Part III — Stochastic Calculus and Applications
Definitions
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

- Brownian motion. Existence and sample path properties.
- Stochastic calculus for continuous processes. Martingales, local martingales, semimartingales, quadratic variation and cross-variation, Itô’s isometry, definition of the stochastic integral, Kunita–Watanabe theorem, and Itô’s formula.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.
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0 Introduction

**Definition** (Gaussian space). Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then a subspace $S \subseteq L^2(\Omega, \mathcal{F}, P)$ is called a Gaussian space if it is a closed linear subspace and every $X \in S$ is a centered Gaussian random variable.

**Definition** (Gaussian white noise). A Gaussian white noise on $\mathbb{R}_+$ is an isometry $WN$ from $L^2(\mathbb{R}_+)$ into some Gaussian space. For $A \subseteq \mathbb{R}_+$, we write $WN(A) = WN(1_A)$. 
1 The Lebesgue–Stieltjes integral

Definition (Signed measure). A signed measure on \([0,T]\) is a difference \(\mu = \mu_+ - \mu_-\) of two positive measures on \([0,T]\) of disjoint support. The decomposition \(\mu = \mu_+ - \mu_-\) is called the Hahn decomposition.

Definition (Total variation). The total variation of a signed measure \(\mu = \mu_+ - \mu_-\) is \(|\mu| = \mu_+ + \mu_-\).

Definition (Total variation). The total variation of a function \(a : [0,T] \rightarrow \mathbb{R}\) is

\[
V_a(t) = |a(0)| + \sup \left\{ \sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_n = T \right\}.
\]

We say \(a\) has bounded variation if \(V_a(T) < \infty\). In this case, we write \(a \in BV\).

Definition (Càdlàg). A function \(a : [0,T] \rightarrow \mathbb{R}\) is càdlàg if it is right-continuous and has left-limits.

Definition (Lebesgue–Stieltjes integral). Let \(a : [0,T] \rightarrow \mathbb{R}\) be càdlàg of bounded variation and let \(\mu\) be the associated signed measure. Then for \(h \in L^1([0,T], |\mu|)\), the Lebesgue–Stieltjes integral is defined by

\[
\int_s^t h(r) \, da(r) = \int_{[s,t]} h(r) \mu(dr),
\]

where \(0 \leq s \leq t \leq T\), and

\[
\int_s^t h(r) \, |da(r)| = \int_{[s,t]} h(r) |\mu|(dr).
\]

We also write

\[
h \cdot a(t) = \int_0^t h(r) \, da(r).
\]

Definition (Finite variation). A càdlàg function \(a : [0, \infty) \rightarrow \mathbb{R}\) is of finite variation if \(a|_{[0,T]} \in BV[0,1]\) for all \(T > 0\).
2 Semi-martingales

Definition (Càdlàg adapted process). A càdlàg adapted process is a map $X : \Omega \times [0, \infty) \to \mathbb{R}$ such that

- (i) $X$ is càdlàg, i.e. $X(\omega, \cdot) : [0, \infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$.
- (ii) $X$ is adapted, i.e. $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for every $t \geq 0$.

Notation. We will write $X \in \mathcal{G}$ to denote that a random variable $X$ is measurable with respect to a $\sigma$-algebra $\mathcal{G}$.

2.1 Finite variation processes

Definition (Finite variation process). A finite variation process is a càdlàg adapted process $A$ such that $A(\omega, \cdot) : [0, \infty) \to \mathbb{R}$ has finite variation for all $\omega \in \Omega$. The total variation process $V$ of a finite variation process $A$ is

$$V_t = \int_0^t |dA_s|.$$ 

Definition $((H \cdot A)_t)$. Let $A$ be a finite variation process and $H$ a process such that for all $\omega \in \Omega$ and $t \geq 0$,

$$\int_0^t H_s(\omega) \, |dA_s(\omega)| < \infty.$$ 

Then define a process $((H \cdot A)_t)_{t \geq 0}$ by

$$(H \cdot A)_t = \int_0^t H_s \, dA_s.$$

Definition (Previsible process). A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is previsible if it is measurable with respect to the previsible $\sigma$-algebra $\mathcal{P}$ generated by the sets $E \times (s, t]$, where $E \in \mathcal{F}_s$ and $s < t$. We call the generating set $\Pi$.

Definition (Simple process). A process $H : \Omega \times [0, \infty) \to \mathbb{R}$ is simple, written $H \in \mathcal{E}$, if

$$H(\omega, t) = \sum_{i=1}^n H_{i-1}(\omega) 1_{(t_{i-1}, t_i]}(t)$$

for random variables $H_{i-1} \in \mathcal{F}_{i-1}$ and $0 = t_0 < \cdots < t_n$.

2.2 Local martingale

Definition (Local martingale). A càdlàg adapted process $X$ is a local martingale if there exists a sequence of stopping times $T_n$ such that $T_n \to \infty$ almost surely, and $X^{T_n}$ is a martingale for every $n$. We say the sequence $T_n$ reduces $X$. 
2.3 Square integrable martingales

Definition ($\mathcal{M}^2$). Let

$$\mathcal{M}^2 = \left\{ X : \Omega \times [0, \infty) \rightarrow \mathbb{R} : X \text{ is càdlàg martingale with } \sup_{t \geq 0} \mathbb{E}(X_t^2) < \infty \right\}.$$ 

$$\mathcal{M}^2_c = \left\{ X \in \mathcal{M}^2 : X(\omega, \cdot) \text{ is continuous for every } \omega \in \Omega \right\}.$$ 

We define an inner product on $\mathcal{M}^2$ by

$$(X, Y)_{\mathcal{M}^2} = \mathbb{E}(X_\infty Y_\infty),$$

which in particular induces a norm

$$\|X\|_{\mathcal{M}^2} = \left( \mathbb{E}(X_\infty^2) \right)^{1/2}.$$ 

We will prove this is indeed an inner product soon. Here recall that for $X \in \mathcal{M}^2$, the martingale convergence theorem implies $X_t \to X_\infty$ almost surely and in $L^2$.

2.4 Quadratic variation

Definition (Uniformly on compact sets in probability). For a sequence of processes $(X^n)$ and a process $X$, we say that $X^n \to X$ u.c.p. iff

$$\mathbb{P} \left( \sup_{s \in [0,t]} |X^n_s - X_s| > \varepsilon \right) \to 0 \text{ as } n \to \infty \text{ for all } t > 0, \varepsilon > 0.$$ 

Definition (Quadratic variation). $\langle M \rangle$ is called the quadratic variation of $M$.

2.5 Covariation

Definition (Covariation). Let $M, N$ be two continuous local martingales. Define the covariation (or simply the bracket) between $M$ and $N$ to be process

$$\langle M, N \rangle_t = \frac{1}{4}(\langle M + N \rangle_t - \langle M - N \rangle_t).$$

2.6 Semi-martingale

Definition (Semi-martingale). A (continuous) adapted process $X$ is a (continuous) semi-martingale if

$$X = X_0 + M + A,$$

where $X_0 \in \mathcal{F}_0$, $M$ is a continuous local martingale with $M_0 = 0$, and $A$ is a continuous finite variation process with $A_0 = 0$.

Definition (Quadratic variation). Let $X = X_0 + M + A$ and $X' = X'_0 + M' + A'$ be (continuous) semi-martingales. Set

$$\langle X \rangle = \langle M \rangle, \quad \langle X, X' \rangle = \langle M, M' \rangle.$$
3 The stochastic integral

3.1 Simple processes

Definition (Simple process). The space of simple processes $E$ consists of functions $H : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ that can be written as

$$H_t(\omega) = \sum_{i=1}^{n} H_{i-1}(\omega)1_{[t_{i-1}, t_i]}(t)$$

for some $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ and bounded random variables $H_i \in F_{t_i}$.

Definition $(H \cdot M)$. For $M \in \mathcal{M}^2$ and $H \in \mathcal{E}$, we set

$$\int_0^t H \, dM = (H \cdot M)_t = \sum_{i=1}^{n} H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

3.2 Itô isometry

Definition ($L^2(M)$). Let $M \in \mathcal{M}^2_c$. Define $L^2(M)$ to be the space of (equivalence classes of) previsible $H : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ such that

$$\|H\|_{L^2(M)} = \|H\|_M = \mathbb{E} \left( \int_0^\infty H_s^2 \, d\langle M \rangle_s \right)^{1/2} < \infty.$$

For $H, K \in L^2(M)$, we set

$$(H, K)_{L^2(M)} = \mathbb{E} \left( \int_0^\infty H_s K_s \, d\langle M \rangle_s \right).$$

Definition (Stochastic integral). $H \cdot M$ is the stochastic integral of $H$ with respect to $M$ and we also write

$$(H \cdot M)_t = \int_0^t H_s \, dM_s.$$

3.3 Extension to local martingales

Definition ($L^2_{bc}(M)$). Let $L^2_{bc}(M)$ be the space of previsible $H$ such that

$$\int_0^t H_s^2 \, d\langle M \rangle_s < \infty \text{ a.s.}$$

for all finite $t > 0$.

3.4 Extension to semi-martingales

Definition (Locally bounded previsible process). A previsible process $H$ is locally bounded if for all $t \geq 0$, we have

$$\sup_{s \leq t} |H_s| < \infty \text{ a.s.}$$
Definition (Stochastic integral). Let $X = X_0 + M + A$ be a continuous semi-martingale, and $H$ a locally bounded previsible process. Then the stochastic integral $H \cdot X$ is the continuous semi-martingale defined by

$$H \cdot X = H \cdot M + H \cdot A,$$

and we write

$$(H \cdot X)_t = \int_0^T H_s \, dX_s.$$
4 Stochastic differential equations

4.1 Existence and uniqueness of solutions

Definition (Stochastic differential equation). Let \( d, m \in \mathbb{N} \), \( b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \) be locally bounded (and measurable). A solution to the stochastic differential equation \( E(\sigma, b) \) given by
\[
\mathrm{d}X_t = b(t, X_t) \, \mathrm{d}t + \sigma(t, X_t) \, \mathrm{d}B_t
\]
consists of
(i) a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) obeying the usual conditions;
(ii) an \( m \)-dimensional Brownian motion \( B \) with \( B_0 = 0 \); and
(iii) an \((\mathcal{F}_t)\)-adapted continuous process \( X \) with values in \( \mathbb{R}^d \) such that
\[
X_t = X_0 + \int_0^t \sigma(s, X_s) \, \mathrm{d}B_s + \int_0^t b(s, X_s) \, \mathrm{d}s.
\]
If \( X_0 = x \in \mathbb{R}^d \), then we say \( X \) is a \( (weak) \) solution to \( E_x(\sigma, b) \). It is a \( strong \) solution if it is adapted with respect to the canonical filtration of \( B \).

Definition (Uniqueness of solutions). For the stochastic differential equation \( E(\sigma, b) \), we say there is
- \textit{uniqueness in law} if for every \( x \in \mathbb{R}^d \), all solutions to \( E_x(\sigma, b) \) have the same distribution.
- \textit{pathwise uniqueness} if when \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) and \( B \) are fixed, any two solutions \( X, X' \) with \( X_0 = X'_0 \) are indistinguishable.

Definition (Lipschitz coefficients). The coefficients \( b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \) are Lipschitz in \( x \) if there exists a constant \( K > 0 \) such that for all \( t \geq 0 \) and \( x, y \in \mathbb{R}^d \), we have
\[
|b(t, x) - b(t, y)| \leq K|x - y| \\
|\sigma(t, x) - \sigma(t, y)| \leq |x - y|
\]

4.2 Examples of stochastic differential equations

Definition (Gaussian orthogonal ensemble). The \textit{Gaussian Orthogonal Ensemble} \( \text{GOE}_N \) is the standard Gaussian measure on \( \mathcal{H}_N \), i.e. \( H \sim \text{GOE}_N \) if
\[
H = \sum_{r=1}^{\dim \mathcal{H}_N} H_i X_i
\]
where each \( X_i \) are iid standard normals.

4.3 Representations of solutions to PDEs

Definition (Uniformly elliptic). We say \( a : \bar{U} \to \mathbb{R}^{d \times d} \) is \textit{uniformly elliptic} if there is a constant \( c > 0 \) such that for all \( \xi \in \mathbb{R}^d \) and \( x \in \bar{U} \), we have
\[
\xi^T a(x) \xi \geq c|\xi|^2.
\]