Part III — Stochastic Calculus and Applications

Theorems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

- **Brownian motion.** Existence and sample path properties.
- **Stochastic calculus for continuous processes.** Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô’s isometry, definition of the stochastic integral, Kunita–Watanabe theorem, and Itô’s formula.
- **Applications to Brownian motion and martingales.** Lévy characterization of Brownian motion, Dubins–Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- **Stochastic differential equations.** Strong and weak solutions, notions of existence and uniqueness, Yamada–Watanabe theorem, strong Markov property, and relation to second order partial differential equations.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.
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0 Introduction

**Proposition.** Let $H$ be any separable Hilbert space. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Gaussian subspace $S \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ and an isometry $I : H \rightarrow S$. In other words, for any $f \in H$, there is a corresponding random variable $I(f) \sim N(0, (f, f)_H)$. Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ and $(f, g)_H = \mathbb{E}[I(f)I(g)]$.

**Proposition.**

- For $A \subseteq \mathbb{R}_+$ with $|A| < \infty$, $WN(A) \sim N(0, |A|)$.
- For disjoint $A, B \subseteq \mathbb{R}_+$, the variables $WN(A)$ and $WN(B)$ are independent.
- If $A = \bigcup_{i=1}^{\infty} A_i$ for disjoint sets $A_i \subseteq \mathbb{R}_+$, with $|A| < \infty, |A_i| < \infty$, then

$$WN(A) = \sum_{i=1}^{\infty} WN(A_i)$$

in $L^2$ and a.s.
1 The Lebesgue–Stieltjes integral

**Theorem.** For any two finite measures $\mu_1, \mu_2$, there is a signed measure $\mu$ with $\mu(A) = \mu_1(A) - \mu_2(A)$.

**Theorem.** There is a bijection

\[
\{ \text{signed measures on } [0,T] \} \leftrightarrow \{ \text{càdlàg functions of bounded variation } a : [0,T] \to \mathbb{R} \}
\]

that sends a signed measure $\mu$ to $a(t) = \mu([0,t])$. To construct the inverse, given $a$, we define

\[
a_\pm = \frac{1}{2}(V_a \pm a).
\]

Then $a_\pm$ are both positive, and $a = a_+ - a_-$. We can then define $\mu_\pm$ by

\[
\mu_\pm[0,t] = a_\pm(t) - a_\pm(0)
\]

\[
\mu = \mu_+ - \mu_-
\]

Moreover, $V_\mu_{[0,t]} = |\mu|_{[0,t]}$.

**Proposition.** Let $a$ be càdlàg and BV on $[0,t]$, and $h$ bounded and left-continuous. Then

\[
\int_0^t h(s) \, da(s) = \lim_{m \to \infty} \sum_{i=1}^m h(t_{i-1}^{(m)}) \left( a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \right)
\]

\[
\int_0^t h(s) |da(s)| = \lim_{m \to \infty} \sum_{i=1}^m h(t_{i-1}^{(m)}) \left| a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \right|
\]

for any sequence of subdivisions $0 = t_0^{(m)} < \cdots < t_n^{(m)} = t$ of $[0,t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$. 

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2 Semi-martingales

2.1 Finite variation processes

**Proposition.** The total variation process $V$ of a càdlàg adapted process $A$ is also càdlàg, finite variation and adapted, and it is also increasing.

**Proposition.** Let $A$ be a finite variation process, and $H$ previsible such that

$$\int_0^t |H(\omega, s)| |dA(\omega, s)| < \infty$$

for all $(\omega, t) \in \Omega \times [0, \infty)$. Then $H \cdot A$ is a finite variation process.

2.2 Local martingale

**Theorem (Optional stopping theorem).** Let $X$ be a càdlàg adapted integrable process. Then the following are equivalent:

(i) $X$ is a martingale, i.e. $X_t \in L^1$ for every $t$, and

$$E(X_t \mid F_s) = X_s$$

for all $t > s$.

(ii) The stopped process $X_T = (X_T^t) = (X_{T \land t})$ is a martingale for all stopping times $T$.

(iii) For all stopping times $T, S$ with $T$ bounded, $X_T \in L^1$ and $E(X_T \mid F_S) = X_{T \land S}$ almost surely.

(iv) For all bounded stopping times $T$, $X_T \in L^1$ and $E(X_T) = E(X_0)$.

For $X$ uniformly integrable, (iii) and (iv) hold for all stopping times.

**Proposition.** Let $X$ be a local martingale and $X_t \geq 0$ for all $t$. Then $X$ is a supermartingale.

**Proposition.** Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then the set

$$\chi = \{E(X \mid \mathcal{G}) : \mathcal{G} \subseteq \mathcal{F} \text{ a sub-\sigma-algebra}\}$$

is uniformly integrable, i.e.

$$\sup_{Y \in \chi} E(|Y| \mathbb{1}_{|Y| > \lambda}) \to 0 \text{ as } \lambda \to \infty.$$

**Theorem (Vitali theorem).** $X_n \to X$ in $L^1$ iff $(X_n)$ is uniformly integrable and $X_n \to X$ in probability.

**Proposition.** The following are equivalent:

(i) $X$ is a martingale.

(ii) $X$ is a local martingale, and for all $t \geq 0$, the set

$$\chi_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$

is uniformly integrable.
Corollary. If $Z \in L^1$ is such that $|X_t| \leq Z$ for all $t$, then $X$ is a martingale. In particular, every bounded local martingale is a martingale.

Proposition. Let $X$ be a continuous local martingale with $X_0 = 0$. Define

$$S_n = \inf\{t \geq 0 : |X_t| = n\}.$$

Then $S_n$ is a stopping time, $S_n \to \infty$ and $X^{S_n}$ is a bounded martingale. In particular, $(S_n)$ reduces $X$.

Theorem. Let $X$ be a continuous local martingale with $X_0 = 0$. If $X$ is also a finite variation process, then $X_t = 0$ for all $t$.

2.3 Square integrable martingales

Theorem (Doob’s inequality). Let $X \in M^2$. Then

$$E\left(\sup_{t \geq 0} X_t^2\right) \leq 4E(X_\infty^2).$$

Theorem. $M^2$ is a Hilbert space and $M^2_c$ is a closed subspace.

2.4 Quadratic variation

Theorem. Let $M$ be a continuous local martingale with $M_0 = 0$. Then there exists a unique (up to indistinguishability) continuous adapted increasing process $(\langle M \rangle_t)_{t \geq 0}$ such that $\langle M \rangle_0 = 0$ and $M_t^2 - \langle M \rangle_t$ is a continuous local martingale. Moreover,

$$\langle M \rangle_t = \lim_{n \to \infty} \langle M \rangle^{(n)}_t, \quad \langle M \rangle^{(n)}_t = \sum_{i=1}^{[2^n t]} (M_{2i-1} - M_{i-1})^2,$$

where the limit u.c.p.

Proposition. Let $M \in M^2_c$. Then $M^2 - \langle M \rangle$ is a uniformly integrable martingale, and

$$\|M - M_0\|_{M^2} = (E\langle M \rangle_\infty)^{1/2}.$$

2.5 Covariation

Proposition.

(i) $\langle M, N \rangle$ is the unique (up to indistinguishability) finite variation process such that $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.

(ii) The mapping $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.

(iii)

$$\langle M, N \rangle_t = \lim_{n \to \infty} \langle M, N \rangle^{(n)}_t \text{ u.c.p.}$$

$$\langle M, N \rangle^{(n)}_t = \sum_{i=1}^{[2^n t]} (M_{2i-1} - M_{(i-1)2^n})(N_{2i-1} - N_{i-1})2^{-n}.$$
(iv) For every stopping time $T$,
\[ \langle M^T, N^T \rangle_t = \langle M^T, N \rangle_t = \langle M, N \rangle_{t \wedge T}. \]

(v) If $M, N \in \mathcal{M}_c^2$, then $M_tN_t - \langle M, N \rangle_t$ is a uniformly integrable martingale, and
\[ \langle M - M_0, N - N_0 \rangle_{\mathcal{M}_c^2} = \mathbb{E}(M, N)_\infty. \]

**Proposition** (Kunita–Watanabe). Let $M, N$ be continuous local martingales and let $H, K$ be two (previsible) processes. Then almost surely
\[ \int_0^\infty |H_s||K_s||d\langle M, N \rangle_s| \leq \left( \int_0^\infty H_s^2 \, d\langle M \rangle_s \right)^{1/2} \left( \int_0^\infty H_s^2 \langle N \rangle_s \right)^{1/2}. \]

### 2.6 Semi-martingale
3 The stochastic integral

3.1 Simple processes

Proposition. If \( M \in \mathcal{M}_c^2 \) and \( H \in \mathcal{E} \), then \( H \cdot M \in \mathcal{M}_c^2 \) and

\[
\|H \cdot M\|_{\mathcal{M}_c^2}^2 = \mathbb{E} \left( \int_0^\infty H_s^2 \, d\langle M \rangle_s \right).
\]

(*)

Proposition. Let \( M \in \mathcal{M}_c^2 \) and \( H \in \mathcal{E} \). Then

\[
\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle
\]

for all \( N \in \mathcal{M}_c^2 \).

3.2 Itô isometry

Proposition. Let \( M \in \mathcal{M}_c^2 \). Then \( \mathcal{E} \) is dense in \( L^2(\mathcal{M}) \).

Theorem. Let \( M \in \mathcal{M}_c^2 \). Then

(i) The map \( H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}_c^2 \) extends uniquely to an isometry \( L^2(\mathcal{M}) \to \mathcal{M}_c^2 \), called the Itô isometry.

(ii) For \( H \in L^2(\mathcal{M}) \), \( H \cdot M \) is the unique martingale in \( \mathcal{M}_c^2 \) such that

\[
\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle
\]

for all \( N \in \mathcal{M}_c^2 \), where the integral on the LHS is the stochastic integral (as above) and the RHS is the finite variation integral.

(iii) If \( T \) is a stopping time, then \( (1_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T \).

Corollary.

\[
\langle H \cdot M, K \cdot N \rangle = H \cdot (K \cdot \langle M, N \rangle) = (HK) \cdot \langle M, N \rangle.
\]

In other words,

\[
\left\langle \int_0^t (-) H_s \, dM_s, \int_0^t (-) K_s \, dN_s \right\rangle_t = \int_0^t H_s K_s \, d\langle M, N \rangle_s.
\]

\( \square \)

Corollary. Since \( H \cdot M \) and \((H \cdot M)(K \cdot N) - (H \cdot M, K \cdot N)\) are martingales starting at 0, we have

\[
\mathbb{E} \left( \int_0^t H \, dM_s \right) = 0
\]

\[
\mathbb{E} \left( \left( \int_0^t H_s \, dM_s \right) \left( \int_0^t K_s \, dN_s \right) \right) = \int_0^t H_s K_s \, d\langle M, N \rangle_s.
\]

\( \square \)

Corollary. Let \( H \in L^2(\mathcal{M}) \), then \( HK \in L^2(\mathcal{M}) \) iff \( K \in L^2(H \cdot M) \), in which case

\[
(KH) \cdot M = K \cdot (H \cdot M).
\]
3.3 Extension to local martingales

**Theorem.** Let $M$ be a continuous local martingale.

(i) For every $H \in L^2_{bc}(M)$, there is a unique continuous local martingale $H \cdot M$ with $(H \cdot M)_0 = 0$ and

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$$

for all $N, M$.

(ii) If $T$ is a stopping time, then

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$  

(iii) If $H \in L^2_{loc}(M)$, $K$ is previsible, then $K \in L^2_{loc}(H \cdot M)$ iff $HK \in L^2_{loc}(M)$, and then

$$K \cdot (H \cdot M) = (KH) \cdot M.$$  

(iv) Finally, if $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$, then the definition is the same as the previous one.

3.4 Extension to semi-martingales

**Proposition.**

(i) $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if $H$ and $K$ are locally bounded.

(iii) $(H \cdot X)^T = H1_{[0,T]} \cdot X = H \cdot X^T$ for every stopping time $T$.

(iv) If $X$ is a continuous local martingale (resp. a finite variation process), then so is $H \cdot X$.

(v) If $H = \sum_{i=1}^n H_{i-1} 1_{(t_{i-1},t_i]}$ and $H_{i-1} \in \mathcal{F}_{t_{i-1}}$ (not necessarily bounded), then

$$(H \cdot X)_t = \sum_{i=1}^n H_{i-1}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}).$$

**Proposition** (Stochastic dominated convergence theorem). Let $X$ be a continuous semi-martingale. Let $H, H^*_n$ be previsible and locally bounded, and let $K$ be previsible and non-negative. Let $t > 0$. Suppose

(i) $H^*_n \to H$ as $n \to \infty$ for all $s \in [0,t]$.

(ii) $|H^*_n| \leq K_s$ for all $s \in [0,t]$ and $n \in \mathbb{N}$.

(iii) $\int_0^t K_s^2 \, d\langle M \rangle_s < \infty$ and $\int_0^t K_s^* \, d|A_s| < \infty$ (note that both conditions are okay if $K$ is locally bounded).

Then

$$\int_0^t H^*_n \, dX_s \to \int_0^t H_s \, dX_s$$ in probability.
Proposition. Let $X$ be a continuous semi-martingale, and let $H$ be an adapted bounded left-continuous process. Then for every subdivision $0 < t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_{m}}^{(m)}$ of $[0,t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$, then

$$\int_0^t H_s \, dX_s = \lim_{m \to \infty} \sum_{i=1}^{n_{m}} H_{t_i^{(m)}}(X_{t_i^{(m)}} - X_{t_{i-1}^{(m)}})$$

in probability.

3.5 Itô formula

Theorem (Integration by parts). Let $X,Y$ be a continuous semi-martingale. Then almost surely,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X,Y \rangle_t$$

The last term is called the Itô correction.

Theorem (Itô’s formula). Let $X^1, \ldots, X^p$ be continuous semi-martingales, and let $f : \mathbb{R}^p \to \mathbb{R}$ be $C^2$. Then, writing $X = (X^1, \ldots, X^p)$, we have, almost surely,

$$f(X_t) = f(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x_i}(X_s) \, dX^i_s + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) \, d\langle X^i, X^j \rangle_s.$$ 

In particular, $f(X)$ is a semi-martingale.

3.6 The Lévy characterization

Theorem (Lévy’s characterization of Brownian motion). Let $(X^1, \ldots, X^d)$ be continuous local martingales. Suppose that $X_0 = 0$ and that $(X^i, X^j)_t = \delta_{ij} t$ for all $i, j = 1, \ldots, d$ and $t \geq 0$. Then $(X^1, \ldots, X^d)$ is a standard $d$-dimensional Brownian motion.

Theorem (Dubins–Schwarz). Let $M$ be a continuous local martingale with $M_0 = 0$ and $\langle M \rangle_\infty = \infty$. Let

$$T_s = \inf \{ t \geq 0 : \langle M \rangle_t > s \},$$

the right-continuous inverse of $\langle M \rangle_t$. Let $B_s = M_{T_s}$ and $\mathcal{G}_s = \mathcal{F}_{T_s}$. Then $T_s$ is a $(\mathcal{F}_t)$ stopping time, $\langle M \rangle_{T_s} = s$ for all $s \geq 0$, $B$ is a $(\mathcal{G}_s)$-Brownian motion, and

$$M_t = B_{\langle M \rangle_t}.$$ 

Lemma. $M$ is constant on $[a,b]$ iff $\langle M \rangle$ being constant on $[a,b]$.

Lemma. $M$ is constant on $[a,b]$ iff $\langle M \rangle$ being constant on $[a,b]$. 

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3.7 Girsanov’s theorem

**Proposition.** Let $M$ be a continuous local martingale with $M_0 = 0$. Then $\mathcal{E}(M) = Z$ satisfies
\[ dZ_t = Z_t \, dM_t, \]
i.e.
\[ Z_t = 1 + \int_0^t Z_s \, dM_s. \]
In particular, $\mathcal{E}(M)$ is a continuous local martingale. Moreover, if $\langle M \rangle$ is uniformly bounded, then $\mathcal{E}(M)$ is a uniformly integrable martingale.

**Theorem** (Girsanov’s theorem). Let $M$ be a continuous local martingale with $M_0 = 0$. Suppose that $\mathcal{E}(M)$ is a uniformly integrable martingale. Define a new probability measure
\[ \frac{dQ}{dP} = \mathcal{E}(M)_\infty \]
Let $X$ be a continuous local martingale with respect to $P$. Then $X - \langle X, M \rangle$ is a continuous local martingale with respect to $Q$. 

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4 Stochastic differential equations

4.1 Existence and uniqueness of solutions

**Theorem** (Yamada–Watanabe). Assume weak existence and pathwise uniqueness holds. Then

(i) Uniqueness in law holds.

(ii) For every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $B$ and any $x \in \mathbb{R}^d$, there is a unique strong solution to $E_x(a,b)$.

**Theorem.** Assume $b, \sigma$ are Lipschitz in $x$. Then there is pathwise uniqueness for the $E(\sigma,b)$ and for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and every $(\mathcal{F}_t)$-Brownian motion $B$, for every $x \in \mathbb{R}^d$, there exists a unique strong solution to $E_x(\sigma,b)$.

**Lemma.** Let $h(t)$ be a function such that

\[ h(t) \leq c \int_0^t h(s) \, ds \]

for some constant $c$. Then

\[ h(t) \leq h(0)e^{ct}. \]

4.2 Examples of stochastic differential equations

**Theorem.** The eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ satisfies

\[ d\lambda_i^t = \left( -\frac{\lambda^i}{2} + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \, dt + \sqrt{\frac{2}{N\beta}} \, dB^i. \]

Here $\beta = 1$, but if we replace symmetric matrices by Hermitian ones, we get $\beta = 2$; if we replace symmetric matrices by symplectic ones, we get $\beta = 4$.

4.3 Representations of solutions to PDEs

**Proposition.** Let $x \in \mathbb{R}^d$, and $X$ a solution to $E_x(\sigma,b)$. Then for every $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ that is $C^1$ in $\mathbb{R}_+$ and $C^2$ in $\mathbb{R}^d$, the process

\[ M^f_t = f(t, X_t) - f(0, X_0) - \int_0^t \left( \frac{\partial}{\partial s} + L \right) f(s, X_s) \, ds \]

is a continuous local martingale.

**Theorem.** Assume $U$ has a smooth boundary (or satisfies the exterior cone condition), $a,b$ are Hölder continuous and $a$ is uniformly elliptic. Then for every Hölder continuous $f : \bar{U} \to \mathbb{R}$ and any continuous $g : \partial U \to \mathbb{R}$, the Dirichlet–Poisson process has a solution.
Theorem. Let $\sigma$ and $b$ be bounded measurable and $\sigma \sigma^T$ uniformly elliptic, $U \subseteq \mathbb{R}^d$ as above. Let $u$ be a solution to the Dirichlet–Poisson problem and $X$ a solution to $E_x(\sigma, b)$ for some $x \in \mathbb{R}^d$. Define the stopping time

$$T_U = \inf \{t \geq 0 : X_t \not\in U \}.$$ 

Then $\mathbb{E}T_U < \infty$ and

$$u(x) = \mathbb{E}_x \left( g(X_{T_U}) + \int_0^{T_U} f(X_s) \, ds \right).$$

In particular, the solution to the PDE is unique.

Theorem. For every $f \in C^2_b(\mathbb{R}^d)$, there exists a solution to the Cauchy problem.

Theorem. Let $u$ be a solution to the Cauchy problem. Let $X$ be a solution to $E_x(\sigma, b)$ for $x \in \mathbb{R}^d$ and $0 \leq s \leq t$. Then

$$\mathbb{E}_x (f(X_t) \mid \mathcal{F}_s) = u(t - s, X_s).$$

In particular,

$$u(t, x) = \mathbb{E}_x(f(X_t)).$$

Theorem (Feynman–Kac formula). Let $f \in C^2_b(\mathbb{R}^d)$ and $V \in C_b(\mathbb{R}^d)$ and suppose that $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{\partial u}{\partial t} = Lu + Vu \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

$$u(0, \cdot) = f \quad \text{on } \mathbb{R}^d,$$

where $Vu = V(x)u(x)$ is given by multiplication.

Then for all $t > 0$ and $x \in \mathbb{R}^d$, and $X$ a solution to $E_x(\sigma, b)$. Then

$$u(t, x) = \mathbb{E}_x \left( f(X_t) \exp \left( \int_0^t V(X_s) \, ds \right) \right).$$