, and the map

$$K \mapsto H^n(X, X \setminus K)$$

gives a direct system of abelian groups indexed by $\mathcal{K}(X)$, where the maps $\rho$ are given by restriction.

Then we have

$$H^*_c(X) \cong \varprojlim_{K \in \mathcal{K}(X)} H^n(X, X \setminus K).$$

**Proof.** We have

$$C^n_c(X) \cong \varprojlim_{K \in \mathcal{K}(X)} C^n(X, X \setminus K),$$

where we have a map

$$\varprojlim_{K \in \mathcal{K}(X)} C^n(X, X \setminus K) \rightarrow C^n_c(X)$$

given in each component of the direct limit by inclusion, and it is easy to see that this is well-defined and bijective.

It is then a general algebraic fact that $H^*$ commutes with inverse limits, and we will not prove it.

**Lemma.** We have

$$H^i_c(\mathbb{R}^d; R) \cong \begin{cases} R & i = d \\ 0 & \text{otherwise} \end{cases}.$$  

**Proof.** Let $B \in \mathcal{K}(\mathbb{R}^d)$ be the balls, namely

$$B = \{nD^d_n, n = 0, 1, 2, \ldots\}.$$ 

Then since every compact set is contained in one of them, we have

$$H^*_c(X) \cong \varprojlim_{K \in \mathcal{K}(\mathbb{R}^d)} H^n(\mathbb{R}^d, \mathbb{R}^d \setminus K) \cong \varprojlim_{nD^d_n \in B} H^n(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d_n; R)$$

We can compute that directly. Since $\mathbb{R}^d$ is contractible, the connecting map

$$H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d_n; R) \rightarrow H^{i-1}(\mathbb{R}^d \setminus nD^d_n; R)$$

in the long exact sequence is an isomorphism. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d_n; R) & \xrightarrow{\partial_{n,n+1}} & H^i(\mathbb{R}^d, \mathbb{R}^d \setminus (n+1)D^d; R) \\
\downarrow \phi & & \downarrow \phi \\
H^{i-1}(\mathbb{R}^d \setminus nD^d_n; R) & \longrightarrow & H^{i-1}(\mathbb{R}^d \setminus (n+1)D^d; R)
\end{array}$$
But all maps here are isomorphisms because the horizontal maps are homotopy equivalences. So we know
\[ \lim_{\to} H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d; R) \cong H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \cong H^{i-1}(\mathbb{R}^d \setminus \{0\}; R). \]
So it follows that
\[ H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \cong H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \cong H^{i-1}(\mathbb{R}^d \setminus \{0\}; R). \]

**Proposition.** Let \( K, L \subseteq X \) be compact. Then there is a long exact sequence
\[
\begin{align*}
H^n(X | K \cap L) &\to H^n(X | K) \oplus H^n(X | L) \to H^n(X | K \cup L) \to \\
H^{n+1}(X | K \cap L) &\to H^{n+1}(X | K) \oplus H^{n+1}(X | L) \to \\
\ldots
\end{align*}
\]
where the unlabelled maps are those induced by inclusion.

**Proof.** We cover \( X \setminus K \cap L \) by
\[ U = \{ X \setminus K, X \setminus L \}. \]
We then draw a huge diagram (here \( * \) denotes the dual, i.e. \( X^* = \text{Hom}(X; R) \), and \( C^* (X | K) = C^* (X, X \setminus K) \)):
\[
\begin{array}{cccccc}
0 &\to &0 &\to &0 &\to \\
\downarrow & &\downarrow & &\downarrow & \\
0 &\to &\left( \frac{C_*(X)}{C_*(X \setminus K \cap L)} \right)^* &\to &C^*(X | K) \oplus C^*(X | L) &\to C^*(X | K \cup L) \to 0 \\
\downarrow & &\downarrow & &\downarrow & \\
0 &\to &\left( \frac{C_*(X)}{C_*(X \setminus K \cap L)} \right)^* &\to &C^*(X | K) \oplus C^*(X | L) &\to C^*(X | K \cup L) \to 0 \\
\downarrow & &\downarrow & &\downarrow & \\
0 &\to &\left( \frac{C_*(X)}{C_*(X \setminus K \cap L)} \right)^* &\to &C^*(X | K) \oplus C^*(X | L) &\to C^*(X | K \cup L) \to 0 \\
\downarrow & &\downarrow & &\downarrow & \\
0 &\to &\text{id + id} &\to &C^*(X | K) \oplus C^*(X | L) &\to C^*(X | K \cup L) \to 0 \\
\end{array}
\]
This is a diagram. Certainly.

The bottom two rows and all columns are exact. By a diagram chase (the nine lemma), we know the top row is exact. Taking the long exact sequence almost gives what we want, except the first term is a funny thing.

We now analyze that object. We look at the left vertical column:
\[ 0 \to \text{Hom}\left( \frac{C_*(X)}{C_*(X \setminus K \cap L)}, \mathbb{R} \right) \to C^*(X) \to \text{Hom}(C_*(X \setminus K \cap L), \mathbb{R}) \to 0 \]
Now by the small simplices theorem, the right hand object gives the same (co)homology as \( C_*(X \setminus K \cap L; \mathbb{R}) \). So we can produce another short exact
sequence:

\[
\begin{array}{cccc}
0 & \to & \text{Hom} \left( \frac{C_{(x)}}{C^m(X \setminus (K \cap L))}, R \right) & \to C^*(X) \\
& & \uparrow & \downarrow \\
& & C^*(X, X \setminus K \cap L) & \to C^*(X) \to \text{Hom}(C_{(x)}(X \setminus K \cap L), R) \\
& & 0 & 0
\end{array}
\]

Now the two right vertical arrows induce isomorphisms when we pass on to homology. So by taking the long exact sequence, the five lemma tells us the left hand map is an isomorphism on homology. So we know

\[
H_* \left( \text{Hom} \left( \frac{C_{(x)}}{C^m(X \setminus (K \cap L))}, R \right) \right) \cong H^*(X \mid K \cap L).
\]

So the long exact of the top row gives what we want.

**Corollary.** Let \( X \) be a manifold, and \( X = A \cup B \), where \( A, B \) are open sets. Then there is a long exact sequence

\[
\begin{array}{cccc}
H^n_c(A \cap B) & \to & H^n_c(A) \oplus H^n_c(B) & \to H^n_c(X) \\
\downarrow & & & \downarrow \\
H^{n+1}_c(A \cap B) & \to & H^{n+1}_c(A) \oplus H^{n+1}_c(B) & \to \cdots
\end{array}
\]

**Proof.** Let \( K \subseteq A \) and \( L \subseteq B \) be compact sets. Then by excision, we have isomorphisms

\[
H^n(X \mid K) \cong H^n(A \mid K),
\]

\[
H^n(X \mid L) \cong H^n(B \mid L),
\]

\[
H^n(X \mid K \cap L) \cong H^n(A \cap B \mid K \cap L).
\]

So the long exact sequence from the previous proposition gives us

\[
\begin{array}{cccc}
H^n(A \cap B \mid K \cap L) & \to & H^n(A \mid K) \oplus H^n(B \mid L) & \to H^n(X \mid K \cup L) \\
\downarrow & & & \downarrow \\
H^{n+1}(A \cap B \mid K \cap L) & \to & H^{n+1}(A \mid K) \oplus H^{n+1}(B \mid L) & \to \cdots
\end{array}
\]

The next step is to take the direct limit over \( K \in K(A) \) and \( L \in K(B) \). We need to make sure that these do indeed give the right compactly supported cohomology. The terms \( H^n(A \mid K) \oplus H^n(B \mid L) \) are exactly right, and the one for \( H^n(A \cap B \mid K \cap L) \) also works because every compact set in \( A \cap B \) is a compact set in \( A \) intersect a compact set in \( B \) (take those to be both the original compact set).

So we get a long exact sequence

\[
\begin{array}{cccc}
H^n_c(A \cap B) & \to & H^n_c(A) \oplus H^n_c(B) & \to \bigcup_{K \in K(A)} H^n_c(X \mid K \cup L) \\
\to & & & \to \\
& & & \to \\
& & & \to \\
& & & \to \bigcup_{L \in K(B)} H^n_c(X \mid K \cap L) \to H^{n+1}_c(A \cap B)
\end{array}
\]
To show that that funny direct limit is really what we want, we have to show that every compact set $C \in X$ lies inside some $K \cup L$, where $K \subseteq A$ and $L \subseteq B$ are compact.

Indeed, as $X$ is a manifold, and $C$ is compact, we can find a finite set of closed balls in $X$, each in $A$ or in $B$, such that their interiors cover $C$. So done. (In general, this will work for any locally compact space)

11.2 Orientation of manifolds

Lemma.

(i) If $R = \mathbb{F}_2$, then every manifold is $R$-orientable.

(ii) If $\{ \varphi_\alpha : \mathbb{R}^d \to U_\alpha \subseteq M \}$ is an open cover of $M$ by Euclidean space such that each homeomorphism

$$\mathbb{R}^d \supseteq \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \xleftarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^d$$

is orientation-preserving, then $M$ is $R$-orientable.

Proof.

(i) $\mathbb{F}_2$ has a unique $\mathbb{F}_2$-module generator.

(ii) For $x \in U_\alpha$, we define $\mu_x$ to be the image of the standard orientation of $\mathbb{R}^d$ via

$$H_d(M \mid x) \xleftarrow{\cong} H_0(U_\alpha \mid x) \xleftarrow{\varphi_\alpha^{-1}} H_d(\varphi_\alpha^{-1}(x)) \xrightarrow{\text{trans.}} H_d(\mathbb{R}^d \mid 0)$$

If this is well-defined, then it is obvious that this is compatible. However, we have to check it is well-defined, because to define this, we need to pick a chart.

If $x \in U_\beta$ as well, we need to look at the corresponding $\mu'_x$ defined using $U_\beta$ instead. But they have to agree by definition of orientation-preserving.

Theorem. Let $M$ be an $R$-oriented manifold and $A \subseteq M$ be compact. Then

(i) There is a unique class $\mu_A \in H_d(M \mid A; R)$ which restricts to $\mu_x \in H_d(M \mid x; R)$ for all $x \in A$.

(ii) $H_i(M \mid A; R) = 0$ for $i > d$.

Proof. Call a compact set $A$ “good” if it satisfies the conclusion of the theorem.

Claim. We first show that if $K, L$ and $K \cap L$ is good, then $K \cup L$ is good.

This is analogous to the proof of the Thom isomorphism theorem, and we will omit this.

Now our strategy is to prove the following in order:

(i) If $A \subseteq \mathbb{R}^d$ is convex, then $A$ is good.

(ii) If $A \subseteq \mathbb{R}^d$, then $A$ is good.
Manifolds and Poincaré duality

11 Algebraic Topology (Theorems with proof)

(iii) If \( A \subseteq M \), then \( A \) is good.

**Claim.** If \( A \subseteq \mathbb{R}^d \) is convex, then \( A \) is good.

Let \( x \in A \). Then we have an inclusion

\[
\mathbb{R}^d \setminus A \hookrightarrow \mathbb{R}^d \setminus \{x\}.
\]

This is in fact a homotopy equivalence by scaling away from \( x \). Thus the map

\[
H_i(\mathbb{R}^d \mid A) \to H_i(\mathbb{R}^d \mid x)
\]

is an isomorphism by the five lemma for all \( i \). Then in degree \( d \), there is some \( \mu_A \) corresponding to \( \mu_x \). This \( \mu_A \) is then has the required property by definition of orientability. The second part of the theorem also follows by what we know about \( H_i(\mathbb{R}^d \mid x) \).

**Claim.** If \( A \subseteq \mathbb{R}^d \), then \( A \) is good.

For \( A \subseteq \mathbb{R}^d \) compact, we can find a finite collection of closed balls \( B_i \) such that

\[
A \subseteq \bigcup_{i=1}^n \bar{B}_i = B.
\]

Moreover, if \( U \supseteq A \) for any open \( U \), then we can in fact take \( B_i \subseteq U \). By induction on the number of balls \( n \), the first claim tells us that any \( B \) of this form is good.

We now let

\[
\mathcal{G} = \{ B \subseteq \mathbb{R}^d : A \subseteq \bar{B}, B \text{ compact and good} \}.
\]

We claim that this is a directed set under inverse inclusion. To see this, for \( B, B' \in \mathcal{G} \), we need to find a \( B'' \in \mathcal{G} \) such that \( B'' \subseteq B, B' \) and \( B'' \) is good and compact. But the above argument tells us we can find one contained in \( B' \cup B'' \).

We now let

\[
B \mapsto H_i(\mathbb{R}^d \mid B),
\]

and there is an induced map

\[
\lim_{B \in \mathcal{G}} H_i(\mathbb{R}^d \mid B) \to H_i(\mathbb{R}^d \mid A),
\]

since each \( H_i(\mathbb{R}^d \mid B) \) maps to \( H_i(\mathbb{R}^d \mid A) \) by inclusion, and these maps are compatible. We claim that this is an isomorphism. We first show that this is surjective. Let \( [c] \in H_i(\mathbb{R}^d \mid A) \). Then the boundary of \( c \in C_i(\mathbb{R}^d) \) is a finite sum of simplices in \( \mathbb{R}^d \setminus A \). So it is a sum of simplices in some compact \( C \subseteq \mathbb{R}^d \setminus A \). But then \( A \subseteq \mathbb{R}^d \setminus C \), and \( \mathbb{R}^d \setminus C \) is an open neighbourhood of \( A \). So we can find a good \( B \) such that

\[
A \subseteq B \subseteq \mathbb{R}^d \setminus C.
\]

Then \( c \in C_i(\mathbb{R}^d \mid B) \) is a cycle. So we know \( [c] \in H_i(\mathbb{R}^d \mid B) \). So the map is surjective. Injectivity is obvious.

An immediate consequence of this is that for \( i > d \), we have \( H_i(\mathbb{R}^d \mid A) = 0 \). Also, if \( i = d \), we know that \( \mu_A \) is given uniquely by the collection \( \{ \mu_B \}_{B \in \mathcal{G}} \) (uniqueness follows from injectivity).
Claim. If $A \subseteq M$, then $A$ is good.

This follows from the fact that any compact $A \subseteq M$ can be written as a finite union of compact $A_\alpha$ with $A_\alpha \subseteq U_\alpha \cong \mathbb{R}^d$. So $A_\alpha$ and their intersections are good. So done.

**Corollary.** If $M$ is compact, then we get a unique class $[M] = \mu_M \in H_0(M; R)$ such that it restricts to $\mu_x$ for each $x \in M$. Moreover, $H^i(M; R) = 0$ for $i > d$.

### 11.3 Poincaré duality

**Theorem** (Poincaré duality). Let $M$ be a $d$-dimensional $R$-oriented manifold. Then there is a map

$$D_M : H^k_c(M; R) \to H_{d-k}(M; R)$$

that is an isomorphism.

**Lemma.** We have

$$d(\sigma \lhd \varphi) = (-1)^d((d\sigma) \lhd \varphi - \sigma \lhd (d\varphi)).$$

**Proof.** Write both sides out.

**Lemma.** If $f : X \to Y$ is a map, and $x \in H_k(X; R)$ and $y \in H^\ell(Y; R)$, then we have

$$f_* (x \lhd y) = f_* (x \lhd f^* (y)) \in H_{k-\ell}(Y; R).$$

In other words, the following diagram commutes:

$$
\begin{array}{ccc}
H_k(X; R) \times H^\ell(Y; R) & \xrightarrow{f_\ast \times \text{id}} & H_k(Y; R) \\
\text{id} \times f^* & & \\
\downarrow & & \downarrow f_\ast \\
H_k(X; R) \times H^\ell(Y; R) & \xrightarrow{f_\ast \times \text{id}} & H_k(Y; R)
\end{array}
$$

**Proof.** We check this on the cochain level. We let $x = \sigma : \Delta^k \to X$. Then we have

$$f_\ast (\sigma \lhd f^\# y) = f_\ast ((f^\# y)(\sigma|_{v_0, \ldots, v_\ell})\sigma|_{v_{\ell}, \ldots, v_k}) = y(f_\ast(\sigma|_{v_0, \ldots, v_\ell}))f_\ast(\sigma|_{v_{\ell}, \ldots, v_k}) = y((f_\ast \sigma)|_{v_0, \ldots, v_\ell})(f_\ast \sigma)|_{v_{\ell}, \ldots, v_k} = (f_\ast \sigma) \lhd y.$$

So done.

**Proof.** We say $M$ is “good” if the Poincaré duality theorem holds for $M$. We now do the most important step in the proof:

**Claim 0.** $\mathbb{R}^d$ is good.
The only non-trivial degrees to check are $\ell = 0, d$, and $\ell = 0$ is straightforward. For $\ell = d$, we have shown that the maps

$$H^\ell_c(\mathbb{R}^d; R) \xrightarrow{\sim} H^d(\mathbb{R}^d \mid 0; R) \xrightarrow{\text{UCT}} \text{Hom}_R(H_0(\mathbb{R}^d \mid 0; R), R)$$

are isomorphisms, where the last map is given by the universal coefficients theorem.

Under these isomorphisms, the map

$$H^\ell_c(\mathbb{R}^d; R) \xrightarrow{D_\ell} H_0(\mathbb{R}^d; R) \xrightarrow{\epsilon} R$$

corresponds to the map $\text{Hom}_K(H_d(\mathbb{R}^d \mid 0; R), R)$ is given by evaluating a function at the fundamental class $\mu_0$. But as $\mu_0 \in H_d(\mathbb{R}^d \mid 0; R)$ is an $R$-module generator, this map is an isomorphism.

**Claim 1.** If $M = U \cup V$ and $U, V, U \cap V$ are good, then $M$ is good.

Again, this is an application of the five lemma with the Mayer-Vietoris sequence. We have

$$
\begin{array}{cccccc}
H^\ell_c(U \cap V) & \xrightarrow{D_{U \cap V}} & H^\ell_c(U) \oplus H^\ell_c(V) & \xrightarrow{D_M} & H^\ell_c(M) & \xrightarrow{D_{U \cap V}} H^{\ell+1}_c(U \cap V) \\
\downarrow D_{U \cap V} & & \downarrow D_{U \oplus V} & & \downarrow D_M & \\
H_{d-\ell}(U \cap V) & \xrightarrow{D_{U \cap V}} H_{d-\ell}(U) \oplus H_{d-\ell}(V) & \xrightarrow{D_M} & H_{d-\ell}(M) & \xrightarrow{D_{U \cap V}} H_{d-\ell-1}(U \cap V)
\end{array}
$$

We are done by the five lemma if this commutes. But unfortunately, it doesn’t. It only commutes up to a sign, but it is sufficient for the five lemma to apply if we trace through the proof of the five lemma.

**Claim 2.** If $U_1 \subseteq U_2 \subseteq \cdots$ with $M = \bigcup_n U_n$, and $U_i$ are all good, then $M$ is good.

Any compact set in $M$ lies in some $U_n$, so the map

$$\varinjlim H^\ell_c(U_n) \to H^\ell_c(U)$$

is an isomorphism. Similarly, since simplices are compact, we also have

$$H_{d-k}(M) = \varinjlim H_{d-k}(U_n).$$

Since the direct limit of open sets is open, we are done.

**Claim 3.** Any open subset of $\mathbb{R}^d$ is good.

Any $U$ is a countable union of open balls (something something rational points something something). For finite unions, we can use Claims 0 and 1 and induction. For countable unions, we use Claim 2.

**Claim 4.** If $M$ has a countable cover by $\mathbb{R}^d$’s it is good.

Same argument as above, where we instead use Claim 3 instead of Claim 0 for the base case.

**Claim 5.** Any manifold $M$ is good.
Any manifold is second-countable by definition, so has a countable open cover by $\mathbb{R}^d$.

**Corollary.** For any compact $d$-dimensional $\mathbb{R}$-oriented manifold $M$, the map

$$[M] \smile \cdot : H^d(M; R) \to H_{d-\ell}(M; R)$$

is an isomorphism.

**Corollary.** Let $M$ be an odd-dimensional compact manifold. Then the Euler characteristic $\chi(M) = 0$.

**Proof.** Pick $R = \mathbb{F}_2$. Then $M$ is $\mathbb{F}_2$-oriented. Since we can compute Euler characteristics using coefficients in $\mathbb{F}_2$, we then have

$$\chi(M) = \sum_{r=0}^{2n+1} (-1)^i \dim \mathbb{F}_2 H^i(M, \mathbb{F}_2).$$

But we know $H^i(M, \mathbb{F}_2) \cong (H_{2n-1-i}(M, \mathbb{F}_2))^\ast \cong H_{2n-1-i}(M, \mathbb{F}_2)$ by Poincaré duality and the universal coefficients theorem.

But the dimensions of these show up in the sum above with opposite signs. So they cancel, and $\chi(M) = 0$.

**Theorem.** Let $M$ be a $d$-dimensional compact $\mathbb{R}$-oriented manifold, and consider the following pairing:

$$\langle \cdot, \cdot \rangle: H^k(M; R) \otimes H^{d-k}(M, R) \to R$$

$$[\varphi] \otimes [\psi] \mapsto \langle \varphi, \psi \rangle [M].$$

If $H_\ast(M; R)$ is free, then $\langle \cdot, \cdot \rangle$ is non-singular, i.e. both adjoints are isomorphisms, i.e. both

$$H^k(M; R) \to \text{Hom}(H^{d-k}(M; R), R)$$

$$[\varphi] \mapsto ([\psi] \mapsto \langle \varphi, \psi \rangle)$$

and the other way round are isomorphisms.

**Proof.** We have

$$\langle \varphi, \psi \rangle = (-1)^{||\varphi||} \langle \psi, \varphi \rangle,$$

as we know

$$\varphi \smile \psi = (-1)^{||\varphi||} \psi \smile \varphi.$$

So if one adjoint is an isomorphism, then so is the other.

To see that they are isomorphisms, we notice that we have an isomorphism

$$H^k(M; R) \xleftarrow{\text{UCT}} \text{Hom}_R(H_k(M; R), R) \xrightarrow{D^m_R} \text{Hom}_R(H^{d-k}(M; R), R).$$

$$[\varphi] \mapsto ([\sigma] \mapsto \varphi(\sigma)) \mapsto ([\psi] \mapsto \varphi([M] \smile \psi)).$$

But we know

$$\varphi([M] \smile \psi) = (\psi \smile \varphi)([M]) = \langle \psi, \varphi \rangle.$$

So this is just the adjoint. So the adjoint is an isomorphism.
11.4 Applications

**Corollary.** Let \( f : M \to N \) be a map between manifolds. If \( \mathbb{F} \) is a field and \( \deg(f) \neq 0 \in \mathbb{F} \), then the induced map
\[
f^* : H^*(N, \mathbb{F}) \to H^*(M, \mathbb{F})
\]
is injective.

**Proof.** Suppose not. Let \( \alpha \in H^k(N, \mathbb{F}) \) be non-zero but \( f^*(\alpha) = 0 \). As
\[
\langle \cdot, \cdot \rangle : H^k(N, \mathbb{F}) \otimes H^{d-k}(N, \mathbb{F}) \to \mathbb{F}
\]
is non-singular, we know there is some \( \beta \in H^{d-k}(N, \mathbb{F}) \) such that
\[
\langle \alpha, \beta \rangle = (\alpha \smile \beta)([N]) = 1.
\]
Then we have
\[
\deg(f) = \deg(f) \cdot 1 = (\alpha \smile \beta)(\deg(f)[N]) = (\alpha \smile \beta)(f^*([M])) = (f^*(\alpha) \smile f^*(\beta))(\langle [M] \rangle) = 0.
\]
This is a contradiction. \( \square \)

11.5 Intersection product

**Lemma.** Let \( X \) be a space and \( V \) a vector bundle over \( X \). If \( V = U \oplus W \), then orientations for any two of \( U, W, V \) give an orientation for the third.

**Proof.** Say \( \dim V = d, \dim U = n, \dim W = m \). Then at each point \( x \in X \), by Künneth’s theorem, we have an isomorphism
\[
H^d(V_x, V_x^#; R) \cong H^n(U_x, U_x^#; R) \otimes H^m(W_x, W_x^#; R) \cong R.
\]
So any local \( R \)-orientation on any two induces one on the third, and it is straightforward to check the local compatibility condition. \( \square \)

**Theorem.** The Poincaré dual of a submanifold is (the extension by zero of) the normal Thom class.

11.6 The diagonal

**Theorem.** We have
\[
\delta = \sum_i (-1)^{|a_i|} a_i \otimes b_i.
\]

**Proof.** We can certainly write
\[
\delta = \sum_i C_{ij} a_i \otimes b_j
\]
for some $C_{ij}$. So we need to figure out what the coefficients $C_{ij}$ are. We try to compute

$$(b_k \otimes a_{\ell} \sim \delta)[M \times M] = \sum C_{ij} (b_k \otimes a_{\ell}) \sim (a_i \otimes b_j)[M \times M]$$

$$= \sum C_{ij} (-1)^{|a_i||a_j|} (b_k \sim a_i) \otimes (a_{\ell} \sim b_j)[M] \otimes [M]$$

$$= \sum C_{ij} (-1)^{|a_i||a_j|} (\delta_{ik} (-1)^{|a_i||b_k|}) \delta_{j\ell}$$

$$= (-1)^{|a_k||a_{\ell}|+|a_{\ell}||b_k|} C_{kl}.$$

But we can also compute this a different way, using the definition of $\delta$:

$$(b_k \otimes a_{\ell} \sim \delta)[M \times M] = (b_k \otimes a_{\ell})(\Delta_*[M]) = (b_k \sim a_{\ell})[M] = (-1)^{|a_{\ell}||b_k|} \delta_{kl}.$$ 

So we see that

$$C_{kl} = \delta_{kl} (-1)^{|a_{\ell}|}.$$ 

**Corollary.** We have

$$\Delta^*(\delta)[M] = \chi(M),$$

the Euler characteristic.

**Proof.** We note that for $a \otimes b \in H^n(M \times M)$, we have

$$\Delta^*(a \otimes b) = \Delta^*(\pi_1^* a \sim \pi_2^* b) = a \sim b$$

because $\pi_1 \circ \Delta = \text{id}$. So we have

$$\Delta^*(\delta) = \sum (-1)^{|a_i|} a_i \sim b_i.$$ 

Thus

$$\Delta^*(\delta)[M] = \sum_i (-1)^{|a_i|} = \sum_k (-1)^k \dim_{\mathbb{Q}} H^k(M; \mathbb{Q}) = \chi(M).$$

**Corollary.** We have

$$e(TM)[M] = \chi(M).$$

**Corollary.** If $M$ has a nowhere-zero vector field, then $\chi(M) = 0$.

**Lemma.** Suppose we have $R$-oriented vector bundles $E \to X$ and $F \to X$ with Thom classes $u_E, u_F$. Then the Thom class for $E \oplus F \to X$ is $u_E \sim u_F$. Thus

$$e(E \oplus F) = e(E) \sim e(F).$$

**Proof.** More precisely, we have projection maps

$$\begin{array}{ccc}
E & \xrightarrow{\pi_E} & E \oplus F \\
\downarrow & & \downarrow \pi_F \\
E & \rightarrow & F
\end{array}$$

We let $U = \pi_E^{-1}(E^\#)$ and $V = \pi_F^{-1}(F^\#)$. Now observe that

$$U \cup V = (E \oplus F)^\#.$$
So if \( \dim E = e, \dim F = f \), then we have a map

\[
H^e(E, E^\#) \otimes H^f(F, F^\#) \xrightarrow{\pi_k \otimes \pi_k} H^e(E \oplus F, U) \otimes H^f(E \oplus F, V) \\
\downarrow \\
H^{e+f}(E \oplus F, (E \oplus F)^\#)
\]

and it is easy to see that the image of \( u_E \otimes u_F \) is the Thom class of \( E \oplus F \) by checking the fibers.

**Corollary.** \( TS^{2n} \) has no proper subbundles.

**Proof.** We know \( e(TS^{2n}) \neq 0 \) as \( e(TS^{2n})[S^2] = \chi(S^{2n}) = 2 \). But it cannot be a proper cup product of two classes, since there is nothing in lower cohomology groups. So \( TS^{2n} \) is not the sum of two subbundles. Hence \( TS^{2n} \) cannot have a proper subbundle \( E \) or else \( TS^{2n} = E \oplus E^\perp \) (for any choice of inner product).

### 11.7 Lefschetz fixed point theorem

**Theorem** (Lefschetz fixed point theorem). Let \( M \) be a compact \( d \)-dimensional \( \mathbb{Z} \)-oriented manifold, and let \( f : M \to M \) be a map such that the graph \( \Gamma_f \) and diagonal \( \Delta \) intersect transversely. Then

\[
\sum_{x \in \text{fix}(f)} \text{sgn} \det(I - D_x f) = \sum_k (-1)^k \text{tr}(f^*: H^k(M; \mathbb{Q}) \to H^k(M; \mathbb{Q})).
\]

**Proof.** We have

\[
[\Gamma_f] \cdot [\Delta(M)] \in H_0(M \times M; \mathbb{Q}).
\]

We now want to calculate \( \varepsilon \) of this. By Poincaré duality, this is equal to

\[
(D_{M \times M}^{-1} \Gamma_f) \cdot (D_{M \times M}^{-1} \Delta(M)) [M \times M] \in \mathbb{Q}.
\]

This is the same as

\[
(D_{M \times M}^{-1} \Delta(M)) ([\Gamma_f]) = \delta(F_*[M]) = (F^* \delta)[M],
\]

where \( F : M \to M \times M \) is given by

\[
F(x) = (x, f(x)).
\]

We now use the fact that

\[
\delta = \sum (-1)^{|a_i|} a_i \otimes b_i.
\]

So we have

\[
F^* \delta = \sum (-1)^{|a_i|} a_i \otimes f^* b_i.
\]

We write

\[
f^* b_i = \sum C_{ij} b_j.
\]

Then we have

\[
(F^* \delta)[M] = \sum_{i, j} (-1)^{|a_i|} C_{ij} (a_i \otimes b_j) [M] = \sum_i (-1)^{|a_i|} C_{ii},
\]

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and $C_{ii}$ is just the trace of $f^*$.

We now compute this product in a different way. As $\Gamma_f$ and $\Delta(M)$ are transverse, we know $\Gamma_f \cap \Delta(M)$ is a 0-manifold, and the orientation of $\Gamma_f$ and $\Delta(M)$ induces an orientation of it. So we have

$$[\Gamma_f] \cdot [\Delta(m)] = [\Gamma_f \cap \Delta(M)] \in H_0(M \times M; \mathbb{Q}).$$

We know this $\Gamma_f \cap \Delta(M)$ has $|\text{fix}(f)|$ many points, so $[\Gamma_f \cap \Delta(M)]$ is the sum of $|\text{fix}(f)|$ many things, which is what we’ve got on the left above. We have to figure out the sign of each term is actually $\text{sgn det}(I - D_x f)$, and this is left as an exercise on the example sheet. \qed