Part III — Riemannian Geometry
Theorems with proof

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This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:

A closer look at geodesics and curvature. Brief review from the Differential Geometry course. Geodesic coordinates and Gauss' lemma. Jacobi fields, completeness and the Hopf–Rinow theorem. Variations of energy, Bonnet–Myers diameter theorem and Synge's theorem.

Hodge theory and Riemannian holonomy. The Hodge star and Laplace–Beltrami operator. The Hodge decomposition theorem (with the 'geometry part' of the proof). Bochner–Weitzenböck formulae. Holonomy groups. Interplays with curvature and de Rham cohomology.

Ricci curvature. Fundamental groups and Ricci curvature. The Cheeger–Gromoll splitting theorem.

Pre-requisites

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course Differential Geometry offered in Michaelmas Term is the ideal pre-requisite.

Contents

1 Basics of Riemannian manifolds

Theorem (Whitney embedding theorem)**.** Every smooth manifold M admits an embedding into $\mathbb{R}^k$ for some k. In other words, M is diffeomorphic to a submanifold of $\mathbb{R}^k$. In fact, we can pick k such that $k \leq 2 \dim M$.

Lemma. Let (N, h) be a Riemannian manifold, and $F : M \to N$ is an immersion, then the pullback $g = F^* h$ defines a metric on M.

2 Riemann curvature

Proposition.

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Proposition.

(i)
$$R_{ij,k\ell} = -R_{ij,\ell k} = -R_{ji,k\ell}.$$

(ii) The *first Bianchi identity*:
$$R^i_{j,k\ell} + R^i_{k,\ell j} + R^i_{\ell,jk} = 0.$$

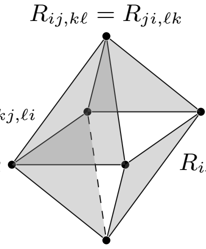
(iii)
$$R_{ij,k\ell} = R_{k\ell,ij}.$$

Proof.

(i) The first equality is obvious as coefficients of a 2-form. For the second equality, we begin with the compatibility of the connection with the metric:
$$\frac{\partial g_{ij}}{\partial x^k} = g(\nabla_k \partial_i, \partial_j) + g(\partial_i, \nabla_k \partial_j).$$
We take a partial derivative, say with respect to x_ℓ, to obtain
$$\frac{\partial^2 g_{ij}}{\partial x^\ell \partial x^k} = g(\nabla_\ell \nabla_k \partial_i, \partial_j) + g(\nabla_k \partial_i, \nabla_\ell \partial_j) + g(\nabla_\ell \partial_i, \nabla_k \partial_j) + g(\partial_i, \nabla_\ell \nabla_k \partial_j).$$
Then we know
$$0 = \frac{\partial^2 g}{\partial x^\ell \partial x^k} - \frac{\partial^2 g}{\partial x_k \partial x_\ell} = g([\nabla_\ell, \nabla_k]\partial_i, \partial_j) + g(\partial_i, [\nabla_\ell, \nabla_k]\partial_j).$$
But we know
$$R(\partial_k, \partial_\ell) = \nabla_{[\partial_k, \partial_\ell]} - [\nabla_k, \nabla_\ell] = -[\nabla_k, \nabla_\ell].$$
Writing $R_{k\ell} = R(\partial_k, \partial_\ell)$, we have
$$0 = g(R_{k\ell}\partial_i, \partial_j) + g(\partial_i, R_{k\ell}\partial_j) = R_{ji,k\ell} + R_{ij,k\ell}.$$
So we are done.

(ii) Recall
$$R^i_{j,k\ell} = (R_{k\ell}\partial_j)^i = ([\nabla_\ell, \nabla_k]\partial_j)^i.$$
So we have
$$\begin{aligned} &R^i_{j,k\ell} + R^i_{k,\ell j} + R^i_{\ell,jk} \\ &= \left[(\nabla_\ell \nabla_k \partial_j - \nabla_k \nabla_\ell \partial_j) + (\nabla_j \nabla_\ell \partial_k - \nabla_\ell \nabla_j \partial_k) + (\nabla_k \nabla_j \partial_\ell - \nabla_j \nabla_k \partial_\ell)\right]^i. \end{aligned}$$
We claim that
$$\nabla_\ell \nabla_k \partial_j - \nabla_\ell \nabla_j \partial_k = 0.$$
Indeed, by definition, we have
$$(\nabla_k \partial_j)^q = \Gamma^q_{kj} = \Gamma^q_{jk} = (\nabla_j \partial_k)^q.$$
The other terms cancel similarly, and we get 0 as promised.

(iii) Consider the following octahedron:



The equalities on each vertex is given by (i). By the first Bianchi identity, for each greyed triangle, the sum of the three vertices is zero.

Now looking at the upper half of the octahedron, adding the two greyed triangles shows us the sum of the vertices in the horizontal square is $(-2)R_{ij,k\ell}$. Looking at the bottom half, we find that the sum of the vertices in the horizontal square is $(-2)R_{k\ell,ij}$. So we must have

$$R_{ij,k\ell} = R_{k\ell,ij}.$$ □

Lemma. Let V be a real vector space of dimension ≥ 2. Suppose $R', R'' : V^{\otimes 4} \to \mathbb{R}$ are both linear in each factor, and satisfies the symmetries we found for the Riemann curvature tensor. We define $K', K'' : \mathrm{Gr}(2, V) \to \mathbb{R}$ as in the sectional curvature. If $K' = K''$, then $R' = R''$.

Proof. For any $X, Y, Z \in V$, we know

$$R'(X + Z, Y, X + Z, Y) = R''(X + Z, Y, X + Z, Y).$$

Using linearity of R' and R'', and cancelling equal terms on both sides, we find

$$R'(Z, Y, X, Y) + R'(X, Y, Z, Y) = R''(Z, Y, X, Y) + R''(X, Y, Z, Y).$$

Now using the symmetry property of R' and R'', this implies

$$R'(X, Y, Z, Y) = R''(X, Y, Z, Y).$$

Similarly, we replace Y with $Y + T$, and then we get

$$R'(X, Y, Z, T) + R'(X, T, Z, Y) = R''(X, Y, Z, Y) + R''"(X, T, Z, Y).$$

We then rearrange and use the symmetries to get

$$R'(X, Y, Z, T) - R''(X, Y, Z, T) = R'(Y, Z, X, T) - R''(Y, Z, X, T).$$

We notice this equation says $R'(X, Y, Z, T) - R''(X, Y, Z, T)$ is invariant under the cyclic permutation $X \to Y \to Z \to X$. So by the first Bianchi identity, we have

$$3(R'(X, Y, Z, T) - R''(X, Y, Z, T)) = 0.$$

So we must have $R' = R''$. □

Corollary. Let (M, g) be a manifold such that for all p, the function $K_p : \mathrm{Gr}(2, T_pM) \to \mathbb{R}$ is a constant map. Let

$$R_p^0(X, Y, Z, T) = g_p(X, Z)g_p(Y, T) - g_p(X, T)g_p(Y, Z).$$

Then

$$R_p = K_p R_p^0.$$

Here K_p is just a real number, since it is constant. Moreover, K_p is a smooth function of p.

Equivalently, in local coordinates, if the metric at a point is δ_{ij}, then we have

$$R_{ij,ij} = -R_{ij,ji} = K_p,$$

and all other entries all zero.

Proof. We apply the previous lemma as follows: we define $R' = K_p R_p^0$ and $R'' = R_p$. It is a straightforward inspection to see that this R^0 does follow the symmetry properties of R_p, and that they define the same sectional curvature. So $R'' = R'$. We know K_p is smooth in p as both g and R are smooth. □

3 Geodesics

3.1 Definitions and basic properties

Proposition. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve. Then there is a uniquely determined operation $\frac{\nabla}{\mathrm{d}t}$ from the space of all lifts of γ to itself, satisfying the following conditions:

(i) For any $c, d \in \mathbb{R}$ and lifts $\tilde{\gamma}^E, \gamma^E$ of γ, we have.

$$\frac{\nabla}{\mathrm{d}t}(c\gamma^E + d\tilde{\gamma}^E) = c\frac{\nabla \gamma^E}{\mathrm{d}t} + d\frac{\nabla \tilde{\gamma}^E}{\mathrm{d}t}$$

(ii) For any lift γ^E of γ and function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$, we have

$$\frac{\nabla}{\mathrm{d}t}(f\gamma^E) = \frac{\mathrm{d}f}{\mathrm{d}t} + f\frac{\nabla \gamma^E}{\mathrm{d}t}.$$

(iii) If there is a local section s of E and a local vector field V on M such that

$$\gamma^E(t) = s(\gamma(t)), \quad \dot{\gamma}(t) = V(\gamma(t)),$$

then we have

$$\frac{\nabla \gamma^E}{\mathrm{d}t} = (\nabla_V s) \circ \gamma.$$

Locally, this is given by

$$\left(\frac{\nabla \gamma^E}{\mathrm{d}t}\right)^i = \dot{a}^i + \Gamma^i_{jk} a^j \dot{x}^k.$$

Proposition. If γ is a geodesic, then $|\dot{\gamma}(t)|_g$ is constant.

Proof. We use the extension $\underline{\dot{\gamma}}$ around $p = \gamma(0)$, and stop writing the underlines. Then we have

$$\dot{\gamma}(g(\dot{\gamma}, \dot{\gamma})) = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) + g(\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}) = 0,$$

which is valid at each $q = \gamma(t)$ on the curve. But at each q, we have

$$\dot{\gamma}(g(\dot{\gamma}, \dot{\gamma})) = \dot{x}^k \frac{\partial}{\partial x_k} g(\dot{\gamma}, \dot{\gamma}) = \frac{\mathrm{d}}{\mathrm{d}t}|\dot{\gamma}(t)|^2_g$$

by the chain rule. So we are done. □

Lemma. Let $p \in M$, and $a \in T_pM$. As before, let $\gamma_p(t, a)$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = p$. Then

$$\gamma_p(\lambda t, a) = \gamma_p(t, \lambda a),$$

and in particular is a geodesic.

Proof. We apply the chain rule to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(\lambda t, a) = \lambda \dot{\gamma}(\lambda t, a)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma(\lambda t, a) = \lambda^2 \ddot{\gamma}(\lambda t, a).$$

So $\gamma(\lambda t, a)$ satisfies the geodesic equations, and have initial velocity λa. Then we are done by uniqueness of ODE solutions. □

Proposition. We have

$$(\mathrm{d}\exp_p)_0 = \mathrm{id}_{T_pM},$$

where we identify $T_0(T_pM) \cong T_pM$ in the natural way.

Proof.

$$(\mathrm{d}\exp_p)_0(v) = \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(tv) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(1, tv) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t, v) = v. \qquad \square$$

Corollary. $\exp_p$ maps an open ball $B(0, \delta) \subseteq T_pM$ to $U \subseteq M$ diffeomorphically for some $\delta > 0$.

Proof. By the inverse mapping theorem. □

Corollary. For any point $p \in M$, there exists a local coordinate chart around p such that

– The coordinates of p are $(0, \cdots, 0)$.

– In local coordinates, the metric at p is $g_{ij}(p) = \delta_{ij}$.

– We have $\Gamma^i_{jk}(p) = 0$.

Proof. The geodesic local coordinates satisfies these property, after identifying T_pM isometrically with $(\mathbb{R}^n, \mathrm{eucl})$. For the last property, we note that the geodesic equations are given by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^k\dot{x}^j = 0.$$

But geodesics through the origin are given by straight lines. So we must have $\Gamma^i_{jk} = 0$. □

Theorem (Gauss' lemma)**.** The geodesic spheres are perpendicular to their radii. More precisely, $\gamma_p(t, a)$ meets every Σ_r orthogonally, whenever this makes sense. Thus we can write the metric in geodesic polars as

$$g = \mathrm{d}r^2 + h(r, \mathbf{v}),$$

where for each r, we have

$$h(r, \mathbf{v}) = g|_{\Sigma_r}.$$

In matrix form, we have

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & h & \\ 0 & & & \end{pmatrix}$$

Proof. We work in geodesic coordinates. It is clear that $g(\partial_r, \partial_r) = 1$.

Consider an arbitrary vector field $X = X(\mathbf{v})$ on S^{n-1}. This induces a vector field on some neighbourhood $B(0, \delta) \subseteq T_pM$ by

$$\tilde{X}(r\mathbf{v}) = X(\mathbf{v}).$$

Pick a direction $\mathbf{v} \in T_pM$, and consider the unit speed geodesic γ in the direction of $\mathbf{v}$. We define

$$G(r) = g(\tilde{X}(r\mathbf{v}), \dot{\gamma}(r)) = g(\tilde{X}, \dot{\gamma}(r)).$$

We begin by noticing that

$$\nabla_{\partial_r} \tilde{X} - \nabla_{\tilde{X}} \partial_r = [\partial_r, \tilde{X}] = 0.$$

Also, we have

$$\frac{\mathrm{d}}{\mathrm{d}r} G(r) = g(\nabla_{\dot{\gamma}} \tilde{X}, \dot{\gamma}) + g(\tilde{X}, \nabla_{\dot{\gamma}} \dot{\gamma}).$$

We know the second term vanishes, since γ is a geodesic. Noting that $\dot{\gamma} = \frac{\partial}{\partial r}$, we know the first term is equal to

$$g(\nabla_{\tilde{X}} \partial_r, \partial_r) = \frac{1}{2}\Big(g(\nabla_{\tilde{X}} \partial_r, \partial_r) + g(\partial_r, \nabla_{\tilde{X}} \partial_r)\Big) = \frac{1}{2} \tilde{X}(g(\partial_r, \partial_r)) = 0,$$

since we know that $g(\partial_r, \partial_r) = 1$ constantly.

Thus, we know $G(r)$ is constant. But $G(0) = 0$ since the metric at 0 is the Euclidean metric. So G vanishes everywhere, and so ∂_r is perpendicular to Σ_g. □

Corollary. Let $a, w \in T_p M$. Then

$$g((\mathrm{d}\exp_p)_a a, (\mathrm{d}\exp_p)_a w) = g(a, w)$$

whenever a lives in the domain of the geodesic local neighbourhood.

3.2 Jacobi fields

Theorem. Let $\gamma : [0, L] \to N$ be a geodesic in a Riemannian manifold (M, g). Then

(i) For any $u, v \in T_{\gamma(0)} M$, there is a unique Jacobi field J along Γ with

$$J(0) = u, \quad \frac{\nabla J}{\mathrm{d}t}(0) = v.$$

If

$$J(0) = 0, \quad \frac{\nabla J}{\mathrm{d}t}(0) = k\dot{\gamma}(0),$$

then $J(t) = kt\dot{\gamma}(t)$. Moreover, if both $J(0)$, $\frac{\nabla J}{\mathrm{d}t}(0)$ are orthogonal to $\dot{\gamma}(0)$, then $J(t)$ is perpendicular to $\dot{\gamma}(t)$ for all $[0, L]$.

In particular, the vector space of all Jacobi fields along γ have dimension $2n$, where $n = \dim M$.

The subspace of those Jacobi fields pointwise perpendicular to $\dot{\gamma}(t)$ has dimensional $2(n-1)$.

(ii) $J(t)$ is independent of the parametrization of $\dot{\gamma}(t)$. Explicitly, if $\tilde{\gamma}(t) = \tilde{\gamma}(\lambda t)$, then $\tilde{J}$ with the same initial conditions as J is given by

$$\tilde{J}(\tilde{\gamma}(t)) = J(\gamma(\lambda t)).$$

Proof.

(i) Pick an orthonormal basis $e_1, \cdots, e_n$ of T_pM, where $p = \gamma(0)$. Then parallel transports $\{X_i(t)\}$ via the Levi-Civita connection preserves the inner product.

We take e_1 to be parallel to $\dot{\gamma}(0)$. By definition, we have
$$X_i(0) = e_i, \quad \frac{\nabla X_i}{\mathrm{d}t} = 0.$$
Now we can write
$$J = \sum_{i=1}^n y_i X_i.$$
Then taking $g(X_i, \cdot\,)$ of (†) , we find that
$$\ddot{y}_i + \sum_{j=2}^n R(\dot{\gamma}, X_j, \dot{\gamma}, X_i) y_j = 0.$$
Then the claims of the theorem follow from the standard existence and uniqueness of solutions of differential equations.

In particular, for the orthogonality part, we know that $J(0)$ and $\frac{\nabla J}{\mathrm{d}t}(0)$ being perpendicular to $\dot{\gamma}$ is equivalent to $y_1(0) = \dot{y}_1(0) = 0$, and then Jacobi's equation gives
$$\ddot{y}_1(t) = 0.$$

(ii) This follows from uniqueness. □

Proposition. Let $\gamma : [a, b] \to M$ be a geodesic, and $f(t, s)$ a variation of $\gamma(t) = f(t, 0)$ such that $f(t, s) = \gamma_s(t)$ is a geodesic for all $|s|$ small. Then
$$J(t) = \frac{\partial f}{\partial s}$$
is a Jacobi field along $\dot{\gamma}$.

Conversely, every Jacobi field along γ can be obtained this way for an appropriate function f.

Proof. The first part is just the exact computation as we had at the beginning of the section, but for the benefit of the reader, we will reproduce the proof again.
$$\begin{aligned}\frac{\nabla^2 J}{\mathrm{d}t} &= \nabla_t \nabla_t \frac{\partial f}{\partial s}\\ &= \nabla_t \nabla_s \frac{\partial f}{\partial t}\\ &= \nabla_s \left(\nabla_t \frac{\partial f}{\partial t}\right) - R(\partial_t, \partial_s)\dot{\gamma}_s.\end{aligned}$$
We notice that the first term vanishes, because $\nabla_t \frac{\partial f}{\partial t} = 0$ by definition of geodesic. So we find
$$\frac{\nabla^2 J}{\mathrm{d}t} = -R(\dot{\gamma}, J)\dot{\gamma},$$
which is the Jacobi equation.

The converse requires a bit more work. We will write $J'(0)$ for the covariant derivative of J along γ. Given a Jacobi field J along a geodesic $\gamma(t)$ for $t \in [0, L]$, we let $\tilde{\gamma}$ be another geodesic such that

$$\tilde{\gamma}(0) = \gamma(0), \quad \dot{\tilde{\gamma}}(0) = J(0).$$

We take parallel vector fields X_0, X_1 along $\tilde{\gamma}$ such that

$$X_0(0) = \dot{\gamma}(0), \quad X_1(0) = J'(0).$$

We put $X(s) = X_0(s) + sX_1(s)$. We put

$$f(t, s) = \exp_{\tilde{\gamma}(s)}(tX(s)).$$

In local coordinates, for each fixed s, we find

$$f(t, s) = \tilde{\gamma}(s) + tX(s) + O(t^2)$$

as $t \to 0$. Then we define

$$\gamma_s(t) = f(t, s)$$

whenever this makes sense. This depends smoothly on s, and the previous arguments say we get a Jacobi field

$$\hat{J}(t) = \frac{\partial f}{\partial s}(t, 0)$$

We now want to check that $\hat{J} = J$. Then we are done. To do so, we have to check the initial conditions. We have

$$\hat{J}(0) = \frac{\partial f}{\partial s}(0, 0) = \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}(0) = J(0),$$

and also

$$\hat{J}'(0) = \frac{\nabla}{\mathrm{d}t}\frac{\partial f}{\partial s}(0, 0) = \frac{\nabla}{\mathrm{d}s}\frac{\partial f}{\partial t}(0, 0) = \frac{\nabla X}{\mathrm{d}s}(0) = X_1(0) = J'(0).$$

So we have $\hat{J} = J$. □

Corollary. Every Jacobi field J along a geodesic γ with $J(0) = 0$ is given by

$$J(t) = (\mathrm{d}\exp_p)_{t\dot{\gamma}(0)}(tJ'(0))$$

for all $t \in [0, L]$.

Proof. Write $\dot{\gamma}(0) = a$, and $J'(0) = w$. By above, we can construct the variation by

$$f(t, s) = \exp_p(t(a + sw)).$$

Then

$$(\mathrm{d}\exp_p)_{t(a+sw)}(tw) = \frac{\partial f}{\partial s}(t, s),$$

which is just an application of the chain rule. Putting $s = 0$ gives the result. □

3.3 Further properties of geodesics

Lemma (Gauss' lemma)**.** Let $a, w \in T_pM$, and

$$\gamma = \gamma_p(t, a) = \exp_p(ta)$$

a geodesic. Then

$$g_{\gamma(t)}((\mathrm{d}\exp_p)_{ta}a, (\mathrm{d}\exp_p)_{ta}w) = g_{\gamma(0)}(a, w).$$

In particular, γ is orthogonal to $\exp_p\{v \in T_pM : |v| = r\}$. Note that the latter need not be a submanifold.

Proof. We fix any $r > 0$, and consider the Jacobi field J satisfying

$$J(0) = 0, \quad J'(0) = \frac{w}{r}.$$

Then by the corollary, we know the Jacobi field is

$$J(t) = (\mathrm{d}\exp_p)_{ta}\left(\frac{tw}{r}\right).$$

We may write

$$\frac{w}{r} = \lambda a + u,$$

with $a \perp u$. Then since Jacobi fields depend linearly on initial conditions, we write

$$J(t) = \lambda t\dot{\gamma}(t) + J_n(t)$$

for a Jacobi field J_n a normal vector field along γ. So we have

$$g(J(r), \dot{\gamma}(r)) = \lambda r|\dot{\gamma}(r)|^2 = g(w, a).$$

But we also have

$$g(w, a) = g(\lambda ar + u, a) = \lambda r|a|^2 = \lambda r|\dot{\gamma}(0)|^2 = \lambda r|\dot{\gamma}(r)|^2.$$

Now we use the fact that

$$J(r) = (\mathrm{d}\exp_p)_{ra}w$$

and

$$\dot{\gamma}(r) = (\mathrm{d}\exp_p)_{ra}a,$$

and we are done. □

Corollary (Local minimizing of length)**.** Let $a \in T_pM$. We define $\varphi(t) = ta$, and $\psi(t)$ a piecewise C^1 curve in T_pM for $t \in [0, 1]$ such that

$$\psi(0) = 0, \quad \psi(1) = a.$$

Then

$$\mathrm{length}(\exp_p \circ \psi) \geq \mathrm{length}(\exp_p \circ \varphi) = |a|.$$

Proof. We may of course assume that ψ never hits 0 again after $t = 0$. We write

$$\psi(t) = \rho(t)\mathbf{u}(t),$$

where $\rho(t) \geq 0$ and $|\mathbf{u}(t)| = 1$. Then

$$\psi' = \rho'\mathbf{u} + \rho\mathbf{u}'.$$

Then using the extended Gauss lemma, and the general fact that if $\mathbf{u}(t)$ is a unit vector for all t, then $\mathbf{u} \cdot \mathbf{u}' = \frac{1}{2}(\mathbf{u} \cdot \mathbf{u})' = 0$, we have

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}x}(\exp_p \circ \psi)(t)\right|^2 &= \left|(\mathrm{d}\exp_p)_{\psi(t)}\psi'(t)\right|^2 \\ &= \rho'(t)^2 + 2g(\rho'(t)\mathbf{u}(t), \rho(t)\mathbf{u}'(t)) + \rho(t)^2|(\mathrm{d}\exp_p)_{\psi(t)}\mathbf{u}'(t)|^2 \\ &= \rho'(t)^2 + \rho(t)^2|(\mathrm{d}\exp_p)_{\psi(t)}\mathbf{u}'(t)|^2, \end{aligned}$$

Thus we have

$$\operatorname{length}(\exp_p \circ \psi) \geq \int_0^1 \rho'(t)\;\mathrm{d}t = \rho(1) - \rho(0) = |a|. \qquad \square$$

Theorem. Let $p \in M$, and let ε be such that $\exp_p|_{B(0,\varepsilon)}$ is a diffeomorphism onto its image, and let U be the image. Then

- For any $q \in U$, there is a unique geodesic $\gamma \in \Omega(p, q)$ with $\ell(\gamma) < \varepsilon$. Moreover, $\ell(\gamma) = d(p, q)$, and is the unique curve that satisfies this property.
- For any point $q \in M$ with $d(p, q) < \varepsilon$, we have $q \in U$.
- If $q \in M$ is any point, $\gamma \in \Omega(p, q)$ has $\ell(\gamma) = d(p, q) < \varepsilon$, then γ is a geodesic.

Proof. Let $q = \exp_p(a)$. Then the path $\gamma(t) = \exp_p(ta)$ is a geodesic from p to q of length $|a| = r < \varepsilon$. This is clearly the only such geodesic, since $\exp_p|_{B(0,\varepsilon)}$ is a diffeomorphism.

Given any other path $\tilde{\gamma} \in \Omega(p, q)$, we want to show $\ell(\tilde{\gamma}) > \ell(\gamma)$. We let

$$\tau = \sup\left\{t \in [0, 1] : \gamma([0, t]) \subseteq \exp_p(\overline{B(0, r)})\right\}.$$

Note that if $\tau \neq 1$, then we must have $\gamma(\tau) \in \Sigma_r$, the geodesic sphere of radius r, otherwise we can continue extending. On the other hand, if $\tau = 1$, then we certainly have $\gamma(\tau) \in \Sigma_r$, since $\gamma(\tau) = q$. Then by local minimizing of length, we have

$$\ell(\tilde{\gamma}) \geq \ell(\tilde{\gamma}_{[0,\tau]}) \geq r.$$

Note that we can always lift $\tilde{\gamma}[0, \tau]$ to a curve from 0 to a in T_pM, since $\exp_p$ is a diffeomorphism in $B(0, \varepsilon)$.

By looking at the proof of the local minimizing of length, and using the same notation, we know that we have equality iff $\tau = 1$ and

$$\rho(t)^2|(\mathrm{d}\exp_p)_{\psi(t)}\psi(t)\mathbf{u}'(t)|^2 = 0$$

for all t. Since $\mathrm{d}\exp_p$ is regular, this requires $\mathbf{u}'(t) = 0$ for all t (since $\rho(t) \neq 0$ when $t \neq 0$, or else we can remove the loop to get a shorter curve). This implies $\tilde{\gamma}$ lifts to a straight line in T_pM, i.e. is a geodesic.

Now given any $q \in M$ with $r = d(p, q) < \varepsilon$, we pick $r' \in [r, \varepsilon)$ and a path $\gamma \in \Omega(p, q)$ such that $\ell(\gamma) = r'$. We again let

$$\tau = \sup\left\{t \in [0, 1] : \gamma([0, t]) \subseteq \exp_p(\overline{B(0, r')})\right\}.$$

If $\tau \neq 1$, then we must have $\gamma(\tau) \in \Sigma_{r'}$, but lifting to T_pM, this contradicts the local minimizing of length.

The last part is an immediate consequence of the previous two. □

Corollary. The distance d on a Riemannian manifold is a metric, and induces the same topology on M as the C^∞ structure.

Corollary. Let $\gamma : [0, 1] \to M$ be a piecewise C^1 minimal geodesic with constant speed. Then γ is in fact a geodesic, and is in particular C^∞.

Proof. We wlog γ is unit speed. Let $t \in [0, 1]$, and pick $\varepsilon > 0$ such that $\exp_p|_{B(0,\varepsilon)}$ is a diffeomorphism. Then by the theorem, $\gamma_{[t,t+\frac{1}{2}\varepsilon]}$ is a geodesic. So γ is C^∞ on $(t, t + \frac{1}{2}\varepsilon)$, and satisfies the geodesic equations there.

Since we can pick ε continuously with respect to t by ODE theorems, any $t \in (0, 1)$ lies in one such neighbourhood. So γ is a geodesic. □

Corollary. Let $\gamma : [0, 1] \subseteq \mathbb{R} \to M$ be a C^2 curve with $|\dot{\gamma}|$ constant. Then this is a geodesic iff it is locally a minimal geodesic, i.e. for any $t \in [0, 1)$, there exists $\delta > 0$ such that

$$d(\gamma(t), \gamma(t + \delta)) = \ell(\gamma|_{[t,t+\delta]}).$$

Proof. This is just carefully applying the previous theorem without getting confused.

To prove $\Rightarrow$, suppose γ is a geodesic, and $t \in [0, 1)$. We wlog γ is unit speed. Then pick U and ε as in the previous theorem, and pick $\delta = \frac{1}{2}\varepsilon$. Then $\gamma|_{[t,t+\delta]}$ is a geodesic with length $< \varepsilon$ between $\gamma(t)$ and $\gamma(t + \delta)$, and hence must have minimal length.

To prove the converse, we note that for each t, the hypothesis tells us $\gamma|_{[t,t+\delta]}$ is a minimizing geodesic, and hence a geodesic, but the previous corollary. By continuity, γ must satisfy the geodesic equation at t. Since t is arbitrary, γ is a geodesic. □

Theorem. Let $\gamma(t) = \exp_p(ta)$ be a geodesic, for $t \in [0, 1]$. Let $q = \gamma(1)$. Assume ta is a regular point for $\exp_p$ for all $t \in [0, 1]$. Then there exists a neighbourhood of γ in $\Omega(p, q)$ such that for all ψ in this neighbourhood, $\ell(\psi) \geq \ell(\gamma)$, with equality iff $\psi = \gamma$ up to reparametrization.

Proof. The idea of the proof is that if ψ is any curve close to γ, then we can use the regularity condition to lift the curve back up to T_pM, and then apply our previous result.

Write $\varphi(t) = ta \in T_pM$. Then by the regularity assumption, for all $t \in [0, 1]$, we know $\exp_p$ is a diffeomorphism of some neighbourhood $W(t)$ of $\varphi(t) = at \in$

T_pM onto the image. By compactness, we can cover $[0, 1]$ by finitely many such covers, say $W(t_1), \cdots, W(t_n)$. We write $W_i = W(t_i)$, and we wlog assume

$$0 = t_0 < t_1 < \cdots < t_k = 1.$$

By cutting things up, we may assume

$$\gamma([t_i, t_{i+1}]) \subseteq W_i.$$

We let

$$U = \bigcup \exp_p(W_i).$$

Again by compactness, there is some $\varepsilon < 0$ such that for all $t \in [t_i, t_{i+1}]$, we have $B(\gamma(t), \varepsilon) \subseteq W_i$.

Now consider any curve ψ of distance ε away from γ. Then $\psi([t_i, t_{i+1}]) \subseteq W_i$. So we can lift it up to T_pM, and the end point of the lift is a. So we are done by local minimization of length. □

3.4 Completeness and the Hopf–Rinow theorem

Theorem. Let (M, g) be geodesically complete. Then any two points can be connected by a minimal geodesic.

Lemma. Let $p, q \in M$. Let

$$S_\delta = \{x \in M : d(x, p) = \delta\}.$$

Then for all sufficiently small δ, there exists $p_0 \in S_\delta$ such that

$$d(p, p_0) + d(p_0, q) = d(p, q).$$

Proof. For $\delta > 0$ small, we know $S_\delta = \Sigma_\delta$ is a geodesic sphere about p, and is compact. Moreover, $d(\,\cdot\,, q)$ is a continuous function. So there exists some $p_0 \in \Sigma_\delta$ that minimizes $d(\,\cdot\,, q)$.

Consider an arbitrary $\gamma \in \Omega(p, q)$. For the sake of sanity, we assume $\delta < d(p, q)$. Then there is some t such that $\gamma(t) \in \Sigma_\delta$, and

$$\ell(\gamma) \geq d(p, \gamma(t)) + d(\gamma(t), q) \geq d(p, p_0) + d(p_0, q).$$

So we know

$$d(p, q) \geq d(p, p_0) + d(p_0, p).$$

The triangle inequality gives the opposite direction. So we must have equality. □

Proof of theorem. We know $\exp_p$ is defined on T_pM. Let $q \in M$. Let $q \in M$. We want a minimal geodesic in $\Omega(p, q)$. By the first lemma, there is some $\delta > 0$ and p_0 such that

$$d(p, p_0) = \delta, \quad d(p, p_0) + d(p_0, q) = d(p, q).$$

Also, there is some $v \in T_pM$ such that $\exp_p v = p_0$. We let

$$\gamma_p(t) = \exp_p\left(t\frac{v}{|v|}\right).$$

We let

$$I = \{t \in \mathbb{R} : d(q, \gamma_p(t)) + t = d(p, q)\}.$$

Then we know

(i) $\delta \in I$

(ii) I is closed by continuity.

Let

$$T = \sup\{I \cap [0, d(p,q)]\}.$$

Since I is closed, this is in fact a maximum. So $T \in I$. We claim that $T = d(p,q)$. If so, then $\gamma_p \in \Omega(p,q)$ is the desired minimal geodesic, and we are done.

Suppose this were not true. Then $T < d(p,q)$. We apply the lemma to $\tilde{p} = \gamma_p(T)$, and q remains as before. Then we can find $\varepsilon > 0$ and some $p_1 \in M$ with the property that

$$\begin{aligned} d(p_1, q) &= d(\gamma_p(T), q) - d(\gamma_p(T), p_1) \\ &= d(\gamma_p(T), q) - \varepsilon \\ &= d(p,q) - T - \varepsilon \end{aligned}$$

Hence we have

$$d(p, p_1) \geq d(p,q) - d(q, p_1) = T + \varepsilon.$$

Let γ_1 be the radial (hence minimal) geodesic from $\gamma_p(T)$ to p_1. Now we know

$$\ell(\gamma_p|_{[0,T]}) + \ell(\gamma_1) = T + \varepsilon.$$

So γ_1 concatenated with $\gamma_p|_{[0,T]}$ is a length-minimizing geodesic from p to p_1, and is hence a geodesic. So in fact p_1 lies on γ_p, say $p_1 = \gamma_p(T+s)$ for some s. Then $T + s \in I$, which is a contradiction. So we must have $T = d(p,q)$, and hence

$$d(q, \gamma_p(T)) + T = d(p,q),$$

hence $d(q, \gamma_p(T)) = 0$, i.e. $q = \gamma_p(T)$. □

Corollary (Hopf–Rinow theorem)**.** For a connected Riemannian manifold (M, g), the following are equivalent:

(i) (M, g) is geodesically complete.

(ii) For all $p \in M$, $\exp_p$ is defined on all T_pM.

(iii) For some $p \in M$, $\exp_p$ is defined on all T_pM.

(iv) Every closed and bounded subset of (M, d) is compact.

(v) (M, d) is complete as a metric space.

Proof. (i) and (ii) are equivalent by definition. (ii) $\Rightarrow$ (iii) is clear, and we proved (iii) $\Rightarrow$ (i).

- (iii) $\Rightarrow$ (iv): Let $K \subseteq M$ be closed and bounded. Then by boundedness, K is contained in $\exp_p(\overline{B(0,R)})$. Let K' be the pre-image of K under $\exp_p$. Then it is a closed and bounded subset of $\mathbb{R}^n$, hence compact. Then K is the continuous image of a compact set, hence compact.
- (iv) $\Rightarrow$ (v): This is a general topological fact.
- (v) $\Rightarrow$ (i): Let $\gamma(t): I \to \mathbb{R}$ be a geodesic, where $I \subseteq \mathbb{R}$. We wlog $|\dot{\gamma}| \equiv 1$. Suppose $I \neq \mathbb{R}$. We wlog $\sup I = a < \infty$. Then $\lim_{t \to a} \gamma(t)$ exist by completeness, and hence $\gamma(a)$ exists. Since geodesics are locally defined near a, we can pick a geodesic in the direction of $\lim_{t \to a} \gamma'(t)$. So we can extend γ further, which is a contradiction. □

3.5 Variations of arc length and energy

Proposition. Let $\gamma_0 : [0, T] \to M$ be a path from p to q such that for all $\gamma \in \Omega(p, q)$ with $\gamma : [0, T] \to M$, we have $E(\gamma) \geq E(\gamma_0)$. Then γ_0 must be a geodesic.

Proof. By the Cauchy-Schwartz inequality, we have

$$\int_0^T |\dot{\gamma}|^2 \;\mathrm{d}t \geq \left(\int_0^T |\dot{\gamma}(t)| \;\mathrm{d}t\right)^2$$

with equality iff $|\dot{\gamma}|$ is constant. In other words,

$$E(\gamma) \geq \frac{\ell(\gamma)^2}{2T}.$$

So we know that if γ_0 minimizes energy, then it must be constant speed. Now given any γ, if we just care about its length, then we may wlog it is constant speed, and then

$$\ell(\gamma) = \sqrt{2E(\gamma)T} \geq \sqrt{2E(\gamma_0)T} = \ell(\gamma_0).$$

So γ_0 minimizes length, and thus γ_0 is a geodesic. □

Theorem (First variation formula)**.**

(i) For any variation H of γ, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s = 0} = g(Y(t), \dot{\gamma}(t))|_0^T - \int_0^T g\left(Y(t), \frac{\nabla}{\mathrm{d}t} \dot{\gamma}(t)\right)\;\mathrm{d}t. \qquad (*)$$

(ii) The critical points, i.e. the γ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s = 0}$$

for all (end-point fixing) variation H of γ, are geodesics.

(iii) If $|\dot{\gamma}_s(t)|$ is constant for each fixed $s \in (-\varepsilon, \varepsilon)$, and $|\dot{\gamma}(t)| \equiv 1$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s = 0} = \left.\frac{\mathrm{d}}{\mathrm{d}s} \ell(\gamma_s)\right|_{s = 0}$$

(iv) If γ is a critical point of the length, then it must be a reparametrization of a geodesic.

Proof. We will assume that we can treat $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ as vector fields on an embedded submanifold, even though H is not necessarily a local embedding.

The result can be proved without this assumption, but will require more technical work.

(i) We have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s} g(\dot{\gamma}_s(t), \dot{\gamma}_s(t)) &= g\left(\frac{\nabla}{\mathrm{d}s}\dot{\gamma}_s(t), \dot{\gamma}_s(t)\right)\\ &= g\left(\frac{\nabla}{\mathrm{d}t}\frac{\partial H}{\partial s}(t,s), \frac{\partial H}{\partial t}(t,s)\right)\\ &= \frac{\partial}{\partial t} g\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right) - g\left(\frac{\partial H}{\partial s}, \frac{\nabla}{\mathrm{d}t}\frac{\partial H}{\partial t}\right).\end{aligned}$$

Comparing with what we want to prove, we see that we get what we want by integrating $\int_0^T \;\mathrm{d}t$, and then putting $s = 0$, and then noting that

$$\left.\frac{\partial H}{\partial s}\right|_{s=0} = Y, \quad \left.\frac{\partial H}{\partial t}\right|_{s=0} = \dot{\gamma}.$$

(ii) If γ is a geodesic, then

$$\frac{\nabla}{\mathrm{d}t}\dot{\gamma}(t) = 0.$$

So the integral on the right hand side of $(*)$ vanishes. Also, we have $Y(0) = 0 = Y(T)$. So the RHS vanishes.

Conversely, suppose γ is a critical point for E. Then choose H with

$$Y(t) = f(t)\frac{\nabla}{\mathrm{d}t}\dot{\gamma}(t)$$

for some $f \in C^\infty[0, T]$ such that $f(0) = f(T) = 0$. Then we know

$$\int_0^T f(t)\left|\frac{\nabla}{\mathrm{d}t}\dot{\gamma}(t)\right|^2\;\mathrm{d}t = 0,$$

and this is true for all f. So we know

$$\frac{\nabla}{\mathrm{d}t}\dot{\gamma} = 0.$$

(iii) This is evident from the previous proposition. Indeed, we fix $[0, T]$, then for all H, we have

$$E(\gamma_s) = \frac{\ell(\gamma_s)^2}{2T},$$

and so

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s)\right|_{s=0} = \frac{1}{T}\ell(\gamma_s)\left.\frac{\mathrm{d}}{\mathrm{d}s}\ell(\gamma_s)\right|_{s=0},$$

and when $s = 0$, the curve is parametrized by arc-length, so $\ell(\gamma_s) = T$.

(iv) By reparametrization, we may wlog $|\dot{\gamma}| \equiv 1$. Then γ is a critical point for ℓ, hence for E, hence a geodesic. □

Theorem (Second variation formula)**.** Let $\gamma(t) : [0, T] \to M$ be a geodesic with $|\dot{\gamma}| = 1$. Let $H(t, s)$ be a variation of γ. Let

$$Y(t, s) = \frac{\partial H}{\partial s}(t, s) = (\mathrm{d}H)_{(t,s)}\frac{\partial}{\partial s}.$$

Then

(i) We have
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}(t,0), \dot\gamma\right)\Big|_0^T + \int_0^T (|Y'|^2 - R(Y, \dot\gamma, Y, \dot\gamma))\;\mathrm{d}t.$$

(ii) Also
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}(t,0), \dot\gamma(t)\right)\Big|_0^T + \int_0^T \left(|Y'|^2 - R(Y, \dot\gamma, Y, \dot\gamma) - g(\dot\gamma, Y')^2\right)\;\mathrm{d}t,$$
where R is the $(4,0)$ curvature tensor, and
$$Y'(t) = \frac{\nabla Y}{\mathrm{d}t}(t,0).$$
Putting
$$Y_n = Y - g(Y, \dot\gamma)\dot\gamma$$
for the normal component of Y, we can write this as
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y_n}{\mathrm{d}s}(t,0), \dot\gamma(t)\right)\Big|_0^T + \int_0^T \left(|Y_n'|^2 - R(Y_n, \dot\gamma, Y_n, \dot\gamma)\right)\;\mathrm{d}t.$$

Proof. We use
$$\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = g(Y(t,s), \dot\gamma_s(t))|_{t=0}^{t=T} - \int_0^T g\left(Y(t,s), \frac{\nabla}{\mathrm{d}t}\dot\gamma_s(t)\right)\;\mathrm{d}t.$$
Taking the derivative with respect to s again gives
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s) = g\left(\frac{\nabla Y}{\mathrm{d}s}, \dot\gamma\right)\Big|_{t=0}^T + g\left(Y, \frac{\nabla}{\mathrm{d}s}\dot\gamma_s\right)\Big|_{t=0}^T - \int_0^T \left(g\left(\frac{\nabla Y}{\mathrm{d}s}, \frac{\nabla}{\mathrm{d}t}\dot\gamma_s\right) + g\left(Y, \frac{\nabla}{\mathrm{d}s}\frac{\nabla}{\mathrm{d}t}\dot\gamma\right)\right)\;\mathrm{d}t.$$
We now use that
$$\begin{aligned}\frac{\nabla}{\mathrm{d}s}\frac{\nabla}{\mathrm{d}t}\dot\gamma_s(t) &= \frac{\nabla}{\mathrm{d}t}\frac{\nabla}{\mathrm{d}s}\dot\gamma_s(t) + R\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right)\dot\gamma_s\\ &= \left(\frac{\nabla}{\mathrm{d}t}\right)^2 Y(t,s) + R\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right)\dot\gamma_s.\end{aligned}$$
We now set $s = 0$, and then the above gives
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}, \dot\gamma\right)\Big|_0^T + g\left(Y, \frac{\nabla\dot\gamma}{\mathrm{d}s}\right)\Big|_0^T - \int_0^T \left[g\left(Y, \left(\frac{\nabla}{\mathrm{d}t}\right)^2 Y\right) + R(\dot\gamma, Y, \dot\gamma, Y)\right]\;\mathrm{d}t.$$

Finally, applying integration by parts, we can write

$$-\int_0^T g\left(Y, \left(\frac{\nabla}{\mathrm{d}t}\right)^2 Y\right)\;\mathrm{d}t = -\,g\left(Y, \frac{\nabla}{\mathrm{d}t}Y\right)\Big|_0^T + \int_0^T \left|\frac{\nabla Y}{\mathrm{d}t}\right|^2\;\mathrm{d}t.$$

Finally, noting that

$$\frac{\nabla}{\mathrm{d}s}\dot\gamma(s) = \frac{\nabla}{\mathrm{d}t}Y(t,s),$$

we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\Big|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}, \dot\gamma\right)\Big|_0^T + \int_0^T \left(|Y'|^2 - R(Y,\dot\gamma,Y,\dot\gamma)\right)\;\mathrm{d}t.$$

It remains to prove the second variation of length. We first differentiate

$$\frac{\mathrm{d}}{\mathrm{d}s}\ell(\gamma_s) = \int_0^T \frac{1}{2\sqrt{g(\dot\gamma_s,\dot\gamma_s)}}\frac{\partial}{\partial s}g(\dot\gamma_s,\dot\gamma_s)\;\mathrm{d}t.$$

Then the second derivative gives

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\Big|_{s=0} = \int_0^T \left[\frac{1}{2}\frac{\partial^2}{\partial s^2}g(\dot\gamma_s,\dot\gamma_s)\Big|_{s=0} - \frac{1}{4}\left(\frac{\partial}{\partial s}g(\dot\gamma_s,\dot\gamma_s)\right)^2\Big|_{s=0}\right]\;\mathrm{d}t,$$

where we used the fact that $g(\dot\gamma,\dot\gamma) = 1$.

We notice that the first term can be identified with the derivative of the energy function. So we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\Big|_{s=0} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\Big|_{s=0} - \int_0^T \left(g\left(\dot\gamma_s, \frac{\nabla}{\mathrm{d}s}\dot\gamma_s\right)\Big|_{s=0}\right)^2\;\mathrm{d}t.$$

So the second part follows from the first. □

3.6 Applications

Theorem (Synge's theorem)**.** Every compact orientable Riemannian manifold (M,g) such that $\dim M$ is even and has $K(g) > 0$ for all planes at $p \in M$ is simply connected.

Lemma. Let M be a compact manifold, and $[\alpha]$ a non-trivial homotopy class of closed curves in M. Then there is a closed minimal geodesic in $[\alpha]$.

Proof. Since M is compact, we can pick some $\varepsilon > 0$ such that for all $p \in M$, the map $\exp_p|_{B(0,p)}$ is a diffeomorphism.

Let $\ell = \inf_{\gamma\in[\alpha]}\ell(\gamma)$. We know that $\ell > 0$, otherwise, there exists a γ with $\ell(\gamma) < \varepsilon$. So γ is contained in some geodesic coordinate neighbourhood, but then α is contractible. So ℓ must be positive.

Then we can find a sequence $\gamma_n \in [\alpha]$ with $\gamma_n : [0,1] \to M$, $|\dot\gamma|$ constant, such that

$$\lim_{n\to\infty}\ell(\gamma_n) = \ell.$$

Choose

$$0 = t_0 < t_1 < \cdots < t_k = 1$$

such that

$$t_{i+1} - t_i < \frac{\varepsilon}{2\ell}.$$

So it follows that

$$d(\gamma_n(t_i), \gamma_n(t_{i+1})) < \varepsilon$$

for all n sufficiently large and all i. Then again, we can replace $\gamma_n|_{[t_i, t_{i+1}]}$ by a radial geodesic without affecting the limit $\lim \ell(\gamma_n)$.

Then we exploit the compactness of M (and the unit sphere) again, and pass to a subsequence of $\{\gamma_n\}$ so that $\gamma_n(t_i), \dot{\gamma}_n(t_i)$ are all convergent for every fixed i as $n \to \infty$. Then the curves converges to some

$$\gamma_n \to \hat{\gamma} \in [\alpha],$$

given by joining the limits $\lim_{n\to\infty} \gamma_n(t_i)$. Then we know that the length converges as well, and so we know $\hat{\gamma}$ is minimal among curves in $[\alpha]$. So $\hat{\gamma}$ is locally minimal, hence a geodesic. So we can take $\gamma = \hat{\gamma}$, and we are done. □

Proof of Synge's theorem. Suppose M satisfies the hypothesis, but $\pi_1(M) \neq \{1\}$. So there is a path α with $[\alpha] \neq 1$, i.e. it cannot be contracted to a point. By the lemma, we pick a representative γ of $[\alpha]$ that is a closed, minimal geodesic.

We now prove the theorem. We may wlog assume $|\dot{\gamma}| = 1$, and t ranges in $[0, T]$. Consider a vector field $X(t)$ for $0 \leq t \leq T$ along $\gamma(t)$ such that

$$\frac{\nabla X}{\mathrm{d}t} = 0, \quad g(X(0), \dot{\gamma}(0)) = 0.$$

Note that since g is a geodesic, we know

$$g(X(t), \dot{\gamma}(t)) = 0,$$

for all $t \in [0, T]$ as parallel transport preserves the inner product. So $X(T) \perp \dot{\gamma}(T) = \dot{\gamma}(0)$ since we have a closed curve.

We consider the map P that sends $X(0) \mapsto X(T)$. This is a linear isometry of $(\dot{\gamma}(0))^\perp$ with itself that preserves orientation. So we can think of P as a map

$$P \in \mathrm{SO}(2n - 1),$$

where $\dim M = 2n$. It is an easy linear algebra exercise to show that every element of $\mathrm{SO}(2n-1)$ must have an eigenvector of eigenvalue 1. So we can find $v \in T_pM$ such that $v \perp \dot{\gamma}(0)$ and $P(v) = v$. We take $X(0) = v$. Then we have $X(T) = v$.

Consider now a variation $H(t, s)$ inducing this $X(t)$. We may assume $|\dot{\gamma}_s|$ is constant. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\ell(\gamma_s)|_{s=0} = 0$$

as γ is minimal. Moreover, since it is a minimum, the second derivative must be positive, or at least non-negative. Is this actually the case?

We look at the second variation formula of length. Using the fact that the loop is closed, the formula reduces to

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\right|_{s=0} = -\int_0^T R(X, \dot{\gamma}, X, \dot{\gamma})\;\mathrm{d}t.$$

But we assumed the sectional curvature is positive. So the second variation is negative! This is a contradiction. □

Proposition.

(i) If $\gamma(t) = \exp_p(ta)$, and $q = \exp_p(\beta a)$ is conjugate to p, then q is a singular value of exp.

(ii) Let J be as in the definition. Then J must be pointwise normal to $\dot{\gamma}$.

Proof.

(i) We wlog $[\alpha, \beta] = [0, 1]$. So $J(0) = 0 = J(1)$. We $a = \dot{\gamma}(0)$ and $w = J'(0)$. Note that a, w are both non-zero, as Jacobi fields are determined by initial conditions. Then $q = \exp_p(a)$.

We have shown earlier that if $J(0) = 0$, then

$$J(t) = (\mathrm{d}\exp_p)_{ta}(tw)$$

for all $0 \le t \le 1$. So it follows $(\mathrm{d}\exp_p)_a(w) = J(1) = 0$. So $(\mathrm{d}\exp_p)_a$ has non-trivial kernel, and hence isn't surjective.

(ii) We claim that any Jacobi field J along a geodesic γ satisfies

$$g(J(t), \dot{\gamma}(t)) = g(J'(0), \dot{\gamma}(0))t + g(J(0), \dot{\gamma}(0)).$$

To prove this, we note that by the definition of geodesic and Jacobi fields, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(J', \dot{\gamma}) = g(J'', \dot{\gamma}(0)) = -g(R(\dot{\gamma}, J), \dot{\gamma}, \dot{\gamma}) = 0$$

by symmetries of R. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(J, \dot{\gamma}) = g(J'(t), \dot{\gamma}(t)) = g(J'(0), \dot{\gamma}(0)).$$

Now integrating gives the desired result.

This result tells us $g(J(t), \dot{\gamma}(t))$ is a linear function of t. But we have

$$g(J(0), \dot{\gamma}(0)) = g(J(1), \dot{\gamma}(1)) = 0.$$

So we know $g(J(t), \dot{\gamma}(t))$ is constantly zero. □

Theorem. Let $\gamma : [0, 1] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma(1) = q$ such that p is conjugate to some $\gamma(t_0)$ for some $t_0 \in (0, 1)$. Then there is a piecewise smooth variation of $f(t, s)$ with $f(t, 0) = \gamma(t)$ such that

$$f(0, s) = p, \quad f(1, s) = q$$

and $\ell(f(\,\cdot\,, s)) < \ell(\gamma)$ whenever $s \neq 0$ is small.

Proof. By the hypothesis, there is a $J(t)$ defined on $t \in [0, 1]$ and $t_0 \in (0, 1)$ such that

$$J(t) \perp \dot{\gamma}(t)$$

for all t, and $J(0) = J(t_0) = 0$ and $J \not\equiv 0$. Then $J'(t_0) \neq 0$.

We define a parallel vector field Z_1 along γ by $Z_1(t_0) = -J'(t_0)$. We pick $\theta \in C^\infty[0, 1]$ such that $\theta(0) = \theta(1) = 0$ and $\theta(t_0) = 1$.

Finally, we define

$$Z = \theta Z_1,$$

and for $\alpha \in \mathbb{R}$, we define

$$Y_\alpha(t) = \begin{cases} J(t) + \alpha Z(t) & 0 \leq t \leq t_0 \\ \alpha Z(t) & t_0 \leq t \leq 1 \end{cases}.$$

We notice that this is not smooth at t_0, but is just continuous. We will postpone the choice of α to a later time.

We know $Y_\alpha(t)$ arises from a piecewise C^∞ variation of γ, say $H_\alpha(t, s)$. The technical claim is that the second variation of length corresponding to $Y_\alpha(t)$ is negative for some α.

We denote by $I(X, Y)_T$ the symmetric bilinear form that gives rise to the second variation of length with fixed end points. If we make the additional assumption that X, Y are normal along γ, then the formula simplifies, and reduces to

$$I(X, Y)_T = \int_0^T (g(X', Y') - R(X, \dot{\gamma}, Y, \dot{\gamma}))\ \mathrm{d}t.$$

Then for $H_\alpha(t, s)$, we have

$$\begin{aligned} \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\right|_{s=0} &= I_1 + I_2 + I_3 \\ I_1 &= I(J, J)_{t_0} \\ I_2 &= 2\alpha I(J, Z)_{t_0} \\ I_3 &= \alpha^2 I(Z, Z)_1. \end{aligned}$$

We look at each term separately.

We first claim that $I_1 = 0$. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t} g(J, J') = g(J', J') + g(J, J''),$$

and $g(J, J'')$ added to the curvature vanishes by the Jacobi equation. Then by integrating by parts and applying the boundary condition, we see that I_1 vanishes.

Also, by integrating by parts, we find

$$I_2 = 2\alpha g(Z, J')|_0^{t_0}.$$

Whence

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ell(\gamma_s)\right|_{s=0} = -2\alpha|J'(t_0)|^2 + \alpha^2 I(Z, Z)_1.$$

Now if $\alpha > 0$ is very very small, then the linear term dominates, and this is negative. Since the first variation vanishes (γ is a geodesic), we know this is a local maximum of length. □

Proof. Consider any $L < \mathrm{diam}(M, g)$. Then by definition (and Hopf–Rinow), we can find $p, q \in M$ such that $d(p, q) = L$, and a minimal geodesic $\gamma \in \Omega(p, q)$ with $\ell(\gamma) = d(p, q)$. We parametrize $\gamma : [0, L] \to M$ so that $|\dot{\gamma}| = 1$.

Now consider any vector field Y along γ such that $Y(p) = 0 = Y(q)$. Since Γ is a minimal geodesic, it is a critical point for ℓ, and the second variation $I(Y, Y)_{[0,L]}$ is non-negative (recall that the second variation has fixed end points).

We extend $\dot{\gamma}(0)$ to an orthonormal basis of T_pM, say $\dot{\gamma}(0) = e_1, e_2, \cdots, e_n$. We further let X_i be the unique vector field such that

$$X_i' = 0, \quad X_i(0) = e_i.$$

In particular, $X_1(t) = \dot{\gamma}(t)$.

For $i = 2, \cdots, n$, we put

$$Y_i(t) = \sin\left(\frac{\pi t}{L}\right) X_i(t).$$

Then after integrating by parts, we find that we have

$$I(Y_i, Y_i)_{[0,L]} = -\int_0^L g(Y_i'' + R(\dot{\gamma}, Y_i)Y_i, \dot{\gamma})\;\mathrm{d}t$$

Using the fact that X_i is parallel, this can be written as

$$= \int_0^L \sin^2 \frac{\pi t}{L}\left(\frac{\pi^2}{L^2} - R(\dot{\gamma}, X_i, \dot{\gamma}, X_i)\right)\;\mathrm{d}t,$$

and since this is length minimizing, we know this is ≥ 0.

We note that we have $R(\dot{\gamma}, X_1, \dot{\gamma}, X_1) = 0$. So we have

$$\sum_{i=2}^n R(\dot{\gamma}, X_i, \dot{\gamma}, X_i) = \operatorname{Ric}(\dot{\gamma}, \dot{\gamma}).$$

So we know

$$\sum_{i=2}^n I(Y_i, Y_i) = \int_0^L \sin^2\frac{\pi t}{L}\left((n-1)\frac{\pi^2}{L} - \operatorname{Ric}(\dot{\gamma}, \dot{\gamma})\right)\;\mathrm{d}t \geq 0.$$

We also know that

$$\operatorname{Ric}(\dot{\gamma}, \dot{\gamma}) \geq \frac{n-1}{r^2}$$

by hypothesis. So this implies that

$$\frac{\pi^2}{L^2} \geq \frac{1}{r^2}.$$

This tells us that

$$L \leq \pi r.$$

Since L is any number less that $\operatorname{diam}(M, g)$, it follows that

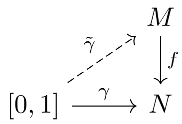
$$\operatorname{diam}(M, g) \leq \pi r.$$

Since M is known to be complete, by Hopf-Rinow theorem, any closed bounded subset is compact. But M itself is closed and bounded! So M is compact.

To understand the fundamental, group, we simply have to consider a universal Riemannian cover $f : \tilde{M} \to M$. We know such a topological universal covering space must exist by general existence theorems. We can then pull back the differential structure and metric along f, since f is a local homeomorphism. So this gives a universal Riemannian cover of M. But they hypothesis of the theorem is local, so it is also satisfied for $\tilde{M}$. So it is also compact. Since $f^{-1}(p)$ is a closed discrete subset of a compact space, it is finite, and we are done. □

Proposition. Let (M, g) and (N, h) be Riemannian manifolds, and suppose M is complete. Suppose there is a smooth surjection $f : M \to N$ that is a local diffeomorphism. Moreover, suppose that for any $p \in M$ and $v \in T_pM$, we have $|\mathrm{d}f_p(v)|_h \geq |v|$. Then f is a covering map.

Proof. By general topology, it suffices to prove that for any smooth curve $\gamma : [0, 1] \to N$, and any $q \in M$ such that $f(q) = \gamma(0)$, there exists a lift of γ starting from from q.



From the hypothesis, we know that $\tilde{\gamma}$ exists on $[0, \varepsilon_0]$ for some "small" $\varepsilon_0 > 0$. We let

$$I = \{0 \leq \varepsilon \leq : \tilde{\gamma} \text{ exists on } [0, \varepsilon]\}.$$

We immediately see this is non-empty, since it contains ε_0. Moreover, it is not difficult to see that I is open in $[0, 1]$, because f is a local diffeomorphism. So it suffices to show that I is closed.

We let $\{t_n\}_{n=1}^{\infty} \subseteq I$ be such that $t_{n+1} > t_n$ for all n, and

$$\lim_{n \to \infty} t_n = \varepsilon_1.$$

Using Hopf-Rinow, either $\{\tilde{\gamma}(t_n)\}$ is contained in some compact K, or it is unbounded. We claim that unboundedness is impossible. We have

$$\begin{aligned}
\ell(\gamma) \geq \ell(\gamma|_{[0,t_n]}) &= \int_0^{t_n} |\dot{\gamma}| \; \mathrm{d}t \\
&= \int_0^{t_n} |\mathrm{d}f_{\tilde{\gamma}(t)}\dot{\tilde{\gamma}}(t)| \; \mathrm{d}t \\
&\geq \int_0^{t_n} |\dot{\tilde{\gamma}}| \; \mathrm{d}t \\
&= \ell(\tilde{\gamma}|_{[0,t_n]}) \\
&\geq d(\tilde{\gamma}(0), \tilde{\gamma}(t_n)).
\end{aligned}$$

So we know this is bounded. So by compactness, we can find some x such that $\tilde{\gamma}(t_{n_\ell}) \to x$ as $\ell \to \infty$. There exists an open $x \in V \subseteq M$ such that $f|_V$ is a diffeomorphism.

Since there are extensions of $\tilde{\gamma}$ to each t_n, eventually we get an extension to within V, and then we can just lift directly, and extend it to ε_1. So $\varepsilon_1 \in I$. So we are done. □

Corollary. Let $f : M \to N$ be a local isometry onto N, and M be complete. Then f is a covering map.

Theorem (Hadamard–Cartan theorem)**.** Let (M^n, g) be a complete Riemannian manifold such that the sectional curvature is always non-positive. Then for every point $p \in M$, the map $\exp_p : T_pM \to M$ is a covering map. In particular, if $\pi_1(M) = 0$, then M is diffeomorphic to $\mathbb{R}^n$.

Lemma. Let $\gamma(t)$ be a geodesic on (M, g) such that $K \leq 0$ along γ. Then γ has no conjugate points.

Proof. We write $\gamma(0) = p$. Let $I(t)$ be a Jacobi field along γ, and $J(0) = 0$. We claim that if J is not identically zero, then J does not vanish everywhere else.

We consider the function

$$f(t) = g(J(t), J(t)) = |J(t)|^2.$$

Then $f(0) = f'(0) = 0$. Consider

$$\frac{1}{2}f''(t) = g(J''(t), J(t)) + g(J'(t), J'(t)) = g(J', J') - R(\dot{\gamma}, J, \dot{\gamma}, J) \geq 0.$$

So f is a convex function, and so we are done. □

Proof of theorem. By the lemma, we know there are no conjugate points. So we know $\exp_p$ is regular everywhere, hence a local diffeomorphism by inverse function theorem. We can use this fact to pull back a metric from M to T_pM such that $\exp_p$ is a local isometry. Since this is a local isometry, we know geodesics are preserved. So geodesics originating from the origin in T_pM are straight lines, and the speed of the geodesics under the two metrics are the same. So we know T_pM is complete under this metric. Also, by Hopf–Rinow, $\exp_p$ is surjective. So we are done. □

4 Hodge theory on Riemannian manifolds

4.1 Hodge star and operators

Proposition. Suppose $\omega_1, \cdots, \omega_n$ is an orthonormal basis of $T_x^* M$. Then we claim that
$$\star(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_{p+1} \wedge \cdots \wedge \omega_n.$$

Proposition. The double Hodge star $\star\star : \bigwedge^p(T_x^* M) \to \bigwedge^p(T_x^* M)$ is equal to $(-1)^{p(n-p)}$.

Proposition.
$$\star \Delta = \Delta \star.$$

Proposition. δ is the formal adjoint of d. Explicitly, for any compactly supported $\alpha \in \Omega^{p-1}$ and $\beta \in \Omega^p$, then
$$\int_M \langle \mathrm{d}\alpha, \beta\rangle_g \, \omega_g = \int_M \langle \alpha, \delta\beta\rangle_g \, \omega_g.$$

Proof. We have
$$\begin{aligned}
0 &= \int_M \mathrm{d}(\alpha \wedge \star\beta)\\
&= \int_M \mathrm{d}\alpha \wedge \star\beta + \int_M (-1)^{p-1} \alpha \wedge \mathrm{d}\star\beta\\
&= \int_M \langle \mathrm{d}\alpha, \beta\rangle_g \, \omega_g + (-1)^{p-1}(-1)^{(n-p+1)(p-1)} \int_M \alpha \wedge \star\star \, \mathrm{d}\star\beta\\
&= \int_M \langle \mathrm{d}\alpha, \beta\rangle_g \, \omega_g + (-1)^{(n-p)(p-1)} \int_M \alpha \wedge \star\star \, \mathrm{d}\star\beta\\
&= \int_M \langle \mathrm{d}\alpha, \beta\rangle_g \, \omega_g - \int_M \alpha \wedge \star\delta\beta\\
&= \int_M \langle \mathrm{d}\alpha, \beta\rangle_g \, \omega_g - \int_M \langle \alpha, \delta\beta\rangle_g \, \omega_g.
\end{aligned}$$
□

Corollary. Δ is formally self-adjoint.

Corollary. Let M be compact. Then
$$\Delta\alpha = 0 \Leftrightarrow \mathrm{d}\alpha = 0 \text{ and } \delta\alpha = 0.$$

Proof. $\Leftarrow$ is clear. For $\Rightarrow$, suppose $\Delta\alpha = 0$. Then we have
$$0 = \langle\langle \alpha, \Delta\alpha\rangle\rangle = \langle\langle \alpha, \mathrm{d}\delta\alpha + \delta\mathrm{d}\alpha\rangle\rangle = \|\delta\alpha\|_g^2 + \|\mathrm{d}\alpha\|_g^2.$$
Since the L^2 norm is non-degenerate, it follows that $\delta\alpha = \mathrm{d}\alpha = 0$. □

4.2 Hodge decomposition theorem

Theorem (Hodge decomposition theorem)**.** Let (M, g) be a compact oriented Riemannian manifold. Then

- For all $p = 0, \cdots, \dim M$, we have $\dim \mathcal{H}^p < \infty$.

– We have
$$\Omega^p(M) = \mathcal{H}^p \oplus \Delta\Omega^p(M).$$
Moreover, the direct sum is orthogonal with respect to the L^2 inner product. We also formally set $\Omega^{-1}(M) = 0$.

Corollary. We have orthogonal decompositions
$$\begin{aligned}\Omega^p(M) &= \mathcal{H}^p \oplus \mathrm{d}\delta\Omega^p(M) \oplus \delta\mathrm{d}\Omega^p(M)\\ &= \mathcal{H}^p \oplus \mathrm{d}\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M).\end{aligned}$$

Proof. Now note that for an α, β, we have
$$\langle\langle \mathrm{d}\delta\alpha, \delta\mathrm{d}\beta\rangle\rangle_g = \langle\langle \mathrm{d}\mathrm{d}\delta\alpha, \mathrm{d}\beta\rangle\rangle_g = 0.$$
So
$$\mathrm{d}\delta\Omega^p(M) \oplus \delta\mathrm{d}\Omega^p(M)$$
is an orthogonal direct sum that clearly contains $\Delta\Omega^p(M)$. But each component is also orthogonal to harmonic forms, because harmonic forms are closed and co-closed. So the first decomposition follows.

To obtain the final decomposition, we simply note that
$$\mathrm{d}\Omega^{p-1}(M) = \mathrm{d}(\mathcal{H}^{p-1} \oplus \Delta\Omega^{p-1}(M)) = \mathrm{d}(\delta\mathrm{d}\Omega^{p-1}(M)) \subseteq \mathrm{d}\delta\Omega^p(M).$$
On the other hand, we certainly have the other inclusion. So the two terms are equal. The other term follows similarly. □

Corollary. Let (M, g) be a compact oriented Riemannian manifold. Then for all $\alpha \in H^p_{\mathrm{dR}}(M)$, there is a unique $\alpha \in \mathcal{H}^p$ such that $[\alpha] = a$. In other words, the obvious map
$$\mathcal{H}^p \to H^p_{\mathrm{dR}}(M)$$
is an isomorphism.

Proof. To see uniqueness, suppose $\alpha_1, \alpha_2 \in \mathcal{H}^p$ are such that $[\alpha_1] = [\alpha_2] \in H^p_{\mathrm{dR}}(M)$. Then
$$\alpha_1 - \alpha_2 = \mathrm{d}\beta$$
for some β. But the left hand side and right hand side live in different parts of the Hodge decomposition. So they must be individually zero. Alternatively, we can compute
$$\|\mathrm{d}\beta\|_g^2 = \langle\langle \mathrm{d}\beta, \alpha_1 - \alpha_2\rangle\rangle_g = \langle\langle \beta, \delta\alpha_1 - \delta\alpha_2\rangle\rangle_g = 0$$
since harmonic forms are co-closed.

To prove existence, let $\alpha \in \Omega^p(M)$ be such that $\mathrm{d}\alpha = 0$. We write
$$\alpha = \alpha_1 + \mathrm{d}\alpha_2 + \delta\alpha_3 \in \mathcal{H}^p \oplus \mathrm{d}\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M).$$
Applying d gives us
$$0 = \mathrm{d}\alpha_1 + \mathrm{d}^2\alpha_2 + \mathrm{d}\delta\alpha_3.$$
We know $\mathrm{d}\alpha_1 = 0$ since α_1 is harmonic, and $\mathrm{d}^2 = 0$. So we must have $\mathrm{d}\delta\alpha_3 = 0$. So
$$\langle\langle \delta\alpha_3, \delta\alpha_3\rangle\rangle_g = \langle\langle \alpha_3, \mathrm{d}\delta\alpha_3\rangle\rangle_g = 0.$$
So $\delta\alpha_3 = 0$. So $[\alpha] = [\alpha_1]$ and α has a representative in $\mathcal{H}^p$. □

Theorem (Compactness theorem)**.** If a sequence $\alpha_n \in \Omega^n(M)$ satisfies $\|\alpha_n\| < C$ and $\|\Delta \alpha_n\| < C$ for all n, then α_n contains a Cauchy subsequence.

Corollary. $\mathcal{H}^p$ is finite-dimensional.

Proof. Suppose not. Then by Gram–Schmidt, we can find an infinite orthonormal sequence e_n such that $\|e_n\| = 1$ and $\|\Delta e_n\| = 0$, and this certainly does not have a Cauchy subsequence. □

Theorem (Regularity theorem)**.** Every weak solution of $\Delta \omega = \alpha$ is of the form

$$\ell(\beta) = \langle \omega, \beta \rangle$$

for $\omega \in \Omega^p(M)$.

Theorem (Hahn–Banach theorem)**.** Let L be a normed vector space, and L_0 be a subspace. We let $f : L_0 \to \mathbb{R}$ be a bounded linear functional. Then f extends to a bounded linear functional $L \to \mathbb{R}$ with the same bound.

Proof of Hodge decomposition theorem. Since $\mathcal{H}^p$ is finite-dimensional, by basic linear algebra, we can decompose

$$\Omega^p(M) = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp.$$

Crucially, we know $(\mathcal{H}^p)^\perp$ is a *closed* subspace. What we want to show is that

$$(\mathcal{H}^p)^\perp = \Delta \Omega^p(M).$$

One inclusion is easy. Suppose $\alpha \in \mathcal{H}^p$ and $\beta \in \Omega^p(M)$. Then we have

$$\langle \alpha, \Delta \beta \rangle = \langle \Delta \alpha, \beta \rangle = 0.$$

So we know that

$$\Delta \Omega^p(M) \subseteq (\mathcal{H}^p)^\perp.$$

The other direction is the hard part. Suppose $\alpha \in (\mathcal{H}^p)^\perp$. We may assume α is non-zero. Since our PDE is a linear one, we may wlog $\|\alpha\| = 1$.

By the regularity theorem, it suffices to prove that $\Delta \omega = \alpha$ has a *weak* solution. We define $\ell : \Delta \Omega^p(M) \to \mathbb{R}$ as follows: for each $\eta \in \Omega^p(M)$, we put

$$\ell(\Delta \eta) = \langle \eta, \alpha \rangle.$$

We check this is well-defined. Suppose $\Delta \eta = \Delta \xi$. Then $\eta - \xi \in \mathcal{H}^p$, and we have

$$\langle \eta, \alpha \rangle - \langle \xi, \alpha \rangle = \langle \eta - \xi, \alpha \rangle = 0$$

since $\alpha \in (\mathcal{H}^p)^\perp$.

We next want to show the boundedness property. We now claim that there exists a positive $C > 0$ such that

$$\ell(\Delta \eta) \leq C \|\Delta \eta\|$$

for all $\eta \in \Omega^p(M)$. To see this, we first note that by Cauchy–Schwartz, we have

$$|\langle \alpha, \eta \rangle| \leq \|\alpha\| \cdot \|\eta\| = \|\eta\|.$$

So it suffices to show that there is a $C > 0$ such that

$$\|\eta\| \leq C\|\Delta\eta\|$$

for every $\eta \in \Omega^p(M)$.

Suppose not. Then we can find a sequence $\eta_k \in (\mathcal{H}^p)^\perp$ such that $\|\eta_k\| = 1$ and $\|\Delta\eta_k\| \to 0$.

But then $\|\Delta\eta_k\|$ is certainly bounded. So by the compactness theorem, we may wlog η_k is Cauchy. Then for any $\psi \in \Omega^p(M)$, the sequence $\langle \psi, \eta_k\rangle$ is Cauchy, by Cauchy–Schwartz, hence convergent.

We define $a : \Omega^p(M) \to \mathbb{R}$ by

$$a(\psi) = \lim_{k\to\infty} \langle \psi, \eta_k\rangle.$$

Then we have

$$a(\Delta\psi) = \lim_{k\to\infty} \langle \eta_k, \Delta\psi\rangle = \lim_{k\to\infty} \langle \Delta\eta_k, \psi\rangle = 0.$$

So we know that a is a weak solution of $\Delta\xi = 0$. By the regularity theorem again, we have

$$a(\psi) = \langle \xi, \psi\rangle$$

for some $\xi \in \Omega^p(M)$. Then $\xi \in \mathcal{H}^p$.

We claim that $\eta_k \to \xi$. Let $\varepsilon > 0$, and pick N such that $n, m > N$ implies $\|\eta_n - \eta_m\| < \varepsilon$. Then

$$\|\eta_n - \xi\|^2 = \langle \eta_n - \xi, \eta_n - \xi\rangle \leq |\langle \eta_m - \xi, \eta_n - \xi\rangle| + \varepsilon\|\eta_n - \xi\|.$$

Taking the limit as $m \to \infty$, the first term vansihes, and this tells us $\|\eta_n - \xi\| \leq \varepsilon$. So $\eta_n \to \xi$.

But this is bad. Since $\eta_k \in (\mathcal{H}^p)^\perp$, and $(\mathcal{H}^p)^\infty$ is closed, we know $\xi \in (\mathcal{H}^p)^\perp$. But also by assumption, we have $\xi \in \mathcal{H}^p$. So $\xi = 0$. But we *also* know $\|\xi\| = \lim \|\eta_k\| = 1$, whcih is a contradiction. So ℓ is bounded.

We then extend ℓ to any bounded linear map on $\Omega^p(M)$. Then we are done. □

4.3 Divergence

Proposition.

$$\operatorname{div}(fX) = \operatorname{tr}(\nabla(fX)) = f\operatorname{div}X + \langle \mathrm{d}f, X\rangle.$$

Theorem. Let $\theta \in \Omega^1(M)$, and let $X_\theta \in \operatorname{Vect}(M)$ be such that $\langle \theta, V\rangle = g(X_\theta, V)$ for all $V \in TM$. Then

$$\delta\theta = -\operatorname{div}X_\theta.$$

Lemma. In local coordinates, for any p-form ψ, we have

$$\mathrm{d}\psi = \sum_{k=1}^n \mathrm{d}x^k \wedge \nabla_k\psi.$$

Proof. We fix a point $x \in M$, and we may wlog we work in normal coordinates at x. By linearity and change of coordinates, we wlog

$$\psi = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p.$$

Now the left hand side is just

$$\mathrm{d}\psi = \sum_{k=p+1}^{n} \frac{\partial f}{\partial x^k}\ \mathrm{d}x^k \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p.$$

But this is also what the RHS is, because $\nabla_k = \partial_k$ at p. □

Lemma. We have

$$(\mathrm{div} X)\ \omega_g = \mathrm{d}(i(X)\ \omega_g),$$

for all $X \in \mathrm{Vect}(M)$.

Proof. Now by unwrapping the definition of $i(X)$, we see that

$$\nabla_Y(i(X)\psi) = i(\nabla_Y X)\psi + i(X)\nabla_Y \psi.$$

From example sheet 3, we know that $\nabla \omega_g = 0$. So it follows that

$$\nabla_Y(i(X)\ \omega_g) = i(\nabla_Y X)\ \omega_g.$$

Therefore we obtain

$$\begin{aligned} &d(i(X)\omega_g) \\ &= \sum_{k=1}^{n} \mathrm{d}x^k \wedge \nabla_k(i(X)\omega_g) \\ &= \sum_{k=1}^{n} \mathrm{d}x^k \wedge i(\nabla_k X)\omega_g \\ &= \sum_{k=1}^{n} \mathrm{d}x^k \wedge i(\nabla_k X)(\sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \\ &= \mathrm{d}x^k(\nabla_k X)\ \omega_g \\ &= (\mathrm{div} X)\ \omega_g. \end{aligned}$$

Note that this requires us to think carefully how wedge products work ($i(X)(\alpha\wedge\beta)$ is not just $\alpha(X)\beta$, or else $\alpha \wedge \beta$ would not be anti-symmetric). □

Corollary (Divergence theorem)**.** For any vector field X, we have

$$\int_M \mathrm{div}(X)\ \omega_g = \int_M \mathrm{d}(i(X)\ \omega_g) = 0.$$

Theorem. Let $\theta \in \Omega^1(M)$, and let $X_\theta \in \mathrm{Vect}(M)$ be such that $\langle \theta, V\rangle = g(X_\theta, V)$ for all $V \in TM$. Then

$$\delta\theta = -\mathrm{div} X_\theta.$$

Proof. By the formal adjoint property of δ, we know that for any $f \in C^\infty(M)$, we have

$$\int_M g(\mathrm{d}f, \theta)\ \omega_g = \int_M f\delta\theta\ \omega_g.$$

So we want to show that

$$\int_M g(\mathrm{d}f, \theta)\ \omega_g = -\int_M f\mathrm{div}X_\omega\ \omega_g.$$

But by the product rule, we have

$$\int_M \mathrm{div}(fX_\theta)\ \omega_g = \int_M g(\mathrm{d}f, \theta)\ \omega_g + \int_M f\mathrm{div}X_\theta\ \omega_g.$$

So the result follows by the divergence theorem. □

Corollary. If θ is a 1-form, and $\{e_k\}$ is a local orthonormal frame field, then

$$\delta\theta = -\sum_{k=1}^n i(e_k)\nabla_{e_i}\theta = -\sum_{k=1}^n \langle \nabla_{e_k}\theta, e_k\rangle.$$

Proof. We note that

$$\begin{aligned} e_i\langle\theta, e_i\rangle &= \langle\nabla_{e_i}\theta, e_i\rangle + \langle\theta, \nabla_{e_i}e_i\rangle \\ e_i g(X_\theta, e_i) &= g(\nabla_{e_i}X_\theta, e_i) + g(X_\theta, \nabla_{e_i}e_i). \end{aligned}$$

By definition of X_θ, this implies that

$$\langle\nabla_{e_i}\theta, e_i\rangle = g(\nabla_{e_i}X_\theta, e_i).$$

So we obtain

$$\delta\theta = -\mathrm{div}X_\theta = -\sum_{i=1}^n g(\nabla_{e_i}X_\theta, e_i) = -\sum_{k=1}^n \langle\nabla_{e_i}\theta, e_i\rangle,$$ □

Proposition. If $\beta \in \Omega^2(M)$, then

$$(\delta\beta)(Y) = -\sum_{k=1}^n (\nabla_{e_k}\beta)(e_k, Y).$$

In other words,

$$\delta\beta = -\sum_{k=1}^n i(e_k)(\nabla_{e_k}\beta).$$

4.4 Introduction to Bochner's method

Theorem (Bochner–Weitzenböck formula)**.** On an oriented Riemannian manifold, we have

$$\Delta = \nabla^*\nabla + \mathrm{Ric}\,.$$

Corollary. Let (M, g) be a compact connected oriented manifold. Then

– If $\mathrm{Ric}(g) > 0$ at each point, then $H^1_{\mathrm{dR}}(M) = 0$.

– If $\mathrm{Ric}(g) \geq 0$ at each point, then $b^1(M) = \dim H^1_{\mathrm{dR}}(M) \leq n$.

– If $\mathrm{Ric}(g) \geq 0$ at each point, and $b^1(M) = n$, then g is flat.

Proof. By Bochner–Weitzenböck, we have

$$\begin{aligned}\langle\langle \Delta\alpha, \alpha\rangle\rangle &= \langle\langle \nabla^*\nabla\alpha, \alpha\rangle\rangle + \int_M \mathrm{Ric}(\alpha,\alpha)\,\omega_g \\ &= \|\nabla\alpha\|_2^2 + \int_M \mathrm{Ric}(\alpha,\alpha)\,\omega_g.\end{aligned}$$

– Suppose $\mathrm{Ric} > 0$. If $\alpha \neq 0$, then the RHS is strictly positive. So the left-hand side is non-zero. So $\Delta\alpha \neq 0$. So $\mathcal{H}^1_M \cong H^1_{\mathrm{dR}}(M) = 0$.

– Suppose α is such that $\Delta\alpha = 0$. Then the above formula forces $\nabla\alpha = 0$. So if we know $\alpha(x)$ for some fixed $x \in M$, then we know the value of α everywhere by parallel transport. Thus α is determined by the initial condition $\alpha(x)$, Thus there are $\leq n = \dim T^*_x M$ linearly independent such α.

– If $b^1(M) = n$, then we can pick a basis $\alpha_1, \cdots, \alpha_n$ of $\mathcal{H}^1_M$. Then as above, these are parallel 1-forms. Then we can pick a dual basis $X_1, \cdots, X_n \in \mathrm{Vect}(M)$. We claim they are also parallel, i.e. $\nabla X_i = 0$. To prove this, we note that

$$\langle \alpha_j, \nabla X_i\rangle + \langle \nabla\alpha_j, X_i\rangle = \nabla\langle\alpha_j, X_i\rangle.$$

But $\langle\alpha_j, X_i\rangle$ is constantly 0 or 1 depending on i and j, So the RHS vanishes. Similarly, the second term on the left vanishes. Since the α_j span, we know we must have $\nabla X_i = 0$.

Now we have

$$R(X_i, X_j)X_k = (\nabla_{[X_i,X_j]} - [\nabla X_i, \nabla_{X_j}])X_k = 0,$$

Since this is valid for all i, j, k, we know R vanishes at each point. So we are done. □

Proposition. In the case of (iii), M is in fact isometric to a flat torus.

Proof sketch. We fix $p \in M$ and consider the map $M \to \mathbb{R}^n$ given by

$$x \mapsto \left(\int_p^x \alpha_i\right)_{i=1,\cdots,n} \in \mathbb{R}^n,$$

where the α_i are as in the previous proof. The integral is taken along any path from p to x, and this is not well-defined. But by Stokes' theorem, and the fact that $\mathrm{d}\alpha_i = 0$, this only depends on the homotopy class of the path.

In fact, $\int_p^x$ depends only on $\gamma \in H_1(M)$, which is finitely generated. Thus, $\int_p^x \alpha_i$ is a well-defined map to $S^1 = \mathbb{R}/\lambda_i\mathbb{Z}$ for some $\lambda_i \neq 0$. Therefore we obtain a map $M \to (S^1)^n = T^n$. Moreover, a bit of inspection shows this is a local diffeomorphism. But since the spaces involved are compact, it follows by some topology arguments that it must be a covering map. But again by compactness, this is a finite covering map. So M must be a torus. So we are done. □

Proposition. Let $e_1, \cdots, e_n$ be an orthonormal frame field, and $\beta \in \Omega^1(T^*M)$. Then we have

$$\nabla^* \beta = -\sum_{i=1}^n i(e_i) \nabla_{e_i} \beta.$$

Proof. Let $\alpha \in \Omega^0(T^*M)$. Then by definition, we have

$$\langle \nabla \alpha, \beta \rangle = \sum_{i=1}^n \langle \nabla_{e_i} \alpha, \beta(e_i) \rangle.$$

Consider the 1-form given by

$$\theta(Y) = \langle \alpha, \beta(Y) \rangle.$$

Then we have

$$\begin{aligned}
\operatorname{div} X_\theta &= \sum_{i=1}^n \langle \nabla_{e_i} X_\theta, e_i \rangle \\
&= \sum_{i=1}^n \nabla_{e_i} \langle X_\theta, e_i \rangle - \langle X_\theta, \nabla_{e_i} e_i \rangle \\
&= \sum_{i=1}^n \nabla_{e_i} \langle \alpha, \beta(e_i) \rangle - \langle \alpha, \beta(\nabla_{e_i} e_i) \rangle \\
&= \sum_{i=1}^n \langle \nabla_{e_i} \alpha, \beta(e_i) \rangle + \langle \alpha, \nabla_{e_i} (\beta(e_i)) \rangle - \langle \alpha, \beta(\nabla_{e_i} e_i) \rangle \\
&= \sum_{i=1}^n \langle \nabla_{e_i} \alpha, \beta(e_i) \rangle + \langle \alpha, (\nabla_{e_i} \beta)(e_i) \rangle.
\end{aligned}$$

So by the divergence theorem, we have

$$\int_M \langle \nabla \alpha, \beta \rangle \; \omega_g = \int_M \sum_{i=1}^n \langle \alpha, (\nabla_{e_i} \beta)(e_i) \rangle \; \omega_g.$$

So the result follows. □

Corollary. For a local orthonormal frame field $e_1, \cdots, e_n$, we have

$$\nabla^* \nabla \alpha = -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \alpha.$$

Lemma. Let $\alpha \in \Omega^1(M)$, $X \in \operatorname{Vect}(M)$. Then

$$\langle \mathrm{d}\delta \alpha, X \rangle = -\sum_{i=1}^n \langle \nabla_X \nabla_{e_i} \alpha, e_i \rangle.$$

Proof.

$$\begin{aligned}
\langle \mathrm{d}\delta \alpha, X \rangle &= X(\delta \alpha) \\
&= -\sum_{i=1}^n X \langle \nabla_{e_i} \alpha, e_i \rangle \\
&= -\sum_{i=1}^n \langle \nabla_X \nabla_{e_i} \alpha, e_i \rangle.
\end{aligned}$$

□

Lemma. For any 2-form β, we have

$$(\delta\beta)(X) = \sum_{k=1}^n -e_k(\beta(e_k, X)) + \beta(e_k, \nabla_{e_k} X).$$

Proof.

$$\begin{aligned}(\delta\beta)(X) &= -\sum_{k=1}^n (\nabla_{e_k}\beta)(e_k, X)\\ &= \sum_{k=1}^n -e_k(\beta(e_k, X)) + \beta(\nabla_{e_k} e_k, X) + \beta(e_k, \nabla_{e_k} X)\\ &= \sum_{k=1}^n -e_k(\beta(e_k, X)) + \beta(e_k, \nabla_{e_k} X).\end{aligned}$$

□

Lemma. For any 1-form α and vector fields X, Y, we have

$$\mathrm{d}\alpha(X, Y) = \langle \nabla_X \alpha, Y\rangle - \langle \nabla_Y \alpha, X\rangle.$$

Proof. Since the connection is torsion-free, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

So we obtain

$$\begin{aligned}\mathrm{d}\alpha(X, Y) &= X\langle \alpha, Y\rangle - Y\langle \alpha, X\rangle - \langle \alpha, [X, Y]\rangle\\ &= \langle \nabla_X \alpha, Y\rangle - \langle \nabla_Y \alpha, X\rangle.\end{aligned}$$

□

Lemma. For any 1-form α and vector field X, we have

$$\langle \delta\mathrm{d}\alpha, X\rangle = -\sum_{k=1}^n \langle \nabla_{e_k}\nabla_{e_k}\alpha, X\rangle + \sum_{k=1}^n \langle \nabla_{e_k}\nabla_X \alpha, e_k\rangle - \sum_{k=1}^n \langle \nabla_{\nabla_{e_k} X}\alpha, e_k\rangle.$$

Proof.

$$\begin{aligned}\langle \delta\mathrm{d}\alpha, X\rangle &= \sum_{k=1}^n \Big[- e_k(\mathrm{d}\alpha(e_k, X)) + \mathrm{d}\alpha(e_k, \nabla_{e_k} X)\Big]\\ &= \sum_{k=1}^n \Big[- e_k(\langle \nabla_{e_k}\alpha, X\rangle - \langle \nabla_X \alpha, e_k\rangle)\\ &\qquad\qquad + \langle \nabla_{e_k}\alpha, \nabla_{e_k} X\rangle - \langle \nabla_{\nabla_{e_k} X}\alpha, e_k\rangle\Big]\\ &= \sum_{k=1}^n \Big[- \langle \nabla_{e_k}\nabla_{e_k}\alpha, X\rangle - \langle \nabla_{e_k}\alpha, \nabla_{e_k} X\rangle + \langle \nabla_{e_k}\nabla_X \alpha, e_k\rangle)\\ &\qquad\qquad + \langle \nabla_{e_k}\alpha, \nabla_{e_k} X\rangle - \langle \nabla_{\nabla_{e_k} X}\alpha, e_k\rangle\Big]\\ &= -\sum_{k=1}^n \langle \nabla_{e_k}\nabla_{e_k}\alpha, X\rangle + \sum_{k=1}^n \langle \nabla_{e_k}\nabla_X \alpha, e_k\rangle - \sum_{k=1}^n \langle \nabla_{\nabla_{e_k} X}\alpha, e_k\rangle.\end{aligned}$$

□

Lemma (Ricci identity)**.** Let M be any Riemannian manifold, and $X, Y, Z \in \mathrm{Vect}(M)$ and $\alpha \in \Omega^1(M)$. Then

$$\langle ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})\alpha, Z\rangle = \langle \alpha, R(X, Y)Z\rangle.$$

Proof. We note that

$$\langle \nabla_{[X,Y]}\alpha, Z\rangle + \langle \alpha, \nabla_{[X,Y]}Z\rangle = [X, Y]\langle \alpha, Z\rangle = \langle [\nabla_X, \nabla_Y]\alpha, Z\rangle + \langle \alpha, [\nabla_X, \nabla_Y]Z\rangle.$$

The second equality follows from writing $[X, Y] = XY - YX$. We then rearrange and use that $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. □

Corollary. For any 1-form α and vector field X, we have

$$\langle \Delta\alpha, X\rangle = \langle \nabla^*\nabla\alpha, X\rangle + \mathrm{Ric}(\alpha)(X).$$

Proof. We have found that

$$\langle \Delta\alpha, X\rangle = \langle \nabla^*\nabla\alpha, X\rangle + \sum_{i=1}^n \langle \alpha, R(e_k, X)e_k\rangle.$$

We have

$$\sum_{i=1}^n \langle \alpha, R(e_k, X)e_k\rangle = \sum_{i=1}^n g(X_\alpha, R(e_k, X)e_k) = \mathrm{Ric}(X_\alpha, X) = \mathrm{Ric}(\alpha)(X).$$

So we are done. □

5 Riemannian holonomy groups

Proposition. If M is simply connected, then $\mathrm{Hol}_x(M)$ is path connected.

Proof. $\mathrm{Hol}_x(M)$ is the image of $\Omega(x, x)$ in $\mathrm{O}(n)$ under the map P, and this map is continuous from the standard theory of ODE's. Simply connected means $\Omega(x, x)$ is path connected. So $\mathrm{Hol}_x(M)$ is path connected. □

Corollary. $\mathrm{Hol}^0(M) \subseteq \mathrm{SO}(n)$.

Proof. $\mathrm{Hol}^0(M)$ is connected, and thus lies in the connected component of the identity in $\mathrm{O}(n)$. □

Proposition (Fundamental principle of Riemannian holonomy)**.** Let (M, g) be a Riemannian manifold, and fix $p, q \in \mathbb{Z}_+$ and $x \in M$. Then the following are equivalent:

(i) There exists a (p, q)-tensor field α on M such that $\nabla\alpha = 0$.

(ii) There exists an element $\alpha_0 \in (T_x M)^{\otimes p} \otimes (T_x^* M)^{\otimes q}$ such that α_0 is invariant under the action of $\mathrm{Hol}_x(M)$.

Proof. To simplify notation, we consider only the case $p = 0$. The general case works exactly the same way, with worse notation. For $\alpha \in (T_x^* M)^q$, we have

$$(\nabla_X \alpha)(X_1, \cdots, X_q) = X(\alpha(X_1, \cdots, X_q)) - \sum_{i=1}^{q} \alpha(X_1, \cdots, \nabla_X X_i, \cdots, X_q).$$

Now consider a loop $\gamma : [0, 1] \to M$ be a loop at x. We choose vector fields X_i along γ for $i = 1, \cdots, q$ such that

$$\frac{\nabla X_i}{\mathrm{d}t} = 0.$$

We write

$$X_i(\gamma(0)) = X_i^0.$$

Now if $\nabla\alpha = 0$, then this tells us

$$\frac{\nabla\alpha}{\mathrm{d}t}(X_1, \cdots, X_q) = 0.$$

By our choice of X_i, we know that $\alpha(X_1, \cdots, X_q)$ is constant along γ. So we know

$$\alpha(X_1^0, \cdots, X_q^0) = \alpha(P_\gamma(X_1^0), \cdots, P_\gamma(X_q^0)).$$

So α is invariant under $\mathrm{Hol}_x(M)$. Then we can take $\alpha_0 = \alpha_x$.

Conversely, if we have such an α_0, then we can use parallel transport to transfer it to everywhere in the manifold. Given any $y \in M$, we define α_y by

$$\alpha_y(X_1, \cdots, X_q) = \alpha_0(P_\gamma(X_1), \cdots, P_\gamma(X_q)),$$

where γ is any path from y to x. This does not depend on the choice of γ precisely because α_0 is invariant under $\mathrm{Hol}_x(M)$.

It remains to check that α is C^∞ with $\nabla\alpha = 0$, which is an easy exercise. □

Theorem. Let (M, g) be a connected and oriented Riemannian manifold, and consider the decomposition of the bundle of k-forms into irreducible representations of the holonomy group,

$$\bigwedge^k T^* M = \bigoplus_i \Lambda_i^k.$$

In other words, each fiber $(\Lambda_i^k)_x \subseteq \bigwedge^k T_x^* M$ is an irreducible representation of $\mathrm{Hol}_x(g)$. Then

(i) For all $\alpha \in \Omega_i^k(M) \equiv \Gamma(\Lambda_i^l)$, we have $\Delta\alpha \in \Omega_i^k(M)$.

(ii) If M is compact, then we have a decomposition

$$H_{\mathrm{dR}}^k(M) = \bigoplus H_{i,\mathrm{dR}}^k(M),$$

where

$$H_{i,\mathrm{dR}}^k(M) = \{[\alpha] : \alpha \in \Omega_i^k(M), \Delta\alpha = 0\}.$$

The dimensions of these groups are known as the *refined Betti numbers*.

6 The Cheeger–Gromoll splitting theorem

Lemma. If M is disconnected at infinity, then M contains a line.

Proof. Note that M is unbounded. Since M is disconnected at infinity, we can find a compact subset $K \subseteq M$ and sequences $p_m, q_m \to \infty$ as $m \to \infty$ (to make this precise, we can pick some fixed point x, and then require $d(x, p_m), d(x, q_m) \to \infty$) such that every $\gamma_m \in \Omega(p_m, q_m)$ passes through K.

In particular, we pick γ_m to be a minimal geodesic from p_m to q_m parametrized by arc-length. Then γ_m passes through K. By reparametrization, we may assume $\gamma_m(0) \in K$.

Since K is compact, we can pass to a subsequence, and wlog $\gamma_m(0) \to x \in K$ and $\dot{\gamma}_m(0) \to a \in T_x M$ (suitably interpreted).

Then we claim the geodesic $\gamma_{x,a}(t)$ is the desired line. To see this, since solutions to ODE's depend smoothly on initial conditions, we can write the line as

$$\ell(t) = \lim_{m \to \infty} \gamma_m(t).$$

Then we know

$$d(\ell(s), \ell(t)) = \lim_{m \to \infty} d(\gamma_m(s), \gamma_m(t)) = |s - t|.$$

So we are done. □

Theorem (Cheeger–Gromoll line-splitting theorem (1971))**.** If (M, g) is a complete connected Riemannian manifold containing a line, and has $\mathrm{Ric}(g) \geq 0$ at each point, then M is isometric to a Riemannian product $(N \times \mathbb{R}, g_0 + \mathrm{d}t^2)$ for some metric g_0 on N.

Corollary. Let (M, g) be a complete connected Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. Then it is isometric to $X \times \mathbb{R}^q$ for some $q \in \mathbb{N}$ and Riemannian manifold X, where X is complete and does not contain any lines.

Lemma. Let (M, g) be a compact Riemannian manifold, and suppose its universal Riemannian cover $(\tilde{M}, \tilde{g})$ is non-compact. Then $(\tilde{M}, \tilde{g})$ contains a line.

Proof. We first find a compact $K \subseteq \tilde{M}$ such that $\pi(K) = M$. Since $\tilde{M}$ must be complete, it is unbounded. Choose p_n, q_n, γ_n like before. Then we can apply deck transformations so that the midpoint lies inside K, and then use compactness of K to find a subsequence so that the midpoint converges. □

Corollary. Let (M, g) be a compact, connected manifold with $\mathrm{Ric}(g) \geq 0$. Then

- The universal Riemannian cover is isometric to the Riemannian product $X \times \mathbb{R}^N$, with X compact, $\pi_1(X) = 1$ and $\mathrm{Ric}(g_X) \geq 0$.
- If there is some $p \in M$ such that $\mathrm{Ric}(g)_p > 0$, then $\pi_1(M)$ is finite.
- Denote by $I(\tilde{M})$ the group of isometries $\tilde{M} \to \tilde{M}$. Then $I(\tilde{M}) = I(X) \times E(\mathbb{R}^q)$, where $E(\mathbb{R}^q)$ is the group of rigid Euclidean motions,

  $$\mathbf{y} \mapsto A\mathbf{y} + \mathbf{b}$$

  where $\mathbf{b} \in \mathbb{R}^n$ and $A \in \mathrm{O}(q)$.

– If $\tilde{M}$ is homogeneous, then so is X.

Proof.

– This is direct from Cheeger–Gromoll and the previous lemma.

– If there is a point with strictly positive Ricci curvature, then the same is true for the universal cover. So we cannot have any non-trivial splitting. So by the previous part, $\tilde{M}$ must be compact. By standard topology, $|\pi_1(M)| = |\pi^{-1}(\{p\})|$.

– We use the fact that $E(\mathbb{R}^q) = I(\mathbb{R}^q)$. Pick a $g \in I(\tilde{M})$. Then we know g takes lines to lines. Now use that all lines in $\tilde{M} \times \mathbb{R}^q$ are of the form $p \times \mathbb{R}$ with $p \in X$ and $\mathbb{R} \subseteq \mathbb{R}^q$ an affine line. Then

$$g(p \times \mathbb{R}) = p' \times \mathbb{R},$$

for some p' and possibly for some other copy of $\mathbb{R}$. By taking unions, we deduce that $g(p \times \mathbb{R}^q) = p' \times \mathbb{R}^q$. We write $h(p) = p'$. Then $h \in I(X)$.

Now for any $X \times \mathbf{a}$ with $\mathbf{a} \in \mathbb{R}^q$, we have $X \times \mathbf{a} \perp p \times \mathbb{R}^q$ for all $p \in X$. So we must have

$$g(X \times \mathbf{a}) = X \times \mathbf{b}$$

for some $\mathbf{b} \in \mathbb{R}^q$. We write $e(\mathbf{a}) = \mathbf{b}$. Then

$$g(p, a) = (h(p), e(a)).$$

Since the metric of X and $\mathbb{R}^q$ are decoupled, it follows that h and e must separately be isometries. □

Proposition. Consider $S^n \times \mathbb{R}$ for $n = 2$ or 3. Then this does not admit any Ricci-flat metric.

Proof. Note that $S^n \times \mathbb{R}$ is disconnected at ∞. So any metric contains a line. Then by Cheeger–Gromoll, $\mathbb{R}$ splits as a Riemannian factor. So we obtain $\mathrm{Ric} = 0$ on the S^n factor. Since we are in $n = 2, 3$, we know S^n is flat, as the Ricci curvature determines the full curvature. So S^n is the quotient of $\mathbb{R}^n$ by a discrete group, and in particular $\pi_1(S^n) \neq 1$. This is a contradiction. □