Advanced Probability 1

1.1 Let $X, Y \in L^1(\mathbb{P})$ and let $\mathcal{G}$ be a σ-algebra. Show that

$$\mathbb{E}(X+Y|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Y|\mathcal{G}) \quad \text{almost surely.}$$

1.2 Let X be a non-negative random variable and let Y be a version of $\mathbb{E}(X|\mathcal{G})$. Show that $\{X > 0\} \subseteq \{Y > 0\}$ almost surely, that is, $1_{\{X>0\}} \leq 1_{\{Y>0\}}$ almost surely. Show further that, for all $A \in \mathcal{G}$, if $\{X > 0\} \subseteq A$ almost surely then $\{Y > 0\} \subseteq A$ almost surely.

1.3 Let $X, Y \in L^2(\mathbb{P})$. Show that if

$$\mathbb{E}(X|Y) = Y \quad \text{and} \quad \mathbb{E}(Y|X) = X \quad \text{almost surely}$$

then $X = Y$ almost surely. Show that this holds also for $X, Y \in L^1(\mathbb{P})$.

1.4 Let X be an integrable random variable and let $x \in \mathbb{R}$. Show that

$$\mathbb{E}(X|X \leq x) \leq \mathbb{E}(X).$$

Here and below, it is to be assumed that the conditioning event has positive probability. Let Y be another random variable, independent of X, and let f be a non-decreasing function such that $f(X+Y)$ is integrable. Show that

$$\mathbb{E}(f(X+Y)|X \leq x) \leq \mathbb{E}(f(X+Y)).$$

Let $S_n = X_1 + \cdots + X_n$, where $X_1, \dots, X_n$ are independent, and let $x_1, \dots, x_n \in \mathbb{R}$. Show that

$$\mathbb{P}(S_n \geq x | X_1 \leq x_1, \dots, X_n \leq x_n) \leq \mathbb{P}(S_n \geq x).$$

1.5 Show that, for any sequence of non-negative random variables $(X_n : n \in \mathbb{N})$ and any σ-algebra $\mathcal{G}$,

$$\mathbb{E}(\liminf X_n|\mathcal{G}) \leq \liminf \mathbb{E}(X_n|\mathcal{G}) \quad \text{almost surely.}$$

1.6 Let X and Y be random variables and let $\lambda \in (0, \infty)$. Show that, if X and $Y - X$ are independent exponential random variables of parameter λ, then Y has density $\lambda^2 y e^{-\lambda y}$ on $(0, \infty)$ and, for all $x \geq 0$, almost surely,

$$\mathbb{P}(X \leq x|Y) = (x/Y) \wedge 1.$$

Show that the converse also holds.

2.1 Let $(X_n)_{n \geq 0}$ be an integrable process, taking values in a countable set $E \subseteq \mathbb{R}$. Show that $(X_n)_{n\geq 0}$ is a martingale in its natural filtration if and only if, for all n and for all $x_0, \dots, x_n \in E$, whenever the conditioning event has positive probability, we have

$$\mathbb{E}(X_{n+1}|X_0 = x_0, \dots, X_n = x_n) = x_n.$$

2.2 Let $(X_n)_{n\geq 0}$ be a martingale and let f be a convex function on $\mathbb{R}$ such that $f(X_n)$ is integrable for all n. Show that $(f(X_n)_{n\geq 0}$ is a submartingale.

2.3 Let $(X_n)_{n\geq 0}$ be a martingale and let T be a stopping time. Consider the following conditions

(a) $T \leq n$ for some $n \geq 0$,
(b) there is a constant $C < \infty$ such that $|X_n| \leq C$ for all $n \leq T$ almost surely,
(c) $\mathbb{E}(T) < \infty$ and there is a constant $C < \infty$ such that $|X_{n+1} - X_n| \leq C$ for all $n < T$ almost surely.

Show that, under each one of these conditions, X_T is integrable and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

2.4 Let $X \in L^2(\mathbb{P})$ and set

$$X_n = \mathbb{E}(X|\mathcal{F}_n)$$

where $(\mathcal{F}_n)_{n\geq 0}$ is a given filtration. Show that, for all $m \leq n$,

$$\|X_m\|_2^2 + \|X_m - X_n\|_2^2 = \|X_n\|_2^2.$$

Hence show there exists $Y \in L^2(\mathbb{P})$ such that $X_n \to Y$ in L^2. Show further that $Y = X$ almost surely if and only if X is $\mathcal{F}_\infty$-measurable, where $\mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geq 0)$.

2.5 Let $(X_n)_{n\geq 0}$ be a martingale, starting from 0. Show that $(X_n)_{n\geq 0}$ is bounded in L^2 if and only if $\sum_n \|X_{n+1} - X_n\|_2^2 < \infty$.

3.1 *Pólya's urn.* At time 0, an urn contains two balls, one black, the other white. Suppose we repeatedly choose a ball at random from the urn and replace it together with a new ball of the same colour. Then, after n steps, there are $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of steps in which a black ball was chosen. Let $M_n = (B_n + 1)/(n+2)$ the proportion of black balls in the urn after n steps. Show that $(M_n)_{n\geq 0}$ is a martingale, relative to a filtration which you should specify. Show also that

$$\mathbb{P}(B_n = k) = (n+1)^{-1}, \quad k = 0, 1, \ldots, n.$$

Deduce that there is a random variable Θ such that $M_n \to \Theta$ almost surely and find the distribution of Θ.

For $\theta \in [0,1]$, set

$$N_n^\theta = \frac{(n+1)!}{B_n!(n-B_n)!}\,\theta^{B_n}(1-\theta)^{n-B_n}.$$

Show that $(N_n^\theta)_{n\geq 0}$ is a martingale.

3.2 *Bayes' urn.* A random number Θ is chosen uniformly in $[0,1]$, and a coin with probability Θ of heads is minted. The coin is tossed repeatedly. Let B_n be the number of heads in n tosses. Show that the process $(B_n)_{n\geq 0}$ has the same distribution as the process $(B_n)_{n\geq 0}$ in Example 3.1. Show that N_n^θ is a conditional density function of Θ given $B_1, \ldots, B_n$.

3.3 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ for all n. Show that the series $\sum_n X_n/n$ converges almost surely.

3.4 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = -1/p_n) = p_n$ and $\mathbb{P}(X_n = 1/q_n) = q_n$, where $p_n = 1/n^2$ and $p_n + q_n = 1$. Set $S_n = X_1 + \cdots + X_n$. Show that $(S_n)_{n\geq 0}$ is a martingale and that S_n/n converges almost surely as $n \to \infty$. Deduce that $S_n \to \infty$ almost surely as $n \to \infty$.

Advanced Probability 2

3.5 Let $(X_n)_{n\geq 0}$ be a Markov chain with state-space E and transition matrix P. Let $f : E \to \mathbb{R}$ be a bounded function. Show that $(f(X_n))_{n\geq 0}$ is a submartingale for all possible initial states $X_0 = x$ if and only if f is subharmonic, that is to say $f \leq Pf$.

3.6 Your winnings per unit stake on game n are ε_n, where $\varepsilon_1, \varepsilon_2, \dots$ are independent random variables with

$$\mathbb{P}(\varepsilon_n = 1) = p, \quad \mathbb{P}(\varepsilon_n = -1) = q,$$

where $p \in (1/2, 1)$ and $q = 1 - p$. Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Your object is to maximize the expected 'interest rate' $\mathbb{E}\log(Z_N/Z_0)$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$. Show that if C is any *previsible* strategy, that is C_n is $\mathcal{F}_{n-1}$-measurable for all n, then $\log Z_n - n\alpha$ is a supermartingale, where

$$\alpha = p\log p + q\log q + \log 2,$$

so that $\mathbb{E}\log(Z_N/Z_0) \leq N\alpha$, but that, for a certain strategy, $\log Z_n - n\alpha$ is a martingale. What is the best strategy?

3.7 Let $(A_n : n \in \mathbb{N})$ be a sequence of independent events all having probability $p > 0$. Fix $N \in \mathbb{N}$ and write T for the first time we achieve a run of N consecutive successes. Thus

$$T = \inf\{n \geq N : 1_{A_n} + \cdots + 1_{A_{n-N+1}} = N\}.$$

Show that $\mathbb{P}(T \geq n) \leq C\alpha^n$ for some constants $C < \infty$ and $\alpha \in (0,1)$ and hence that $\mathbb{E}(T) < \infty$.

3.8 *Wald's identities.* Let $(S_n)_{n\geq 0}$ be a random walk in $\mathbb{R}$, starting from 0, with steps of mean μ and variance $\sigma^2 \in (0,\infty)$. Fix $a, b \in \mathbb{R}$ with $a < 0 < b$ and set

$$T = \inf\{n \geq 0 : S_n \leq a \text{ or } S_n \geq b\}.$$

Show that $\mathbb{E}(T) < \infty$ and $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$. Show further that, in the case $\mu = 0$, we have $\mathbb{E}(S_T^2) = \sigma^2\mathbb{E}(T)$. Show also that, for any $\lambda \in \mathbb{R}$ such that $\mathbb{E}(e^{\lambda S_1}) = 1$, we have $\mathbb{E}(e^{\lambda S_T}) = 1$. In the case of the simple random walk on $\mathbb{Z}$, and for $a, b \in \mathbb{Z}$, use these identities to find $\mathbb{E}(T)$ and $\mathbb{P}(S_T = a)$.

3.9 *Azuma–Hoeffding Inequality.* Let Y be a random variable of mean zero, such that $|Y| \leq c$ for some constant $c < \infty$. Use the convexity of $y \mapsto e^{\theta y}$ on $[-c, c]$ to show that, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta Y}) \leq \cosh\theta c \leq e^{\theta^2 c^2/2}.$$

Now let $(M_n)_{n\geq 0}$ be a martingale, starting from 0, such that $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$, for some constants $c_n < \infty$. Set $v_n = c_1^2 + \cdots + c_n^2$. Show that, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta M_n}) \leq e^{\theta^2 v_n/2}.$$

Show further that, for all $x \geq 0$,

$$\mathbb{P}\left(\sup_{k \leq n} M_k \geq x\right) \leq e^{-x^2/(2v_n)}.$$

3.10 Let f be a Lipschitz function on $[0,1]$ of constant K. Thus, for all $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

Denote by f_n the simplest piecewise linear function agreeing with f on $D_n = \{k2^{-n} : k = 0, 1, \ldots, 2^n\}$. Then f_n has a derivative f'_n on $[0,1] \setminus D_n$. Set $M_n = f'_n 1_{[0,1]\setminus D_n}$. Show that M_n converges almost everywhere and in L^1 and deduce that there is a bounded Borel function f' on $[0,1]$ such that

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

4.1 Show that the σ-algebra on $C([0,\infty), \mathbb{R})$ generated by the coordinate functions is the same as its Borel σ-algebra for the topology of uniform convergence on compacts.

4.2 Let S and T be stopping times and let $(X_t)_{t\geq 0}$ be a cadlag adapted process, associated to a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that $S \wedge T$ is a stopping time, that $\mathcal{F}_T$ is a σ-algebra, and that $\mathcal{F}_S \subseteq \mathcal{F}_T$ if $S \leq T$. Show also that $X_T 1_{\{T<\infty\}}$ is an $\mathcal{F}_T$-measurable random variable.

4.3 Let T be an exponential random variable of parameter 1. Set $X_t = e^t 1_{t<T}$. Describe the natural filtration of $(X_t)_{t\geq 0}$. Show that $\mathbb{E}(X_t 1_{\{T>r\}}) = \mathbb{E}(X_s 1_{\{T>r\}})$ for $r \leq s \leq t$, and hence deduce that $(X_t)_{t\geq 0}$ is a cadlag martingale. Determine whether $(X_t)_{t\geq 0}$ is uniformly integrable.

4.4 Let T be a random variable in $[0,\infty)$ having a positive and continuous density function f on $[0,\infty)$. Define the *hazard function* A on $[0,\infty)$ by

$$A(t) = \int_0^t \frac{f(s)ds}{1 - F(s)}$$

where F is the distribution function of T. Show that $A(T)$ is an exponential random variable of parameter 1. Set $X_t = 1_{\{t\geq T\}} - A(T \wedge t)$. Show that $(X_t)_{t\geq 0}$ is a cadlag martingale.

4.5 Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, satisfying the usual conditions, and let $(\xi_t)_{t\geq 0}$ be an adapted integrable process such that $\mathbb{E}(\xi_t|\mathcal{F}_s) = \xi_s$ almost surely, for all $s, t \geq 0$ with $s \leq t$. Show that there is a cadlag martingale $(X_t)_{t\geq 0}$ such that $\xi_t = X_t$ almost surely, for all $t \geq 0$.

Advanced Probability 3

5.1 Assuming Prohorov's theorem, prove that if $(\mu_n : n \in \mathbb{N})$ is a tight sequence of finite measures on $\mathbb{R}$ and if

$$\sup_n \mu_n(\mathbb{R}) < \infty$$

then there is a subsequence (n_k) and a finite measure μ on $\mathbb{R}$ such that $\mu_{n_k} \to \mu$ weakly.

5.2 *Weak law of large numbers.* Let $(X_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed, integrable random variables. Set $S_n = X_1 + \cdots + X_n$. Use characteristic functions to show that $S_n/n \to \mathbb{E}(X_1)$ weakly.

5.3 Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables and suppose that $X_n \to X$ weakly. Show that, if X is almost surely constant, then also X_n converges to X in probability. Is the condition that X is almost surely constant necessary?

6.1 Let X be an integrable random variable in $\mathbb{R}$ which is not almost surely constant. Show that, for all $\lambda \geq 0$,

$$\mathbb{E}(e^{\lambda X} | X \leq K) \uparrow \mathbb{E}(e^{\lambda X}) \quad \text{as } K \to \infty.$$

Suppose that $M(\lambda) = \mathbb{E}(e^{\lambda X}) < \infty$ for all $\lambda \geq 0$. Show that M has a continuous derivative on $[0, \infty)$ and has derivatives of all orders on $(0, \infty)$. Set $\psi(\lambda) = \log M(\lambda)$ and define a new probability measure $\mathbb{P}_\lambda$ by

$$\mathbb{P}_\lambda(A) = \mathbb{E}(e^{\lambda X - \psi(\lambda)} 1_A)$$

Show that X has mean $\psi'(\lambda)$ and variance $\psi''(\lambda)$ under $\mathbb{P}_\lambda$. Hence show that the derivative ψ' define a homeomorphism on $[0, \infty)$ and determine its range.

6.2 Let $(X_t)_{t \geq 0}$ be a Poisson process of rate 1. Show that, for all $a \geq 1$,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}(X_t \geq at) = -a \log a + a - 1.$$

7.1 Let $(B_t)_{t \geq 0}$ be a continuous random process, starting from 0, and let $c \in (0, \infty)$. Show that the following are equivalent

(a) $(B_t)_{t \geq 0}$ is a Brownian motion,
(b) $(B_t)_{t \geq 0}$ is a zero-mean Gaussian process with covariance $\mathbb{E}(B_s B_t) = s \wedge t$ for $s, t \geq 0$,
(c) $(c^{-1} B_{c^2 t})_{t \geq 0}$ is a Brownian motion,
(d) $(t B_{1/t})_{t \geq 0}$ is a Brownian motion,

where in (d) the process is defined to take the value 0 when $t = 0$, and for all $t > 0$ if this is necessary to make it continuous.

7.2 Let $(B_t)_{t \geq 0}$ be a Brownian motion, starting from 0. Define for $a \in \mathbb{R}$

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Show that T_a is a stopping time. Show that, for $a > 0$ and $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = 2\mathbb{P}(B_t \geq a).$$

Hence show that $T_a < \infty$ almost surely and find a density function for T_a.

7.3 Let $(B_t)_{t\geq 0}$ be a Brownian motion, starting from 0. Set $Q_t = B_t^2 - t$. Show that $(B_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ are continuous martingales. For $a, b > 0$, show that $\mathbb{P}(T_{-a} < T_b) = b/(a+b)$ and find $\mathbb{E}(T_{-a} \wedge T_b)$.

7.4 Let $(B_t)_{t\geq 0}$ be a Brownian motion, starting from 0. Show that

$$\limsup_{t\to 0} B_t/t = -\liminf_{t\to 0} B_t/t = \infty \quad \text{almost surely.}$$

For $a > 0$, set

$$L = \sup\{t > 0 : B_t = at\}.$$

Show that L has the same distribution as T_a^{-1}. Define

$$S = \sup\{t \leq 1 : B_t = 0\}, \quad T = \inf\{t \geq 1 : B_t = 0\}.$$

Show that S and T^{-1} have the same distribution. Hence show that

$$\mathbb{P}(S \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}.$$

7.5 Let $(B_t)_{t\geq 0}$ be a Brownian motion, starting from 0. Find the joint distribution of $(B_t, \max_{s\leq t} B_s)$.

7.6 Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ and let U be an orthogonal $d \times d$ matrix. Show that $(UB_t)_{t\geq 0}$ is also a Brownian motion in $\mathbb{R}^d$.

7.7 Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^3$. You may assume that $B_t \neq 0$ for all $t > 0$ almost surely. Set $R_t = 1/|B_t|$. Show that

(i) $(R_t : t \geq 1)$ is bounded in L^2,
(ii) $\mathbb{E}(R_t) \to 0$ as $t \to \infty$,
(iii) R_t is a supermartingale.

Deduce that $|B_t| \to \infty$ almost surely as $t \to \infty$.

7.8 *Blumenthal's zero-one law.* Let $(B_t)_{t\geq 0}$ be a Brownian motion, starting from 0. Set

$$\mathcal{F}_t^B = \sigma(B_s : s \leq t), \quad \mathcal{F}_{0+}^B = \cap_{t>0} \mathcal{F}_t^B.$$

Show that $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_{0+}^B$.

7.9 Let $(B_t)_{t\geq 0}$ be a Brownian motion, starting from 0. Show that, almost surely,

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \limsup_{t\uparrow\infty} \frac{B_t}{\sqrt{t}} = -\liminf_{t\uparrow\infty} \frac{B_t}{\sqrt{t}} = \infty.$$

Advanced Probability 4

7.10 Let μ denote Wiener measure on $W = \{x \in C([0,1],\mathbb{R}) : x_0 = 0\}$. For $a \in \mathbb{R}$, define a new probability measure μ_a on W by

$$d\mu_a/d\mu(x) = \exp(ax_1 - a^2/2).$$

Show that under μ_a the coordinate process remains Gaussian, and identify its distribution. Deduce that $\mu(A) > 0$ for every non-empty open set $A \subseteq W$.

7.11 Let $B = (B_t)_{0\le t\le 1}$ be a Brownian motion, starting from 0. Denote by μ the law of B on $W = C([0,1],\mathbb{R})$. For each $y \in \mathbb{R}$, set

$$Z_t^y = yt + (B_t - tB_1)$$

and denote by μ^y the law of $Z^y = (Z_t^y)_{0\le t\le 1}$ on W. Show that, for any bounded measurable function $F : W \to \mathbb{R}$ and for $f(y) = \mu^y(F)$ we have

$$\mathbb{E}(F(B)|B_1) = f(B_1) \quad \text{almost surely.}$$

7.12 Let D be a bounded open set in $\mathbb{R}^n$ and let $h : \bar{D} \to \mathbb{R}$ be a bounded continuous function, harmonic in D. Show that, for all $x \in D$,

$$\inf_{y\in\partial D} h(y) \le h(x) \le \sup_{y\in\partial D} h(y).$$

7.13 (i) Let $(B_t)_{t\ge 0}$ be a Brownian motion in $\mathbb{R}^2$, starting from (x, y). Compute the distribution of B_T, where

$$T = \inf\{t \ge 0 : B_t \notin H\}$$

and where H is the upper half plane $\{(x, y) : y > 0\}$.

(ii) Show that, for any bounded continuous function $u : \bar{H} \to \mathbb{R}$, harmonic in H, with $u(x, 0) = f(x)$ for all $x \in \mathbb{R}$, we have

$$u(x,y) = \int_{\mathbb{R}} f(s)\frac{1}{\pi}\,\frac{y}{(x-s)^2 + y^2}\,ds.$$

(iii) Let D be any open set in $\mathbb{R}^2$ for which there exists a continuous homeomorphism $g : \bar{H} \to \bar{D}$, which is conformal in H. Show that, if u is harmonic in D, then $u \circ g$ is harmonic in H.

(iv) Find an explicit integral representation for bounded continuous functions $u : \bar{D} \to \mathbb{R}$, harmonic in D, in terms of their values on the boundary of D.

(v) Determine the exit distribution of Brownian motion from D.

8.1 Let $(E, \mathcal{E}, K)$ be a finite measure space and let $g \in L^1(K)$. Let $\tilde{M} = M - \mu$ be a compensated Poisson random measure on $(0, \infty) \times E$, where the compensator μ is determined by $\mu((0,t] \times A) = tK(A)$ for $A \in \mathcal{E}$. Set

$$\tilde{M}_t(g) = \begin{cases} \int_{(0,t]\times E} g(y)\tilde{M}(ds, dy), & \text{if } M((0,t] \times E) < \infty \text{ for all } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $(\tilde{M}_t(g))_{t\geq 0}$ is a cadlag martingale with stationary independent increments. Show further that, for all $t \geq 0$,

$$\mathbb{E}(\tilde{M}_t^2(g)) = t \int_E g(y)^2 K(dy)$$

and

$$\mathbb{E}(e^{iu\tilde{M}_t(g)}) = \exp\left\{ t \int_E (e^{iug(y)} - 1 - iug(y))K(dy) \right\}.$$

9.1 Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic exponent ψ. Show that, for all $u \in \mathbb{R}$, the following process is a martingale

$$M_t^u = \exp\{iuX_t - t\psi(u)\}.$$

9.2 Say that a Lévy process $(X_t)_{t\geq 0}$ satisfies the scaling relation with exponent $\alpha \in (0, \infty)$ if

$$(cX_{c^{-\alpha}t})_{t\geq 0} \sim (X_t)_{t\geq 0}, \quad c \in (0, \infty).$$

For example, Brownian motion satisfies the scaling relation with exponent 2. Find, for each $\alpha \in (0, 2)$, a Lévy process having a scaling relation with exponent α.

9.3 Let $(X_t)_{t\geq 0}$ be the Lévy process corresponding to the Lévy triple (a, b, K). Show that, if K consists of finitely many atoms, then $(X_t)_{t\geq 0}$ can be written as a linear combination of a Brownian motion, a uniform drift and finitely many Poisson processes.