Part III — Stochastic Calculus and Applications
Definitions

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Lent 2018

- *Brownian motion.* Existence and sample path properties.
- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita–Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins–Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, Yamada–Watanabe theorem, strong Markov property, and relation to second order partial differential equations.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.

Contents

0 Introduction

Definition (Gaussian space)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then a subspace $S \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Gaussian space* if it is a closed linear subspace and every $X \in S$ is a centered Gaussian random variable.

Definition (Gaussian white noise)**.** A *Gaussian white noise* on $\mathbb{R}_+$ is an isometry WN from $L^2(\mathbb{R}_+)$ into some Gaussian space. For $A \subseteq \mathbb{R}_+$, we write $WN(A) = WN(\mathbf{1}_A)$.

1 The Lebesgue–Stieltjes integral

Definition (Signed measure)**.** A *signed measure* on $[0, T]$ is a difference $\mu = \mu_+ - \mu_-$ of two positive measures on $[0, T]$ of disjoint support. The decomposition $\mu = \mu_+ - \mu_-$ is called the *Hahn decomposition*.

Definition (Total variation)**.** The total variation of a signed measure $\mu = \mu_+ - \mu_-$ is $|\mu| = \mu_+ + \mu_-$.

Definition (Total variation)**.** The *total variation* of a function $a : [0, T] \to \mathbb{R}$ is

$$V_a(t) = |a(0)| + \sup \left\{ \sum_{i=1}^n |a(t_i) - a(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_n = T \right\}.$$

We say a has *bounded variation* if $V_a(T) < \infty$. In this case, we write $a \in BV$.

Definition (Càdlàg)**.** A function $a : [0, T] \to \mathbb{R}$ is *càdlàg* if it is right-continuous and has left-limits.

Definition (Lebesgue–Stieltjes integral)**.** Let $a : [0, T] \to \mathbb{R}$ be càdlàg of bounded variation and let μ be the associated signed measure. Then for $h \in L^1([0, T], |\mu|)$, the *Lebesgue–Stieltjes integral* is defined by

$$\int_s^t h(r)\, \mathrm{d}a(r) = \int_{(s,t]} h(r)\mu(\mathrm{d}r),$$

where $0 \leq s \leq t \leq T$, and

$$\int_s^t h(r)\, |\mathrm{d}a(r)| = \int_{(s,t]} h(r)|\mu|(\mathrm{d}r).$$

We also write

$$h \cdot a(t) = \int_0^t h(r)\, \mathrm{d}a(r).$$

Definition (Finite variation)**.** A càdlàg function $a : [0, \infty) \to \mathbb{R}$ is of finite variation if $a|_{[0,T]} \in BV[0, 1]$ for all $T > 0$.

2 Semi-martingales

Definition (Càdlàg adapted process)**.** A *càdlàg adapted process* is a map $X : \Omega \times [0, \infty) \to \mathbb{R}$ such that

(i) X is càdlàg, i.e. $X(\omega, \cdot\,) : [0, \infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$.

(ii) X is adapted, i.e. $X_t = X(\,\cdot\,, t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$.

Notation. We will write $X \in \mathcal{G}$ to denote that a random variable X is measurable with respect to a σ-algebra $\mathcal{G}$.

2.1 Finite variation processes

Definition (Finite variation process)**.** A *finite variation process* is a càdlàg adapted process A such that $A(\omega, \cdot\,) : [0, \infty) \to \mathbb{R}$ has finite variation for all $\omega \in \Omega$. The *total variation process* V of a finite variation process A is

$$V_t = \int_0^T |\mathrm{d}A_s|.$$

Definition ($(H \cdot A)_t$)**.** Let A be a finite variation process and H a process such that for all $\omega \in \Omega$ and $t \geq 0$,

$$\int_0^t H_s(\omega)|\, |\mathrm{d}A_s(\omega)| < \infty.$$

Then define a process $((H \cdot A)_t)_{t \geq 0}$ by

$$(H \cdot A)_t = \int_0^t H_s \,\mathrm{d}A_s.$$

Definition (Previsible process)**.** A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is *previsible* if it is measurable with respect to the *previsible σ-algebra* $\mathcal{P}$ generated by the sets $E \times (s, t]$, where $E \in \mathcal{F}_s$ and $s < t$. We call the generating set Π.

Definition (Simple process)**.** A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is *simple*, written $H \in \mathcal{E}$, if

$$H(\omega, t) = \sum_{i=1}^n H_{i-1}(\omega)\mathbf{1}_{(t_{i-1}, t_i]}(t)$$

for random variables $H_{i-1} \in \mathcal{F}_{i-1}$ and $0 = t_0 < \cdots < t_n$.

2.2 Local martingale

Definition (Local martingale)**.** A càdlàg adapted process X is a *local martingale* if there exists a sequence of stopping times T_n such that $T_n \to \infty$ almost surely, and X^{T_n} is a martingale for every n. We say the sequence T_n *reduces* X.

2.3 Square integrable martingales

Definition ($\mathcal{M}^2$)**.** Let

$$\mathcal{M}^2 = \left\{X : \Omega \times [0, \infty) \to \mathbb{R} : X \text{ is cádlág martingale with } \sup_{t \geq 0} \mathbb{E}(X_t^2) < \infty\right\}.$$
$$\mathcal{M}_c^2 = \left\{X \in \mathcal{M}^2 : X(\omega, \cdot\,) \text{ is continuous for every } \omega \in \Omega\right\}$$

We define an inner product on $\mathcal{M}^2$ by

$$(X, Y)_{\mathcal{M}^2} = \mathbb{E}(X_\infty Y_\infty),$$

which in aprticular induces a norm

$$\|X\|_{\mathcal{M}^2} = \left(\mathbb{E}(X_\infty^2)\right)^{1/2}.$$

We will prove this is indeed an inner product soon. Here recall that for $X \in \mathcal{M}^2$, the martingale convergence theorem implies $X_t \to X_\infty$ almost surely and in L^2.

2.4 Quadratic variation

Definition (Uniformly on compact sets in probability)**.** For a sequence of processes (X^n) and a process X, we say that $X^n \to X$ u.c.p. iff

$$\mathbb{P}\left(\sup_{s \in [0, t]} |X_s^n - X_s| > \varepsilon\right) \to 0 \text{ as } n \to \infty \text{ for all } t > 0, \varepsilon > 0.$$

Definition (Quadratic variation)**.** $\langle M\rangle$ is called the *quadratic variation* of M.

2.5 Covariation

Definition (Covariation)**.** Let M, N be two continuous local martingales. Define the *covariation* (or simply the *bracket*) between M and N to be process

$$\langle M, N\rangle_t = \frac{1}{4}(\langle M + N\rangle_t - \langle M - N\rangle_t).$$

2.6 Semi-martingale

Definition (Semi-martingale)**.** A (continuous) adapted process X is a *(continuous) semi-martingale* if

$$X = X_0 + M + A,$$

where $X_0 \in \mathcal{F}_0$, M is a continuous local martingale with $M_0 = 0$, and A is a continuous finite variation process with $A_0 = 0$.

Definition (Quadratic variation)**.** Let $X = X_0 + M + A$ and $X' = X_0' + M' + A'$ be (continuous) semi-martingales. Set

$$\langle X\rangle = \langle M\rangle, \quad \langle X, X'\rangle = \langle M, M'\rangle.$$

3 The stochastic integral

3.1 Simple processes

Definition (Simple process)**.** The space of *simple processes* $\mathcal{E}$ consists of functions $H : \Omega \times [0, \infty) \to \mathbb{R}$ that can be written as

$$H_t(\omega) = \sum_{i=1}^{n} H_{i-1}(\omega) \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

for some $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ and bounded random variables $H_i \in \mathcal{F}_{t_i}$.

Definition $(H \cdot M)$**.** For $M \in \mathcal{M}^2$ and $H \in \mathcal{E}$, we set

$$\int_0^t H \, \mathrm{d}M = (H \cdot M)_t = \sum_{i=1}^{n} H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

3.2 Itô isometry

Definition $(L^2(M))$**.** Let $M \in \mathcal{M}_c^2$. Define $L^2(M)$ to be the space of (equivalence classes of) previsible $H : \Omega \times [0, \infty) \to \mathbb{R}$ such that

$$\|H\|_{L^2(M)} = \|H\|_{\mathcal{M}} = \mathbb{E}\left(\int_0^\infty H_s^2 \, \mathrm{d}\langle M \rangle_s \right)^{1/2} < \infty.$$

For $H, K \in L^2(M)$, we set

$$(H, K)_{L^2(M)} = \mathbb{E}\left(\int_0^\infty H_s K_s \, \mathrm{d}\langle M \rangle_s \right).$$

Definition (Stochastic integral)**.** $H \cdot M$ is the *stochastic integral* of H with respect to M and we also write

$$(H \cdot M)_t = \int_0^t H_s \, \mathrm{d}M_s.$$

3.3 Extension to local martingales

Definition $(L_{bc}^2(M))$**.** Let $L_{bc}^2(M)$ be the space of previsible H such that

$$\int_0^t H_s^2 \, \mathrm{d}\langle M \rangle_s < \infty \text{ a.s.}$$

for all finite $t > 0$.

3.4 Extension to semi-martingales

Definition (Locally boounded previsible process)**.** A previsible process H is *locally bounded* if for all $t \geq 0$, we have

$$\sup_{s \leq t} |H_s| < \infty \text{ a.s.}$$

Definition (Stochastic integral)**.** Let $X = X_0 + M + A$ be a continuous semi-martingale, and H a locally bounded previsible process. Then the *stochastic integral* $H \cdot X$ is the continuous semi-martingale defined by

$$H \cdot X = H \cdot M + H \cdot A,$$

and we write

$$(H \cdot X)_t = \int_0^T H_s \, \mathrm{d}X_s.$$

3.5 Itô formula

3.6 The Lévy characterization

3.7 Girsanov's theorem

Definition (Stochastic exponential)**.** Let M be a continuous local martingale. Then the *stochastic exponential* (or *Doléans–Dade exponential*) of M is

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$$

4 Stochastic differential equations

4.1 Existence and uniqueness of solutions

Definition (Stochastic differential equation)**.** Let $d, m \in \mathbb{N}$, $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be locally bounded (and measurable). A solution to the stochastic differential equation $E(\sigma, b)$ given by

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t$$

consists of

(i) a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ obeying the usual conditions;

(ii) an m-dimensional Brownian motion B with $B_0 = 0$; and

(iii) an $(\mathcal{F}_t)$-adapted continuous process X with values in $\mathbb{R}^d$ such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s + \int_0^t b(s, X_s)\,\mathrm{d}s.$$

If $X_0 = x \in \mathbb{R}^d$, then we say X is a *(weak) solution* to $E_x(\sigma, b)$. It is a *strong* solution if it is adapted with respect to the canonical filtration of B.

Definition (Uniqueness of solutions)**.** For the stochastic differential equation $E(\sigma, b)$, we say there is

- *uniqueness in law* if for every $x \in \mathbb{R}^d$, all solutions to $E_x(\sigma, b)$ have the same distribution.
- *pathwise uniqueness* if when $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B are fixed, any two solutions X, X' with $X_0 = X'_0$ are indistinguishable.

Definition (Lipschitz coefficients)**.** The coefficients $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are Lipschitz in x if there exists a constant $K > 0$ such that for all $t \geq 0$ and $x, y \in \mathbb{R}^d$, we have

$$|b(t, x) - b(t, y)| \leq K|x - y|$$
$$|\sigma(t, x) - \sigma(t, y)| \leq |x - y|$$

4.2 Examples of stochastic differential equations

Definition (Gaussian orthogonal ensemble)**.** The *Gaussian Orthogonal Ensemble* GOE_N is the standard Gaussian measure on $\mathcal{H}_N$, i.e. $H \sim \mathrm{GOE}_N$ if

$$H = \sum_{r=1}^{\dim \mathcal{H}_n} H^i X^i$$

where each X^i are iid standard normals.

4.3 Representations of solutions to PDEs

Definition (Uniformly elliptic)**.** We say $a : \bar{U} \to \mathbb{R}^{d \times d}$ is *uniformly elliptic* if there is a constant $c > 0$ such that for all $\xi \in \mathbb{R}^d$ and $x \in \bar{U}$, we have

$$\xi^T a(x) \xi \geq c|\xi|^2.$$