Part III — Riemannian Geometry
Theorems

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This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:

A closer look at geodesics and curvature. Brief review from the Differential Geometry course. Geodesic coordinates and Gauss' lemma. Jacobi fields, completeness and the Hopf–Rinow theorem. Variations of energy, Bonnet–Myers diameter theorem and Synge's theorem.

Hodge theory and Riemannian holonomy. The Hodge star and Laplace–Beltrami operator. The Hodge decomposition theorem (with the 'geometry part' of the proof). Bochner–Weitzenböck formulae. Holonomy groups. Interplays with curvature and de Rham cohomology.

Ricci curvature. Fundamental groups and Ricci curvature. The Cheeger–Gromoll splitting theorem.

Pre-requisites

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course Differential Geometry offered in Michaelmas Term is the ideal pre-requisite.

Contents

1 Basics of Riemannian manifolds

Theorem (Whitney embedding theorem)**.** Every smooth manifold M admits an embedding into $\mathbb{R}^k$ for some k. In other words, M is diffeomorphic to a submanifold of $\mathbb{R}^k$. In fact, we can pick k such that $k \leq 2 \dim M$.

Lemma. Let (N, h) be a Riemannian manifold, and $F : M \to N$ is an immersion, then the pullback $g = F^* h$ defines a metric on M.

2 Riemann curvature

Proposition.

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Proposition.

(i)
$$R_{ij,k\ell} = -R_{ij,\ell k} = -R_{ji,k\ell}.$$

(ii) The *first Bianchi identity*:
$$R^i_{j,k\ell} + R^i_{k,\ell j} + R^i_{\ell,jk} = 0.$$

(iii)
$$R_{ij,k\ell} = R_{k\ell,ij}.$$

Lemma. Let V be a real vector space of dimension ≥ 2. Suppose $R', R'' : V^{\otimes 4} \to \mathbb{R}$ are both linear in each factor, and satisfies the symmetries we found for the Riemann curvature tensor. We define $K', K'' : \mathrm{Gr}(2, V) \to \mathbb{R}$ as in the sectional curvature. If $K' = K''$, then $R' = R''$.

Corollary. Let (M, g) be a manifold such that for all p, the function $K_p : \mathrm{Gr}(2, T_pM) \to \mathbb{R}$ is a constant map. Let

$$R^0_p(X, Y, Z, T) = g_p(X, Z)g_p(Y, T) - g_p(X, T)g_p(Y, Z).$$

Then
$$R_p = K_p R^0_p.$$

Here K_p is just a real number, since it is constant. Moreover, K_p is a smooth function of p.

Equivalently, in local coordinates, if the metric at a point is δ_{ij}, then we have

$$R_{ij,ij} = -R_{ij,ji} = K_p,$$

and all other entries all zero.

3 Geodesics

3.1 Definitions and basic properties

Proposition. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve. Then there is a uniquely determined operation $\frac{\nabla}{\mathrm{d}t}$ from the space of all lifts of γ to itself, satisfying the following conditions:

(i) For any $c, d \in \mathbb{R}$ and lifts $\tilde{\gamma}^E, \gamma^E$ of γ, we have.

$$\frac{\nabla}{\mathrm{d}t}(c\gamma^E + d\tilde{\gamma}^E) = c\frac{\nabla\gamma^E}{\mathrm{d}t} + d\frac{\nabla\tilde{\gamma}^E}{\mathrm{d}t}$$

(ii) For any lift γ^E of γ and function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$, we have

$$\frac{\nabla}{\mathrm{d}t}(f\gamma^E) = \frac{\mathrm{d}f}{\mathrm{d}t} + f\frac{\nabla\gamma^E}{\mathrm{d}t}.$$

(iii) If there is a local section s of E and a local vector field V on M such that

$$\gamma^E(t) = s(\gamma(t)), \quad \dot{\gamma}(t) = V(\gamma(t)),$$

then we have

$$\frac{\nabla\gamma^E}{\mathrm{d}t} = (\nabla_V s) \circ \gamma.$$

Locally, this is given by

$$\left(\frac{\nabla\gamma^E}{\mathrm{d}t}\right)^i = \dot{a}^i + \Gamma^i_{jk} a^j \dot{x}^k.$$

Proposition. If γ is a geodesic, then $|\dot{\gamma}(t)|_g$ is constant.

Lemma. Let $p \in M$, and $a \in T_pM$. As before, let $\gamma_p(t, a)$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = p$. Then

$$\gamma_p(\lambda t, a) = \gamma_p(t, \lambda a),$$

and in particular is a geodesic.

Proposition. We have

$$(\mathrm{d}\exp_p)_0 = \mathrm{id}_{T_pM},$$

where we identify $T_0(T_pM) \cong T_pM$ in the natural way.

Corollary. $\exp_p$ maps an open ball $B(0, \delta) \subseteq T_pM$ to $U \subseteq M$ diffeomorphically for some $\delta > 0$.

Corollary. For any point $p \in M$, there exists a local coordinate chart around p such that

- The coordinates of p are $(0, \cdots, 0)$.
- In local coordinates, the metric at p is $g_{ij}(p) = \delta_{ij}$.
- We have $\Gamma^i_{jk}(p) = 0$.

Theorem (Gauss' lemma)**.** The geodesic spheres are perpendicular to their radii. More precisely, $\gamma_p(t, a)$ meets every Σ_r orthogonally, whenever this makes sense. Thus we can write the metric in geodesic polars as

$$g = \mathrm{d}r^2 + h(r, \mathbf{v}),$$

where for each r, we have

$$h(r, \mathbf{v}) = g|_{\Sigma_r}.$$

In matrix form, we have

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & h & \\ 0 & & & \end{pmatrix}$$

Corollary. Let $a, w \in T_pM$. Then

$$g((\mathrm{d}\exp_p)_a a, (\mathrm{d}\exp_p)_a w) = g(a, w)$$

whenever a lives in the domain of the geodesic local neighbourhood.

3.2 Jacobi fields

Theorem. Let $\gamma : [0, L] \to N$ be a geodesic in a Riemannian manifold (M, g). Then

(i) For any $u, v \in T_{\gamma(0)}M$, there is a unique Jacobi field J along Γ with

$$J(0) = u, \quad \frac{\nabla J}{\mathrm{d}t}(0) = v.$$

If

$$J(0) = 0, \quad \frac{\nabla J}{\mathrm{d}t}(0) = k\dot{\gamma}(0),$$

then $J(t) = kt\dot{\gamma}(t)$. Moreover, if both $J(0), \frac{\nabla J}{\mathrm{d}t}(0)$ are orthogonal to $\dot{\gamma}(0)$, then $J(t)$ is perpendicular to $\dot{\gamma}(t)$ for all $[0, L]$.

In particular, the vector space of all Jacobi fields along γ have dimension $2n$, where $n = \dim M$.

The subspace of those Jacobi fields pointwise perpendicular to $\dot{\gamma}(t)$ has dimensional $2(n - 1)$.

(ii) $J(t)$ is independent of the parametrization of $\dot{\gamma}(t)$. Explicitly, if $\tilde{\gamma}(t) = \tilde{\gamma}(\lambda t)$, then $\tilde{J}$ with the same initial conditions as J is given by

$$\tilde{J}(\tilde{\gamma}(t)) = J(\gamma(\lambda t)).$$

Proposition. Let $\gamma : [a, b] \to M$ be a geodesic, and $f(t, s)$ a variation of $\gamma(t) = f(t, 0)$ such that $f(t, s) = \gamma_s(t)$ is a geodesic for all $|s|$ small. Then

$$J(t) = \frac{\partial f}{\partial s}$$

is a Jacobi field along $\dot{\gamma}$.

Conversely, every Jacobi field along γ can be obtained this way for an appropriate function f.

Corollary. Every Jacobi field J along a geodesic γ with $J(0) = 0$ is given by

$$J(t) = (\mathrm{d}\exp_p)_{t\dot\gamma(0)}(tJ'(0))$$

for all $t \in [0, L]$.

3.3 Further properties of geodesics

Lemma (Gauss' lemma)**.** Let $a, w \in T_pM$, and

$$\gamma = \gamma_p(t, a) = \exp_p(ta)$$

a geodesic. Then

$$g_{\gamma(t)}((\mathrm{d}\exp_p)_{ta}a, (\mathrm{d}\exp_p)_{ta}w) = g_{\gamma(0)}(a, w).$$

In particular, γ is orthogonal to $\exp_p\{v \in T_pM : |v| = r\}$. Note that the latter need not be a submanifold.

Corollary (Local minimizing of length)**.** Let $a \in T_pM$. We define $\varphi(t) = ta$, and $\psi(t)$ a piecewise C^1 curve in T_pM for $t \in [0, 1]$ such that

$$\psi(0) = 0, \quad \psi(1) = a.$$

Then

$$\mathrm{length}(\exp_p \circ \psi) \geq \mathrm{length}(\exp_p \circ \varphi) = |a|.$$

Theorem. Let $p \in M$, and let ε be such that $\exp_p|_{B(0,\varepsilon)}$ is a diffeomorphism onto its image, and let U be the image. Then

- For any $q \in U$, there is a unique geodesic $\gamma \in \Omega(p, q)$ with $\ell(\gamma) < \varepsilon$. Moreover, $\ell(\gamma) = d(p, q)$, and is the unique curve that satisfies this property.
- For any point $q \in M$ with $d(p, q) < \varepsilon$, we have $q \in U$.
- If $q \in M$ is any point, $\gamma \in \Omega(p, q)$ has $\ell(\gamma) = d(p, q) < \varepsilon$, then γ is a geodesic.

Corollary. The distance d on a Riemannian manifold is a metric, and induces the same topology on M as the C^∞ structure.

Corollary. Let $\gamma : [0, 1] \to M$ be a piecewise C^1 minimal geodesic with constant speed. Then γ is in fact a geodesic, and is in particular C^∞.

Corollary. Let $\gamma : [0, 1] \subseteq \mathbb{R} \to M$ be a C^2 curve with $|\dot\gamma|$ constant. Then this is a geodesic iff it is locally a minimal geodesic, i.e. for any $t \in [0, 1)$, there exists $\delta > 0$ such that

$$d(\gamma(t), \gamma(t + \delta)) = \ell(\gamma|_{[t,t+\delta]}).$$

Theorem. Let $\gamma(t) = \exp_p(ta)$ be a geodesic, for $t \in [0, 1]$. Let $q = \gamma(1)$. Assume ta is a regular point for $\exp_p$ for all $t \in [0, 1]$. Then there exists a neighbourhood of γ in $\Omega(p, q)$ such that for all ψ in this neighbourhood, $\ell(\psi) \geq \ell(\gamma)$, with equality iff $\psi = \gamma$ up to reparametrization.

3.4 Completeness and the Hopf–Rinow theorem

Theorem. Let (M, g) be geodesically complete. Then any two points can be connected by a minimal geodesic.

Lemma. Let $p, q \in M$. Let

$$S_\delta = \{x \in M : d(x, p) = \delta\}.$$

Then for all sufficiently small δ, there exists $p_0 \in S_\delta$ such that

$$d(p, p_0) + d(p_0, q) = d(p, q).$$

Corollary (Hopf–Rinow theorem)**.** For a connected Riemannian manifold (M, g), the following are equivalent:

(i) (M, g) is geodesically complete.

(ii) For all $p \in M$, $\exp_p$ is defined on all T_pM.

(iii) For some $p \in M$, $\exp_p$ is defined on all T_pM.

(iv) Every closed and bounded subset of (M, d) is compact.

(v) (M, d) is complete as a metric space.

3.5 Variations of arc length and energy

Proposition. Let $\gamma_0 : [0, T] \to M$ be a path from p to q such that for all $\gamma \in \Omega(p, q)$ with $\gamma : [0, T] \to M$, we have $E(\gamma) \geq E(\gamma_0)$. Then γ_0 must be a geodesic.

Theorem (First variation formula)**.**

(i) For any variation H of γ, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s=0} = g(Y(t), \dot{\gamma}(t))|_0^T - \int_0^T g\left(Y(t), \frac{\nabla}{\mathrm{d}t}\dot{\gamma}(t)\right)\,\mathrm{d}t. \qquad (*)$$

(ii) The critical points, i.e. the γ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s=0}$$

for all (end-point fixing) variation H of γ, are geodesics.

(iii) If $|\dot{\gamma}_s(t)|$ is constant for each fixed $s \in (-\varepsilon, \varepsilon)$, and $|\dot{\gamma}(t)| \equiv 1$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} E(\gamma_s)\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s} \ell(\gamma_s)\right|_{s=0}$$

(iv) If γ is a critical point of the length, then it must be a reparametrization of a geodesic.

Theorem (Second variation formula)**.** Let $\gamma(t) : [0, T] \to M$ be a geodesic with $|\dot{\gamma}| = 1$. Let $H(t, s)$ be a variation of γ. Let

$$Y(t, s) = \frac{\partial H}{\partial s}(t, s) = (\mathrm{d}H)_{(t,s)} \frac{\partial}{\partial s}.$$

Then

(i) We have

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2} E(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}(t, 0), \dot{\gamma}\right)\Big|_0^T + \int_0^T (|Y'|^2 - R(Y, \dot{\gamma}, Y, \dot{\gamma}))\;\mathrm{d}t.$$

(ii) Also

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2} \ell(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y}{\mathrm{d}s}(t, 0), \dot{\gamma}(t)\right)\Big|_0^T + \int_0^T \left(|Y'|^2 - R(Y, \dot{\gamma}, Y, \dot{\gamma}) - g(\dot{\gamma}, Y')^2\right)\;\mathrm{d}t,$$

where R is the $(4, 0)$ curvature tensor, and

$$Y'(t) = \frac{\nabla Y}{\mathrm{d}t}(t, 0).$$

Putting

$$Y_n = Y - g(Y, \dot{\gamma})\dot{\gamma}$$

for the normal component of Y, we can write this as

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2} \ell(\gamma_s)\right|_{s=0} = g\left(\frac{\nabla Y_n}{\mathrm{d}s}(t, 0), \dot{\gamma}(t)\right)\Big|_0^T + \int_0^T \left(|Y_n'|^2 - R(Y_n, \dot{\gamma}, Y_n, \dot{\gamma})\right)\;\mathrm{d}t.$$

3.6 Applications

Theorem (Synge's theorem)**.** Every compact orientable Riemannian manifold (M, g) such that $\dim M$ is even and has $K(g) > 0$ for all planes at $p \in M$ is simply connected.

Lemma. Let M be a compact manifold, and $[\alpha]$ a non-trivial homotopy class of closed curves in M. Then there is a closed minimal geodesic in $[\alpha]$.

Proposition.

(i) If $\gamma(t) = \exp_p(ta)$, and $q = \exp_p(\beta a)$ is conjugate to p, then q is a singular value of exp.

(ii) Let J be as in the definition. Then J must be pointwise normal to $\dot{\gamma}$.

Theorem. Let $\gamma : [0, 1] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma(1) = q$ such that p is conjugate to some $\gamma(t_0)$ for some $t_0 \in (0, 1)$. Then there is a piecewise smooth variation of $f(t, s)$ with $f(t, 0) = \gamma(t)$ such that

$$f(0, s) = p, \quad f(1, s) = q$$

and $\ell(f(\,\cdot\,, s)) < \ell(\gamma)$ whenever $s \neq 0$ is small.

Proposition. Let (M, g) and (N, h) be Riemannian manifolds, and suppose M is complete. Suppose there is a smooth surjection $f : M \to N$ that is a local diffeomorphism. Moreover, suppose that for any $p \in M$ and $v \in T_pM$, we have $|\mathrm{d}f_p(v)|_h \geq |v|$. Then f is a covering map.

Corollary. Let $f : M \to N$ be a local isometry onto N, and M be complete. Then f is a covering map.

Theorem (Hadamard–Cartan theorem)**.** Let (M^n, g) be a complete Riemannian manifold such that the sectional curvature is always non-positive. Then for every point $p \in M$, the map $\exp_p : T_pM \to M$ is a covering map. In particular, if $\pi_1(M) = 0$, then M is diffeomorphic to $\mathbb{R}^n$.

Lemma. Let $\gamma(t)$ be a geodesic on (M, g) such that $K \leq 0$ along γ. Then γ has no conjugate points.

4 Hodge theory on Riemannian manifolds

4.1 Hodge star and operators

Proposition. Suppose $\omega_1, \cdots, \omega_n$ is an orthonormal basis of $T_x^* M$. Then we claim that

$$\star(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_{p+1} \wedge \cdots \wedge \omega_n.$$

Proposition. The double Hodge star $\star\star : \bigwedge^p(T_x^* M) \to \bigwedge^p(T_x^* M)$ is equal to $(-1)^{p(n-p)}$.

Proposition.

$$\star \Delta = \Delta \star.$$

Proposition. δ is the formal adjoint of d. Explicitly, for any compactly supported $\alpha \in \Omega^{p-1}$ and $\beta \in \Omega^p$, then

$$\int_M \langle \mathrm{d}\alpha, \beta \rangle_g \, \omega_g = \int_M \langle \alpha, \delta\beta \rangle_g \, \omega_g.$$

Corollary. Δ is formally self-adjoint.

Corollary. Let M be compact. Then

$$\Delta\alpha = 0 \Leftrightarrow \mathrm{d}\alpha = 0 \text{ and } \delta\alpha = 0.$$

4.2 Hodge decomposition theorem

Theorem (Hodge decomposition theorem)**.** Let (M, g) be a compact oriented Riemannian manifold. Then

- For all $p = 0, \cdots, \dim M$, we have $\dim \mathcal{H}^p < \infty$.
- We have
  $$\Omega^p(M) = \mathcal{H}^p \oplus \Delta\Omega^p(M).$$
  Moreover, the direct sum is orthogonal with respect to the L^2 inner product. We also formally set $\Omega^{-1}(M) = 0$.

Corollary. We have orthogonal decompositions

$$\begin{aligned} \Omega^p(M) &= \mathcal{H}^p \oplus \mathrm{d}\delta\Omega^p(M) \oplus \delta\mathrm{d}\Omega^p(M) \\ &= \mathcal{H}^p \oplus \mathrm{d}\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M). \end{aligned}$$

Corollary. Let (M, g) be a compact oriented Riemannian manifold. Then for all $\alpha \in H^p_{\mathrm{dR}}(M)$, there is a unique $\alpha \in \mathcal{H}^p$ such that $[\alpha] = a$. In other words, the obvious map

$$\mathcal{H}^p \to H^p_{\mathrm{dR}}(M)$$

is an isomorphism.

Theorem (Compactness theorem)**.** If a sequence $\alpha_n \in \Omega^n(M)$ satisfies $\|\alpha_n\| < C$ and $\|\Delta\alpha_n\| < C$ for all n, then α_n contains a Cauchy subsequence.

Corollary. $\mathcal{H}^p$ is finite-dimensional.

Theorem (Regularity theorem)**.** Every weak solution of $\Delta\omega = \alpha$ is of the form

$$\ell(\beta) = \langle \omega, \beta \rangle$$

for $\omega \in \Omega^p(M)$.

Theorem (Hahn–Banach theorem)**.** Let L be a normed vector space, and L_0 be a subspace. We let $f : L_0 \to \mathbb{R}$ be a bounded linear functional. Then f extends to a bounded linear functional $L \to \mathbb{R}$ with the same bound.

4.3 Divergence

Proposition.

$$\mathrm{div}(fX) = \mathrm{tr}(\nabla(fX)) = f\mathrm{div}X + \langle \mathrm{d}f, X \rangle.$$

Theorem. Let $\theta \in \Omega^1(M)$, and let $X_\theta \in \mathrm{Vect}(M)$ be such that $\langle \theta, V \rangle = g(X_\theta, V)$ for all $V \in TM$. Then

$$\delta\theta = -\mathrm{div}X_\theta.$$

Lemma. In local coordinates, for any p-form ψ, we have

$$\mathrm{d}\psi = \sum_{k=1}^n \mathrm{d}x^k \wedge \nabla_k \psi.$$

Lemma. We have

$$(\mathrm{div}X)\ \omega_g = \mathrm{d}(i(X)\ \omega_g),$$

for all $X \in \mathrm{Vect}(M)$.

Corollary (Divergence theorem)**.** For any vector field X, we have

$$\int_M \mathrm{div}(X)\ \omega_g = \int_M \mathrm{d}(i(X)\ \omega_g) = 0.$$

Theorem. Let $\theta \in \Omega^1(M)$, and let $X_\theta \in \mathrm{Vect}(M)$ be such that $\langle \theta, V \rangle = g(X_\theta, V)$ for all $V \in TM$. Then

$$\delta\theta = -\mathrm{div}X_\theta.$$

Corollary. If θ is a 1-form, and $\{e_k\}$ is a local orthonormal frame field, then

$$\delta\theta = -\sum_{k=1}^n i(e_k)\nabla_{e_i}\theta = -\sum_{k=1}^n \langle \nabla_{e_k}\theta, e_k \rangle.$$

Proposition. If $\beta \in \Omega^2(M)$, then

$$(\delta\beta)(Y) = -\sum_{k=1}^n (\nabla_{e_k}\beta)(e_k, Y).$$

In other words,

$$\delta\beta = -\sum_{k=1}^n i(e_k)(\nabla_{e_k}\beta).$$

4.4 Introduction to Bochner's method

Theorem (Bochner–Weitzenböck formula)**.** On an oriented Riemannian manifold, we have

$$\Delta = \nabla^* \nabla + \mathrm{Ric}\,.$$

Corollary. Let (M, g) be a compact connected oriented manifold. Then

- If $\mathrm{Ric}(g) > 0$ at each point, then $H^1_{\mathrm{dR}}(M) = 0$.
- If $\mathrm{Ric}(g) \geq 0$ at each point, then $b^1(M) = \dim H^1_{\mathrm{dR}}(M) \leq n$.
- If $\mathrm{Ric}(g) \geq 0$ at each point, and $b^1(M) = n$, then g is flat.

Proposition. In the case of (iii), M is in fact isometric to a flat torus.

Proposition. Let $e_1, \cdots, e_n$ be an orthonormal frame field, and $\beta \in \Omega^1(T^* M)$. Then we have

$$\nabla^* \beta = -\sum_{i=1}^n i(e_i) \nabla_{e_i} \beta.$$

Corollary. For a local orthonormal frame field $e_1, \cdots, e_n$, we have

$$\nabla^* \nabla \alpha = -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \alpha.$$

Lemma. Let $\alpha \in \Omega^1(M)$, $X \in \mathrm{Vect}(M)$. Then

$$\langle \mathrm{d}\delta\alpha, X\rangle = -\sum_{i=1}^n \langle \nabla_X \nabla_{e_i} \alpha, e_i\rangle.$$

Lemma. For any 2-form β, we have

$$(\delta\beta)(X) = \sum_{k=1}^n -e_k(\beta(e_k, X)) + \beta(e_k, \nabla_{e_k} X).$$

Lemma. For any 1-form α and vector fields X, Y, we have

$$\mathrm{d}\alpha(X, Y) = \langle \nabla_X \alpha, Y\rangle - \langle \nabla_Y \alpha, X\rangle.$$

Lemma. For any 1-form α and vector field X, we have

$$\langle \delta \mathrm{d}\alpha, X\rangle = -\sum_{k=1}^n \langle \nabla_{e_k} \nabla_{e_k} \alpha, X\rangle + \sum_{k=1}^n \langle \nabla_{e_k} \nabla_X \alpha, e_k\rangle - \sum_{k=1}^n \langle \nabla_{\nabla_{e_k} X} \alpha, e_k\rangle.$$

Lemma (Ricci identity)**.** Let M be any Riemannian manifold, and $X, Y, Z \in \mathrm{Vect}(M)$ and $\alpha \in \Omega^1(M)$. Then

$$\langle ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})\alpha, Z\rangle = \langle \alpha, R(X, Y) Z\rangle.$$

Corollary. For any 1-form α and vector field X, we have

$$\langle \Delta\alpha, X\rangle = \langle \nabla^* \nabla \alpha, X\rangle + \mathrm{Ric}(\alpha)(X).$$

5 Riemannian holonomy groups

Proposition. If M is simply connected, then $\mathrm{Hol}_x(M)$ is path connected.

Corollary. $\mathrm{Hol}^0(M) \subseteq \mathrm{SO}(n)$.

Proposition (Fundamental principle of Riemannian holonomy)**.** Let (M, g) be a Riemannian manifold, and fix $p, q \in \mathbb{Z}_+$ and $x \in M$. Then the following are equivalent:

(i) There exists a (p, q)-tensor field α on M such that $\nabla\alpha = 0$.

(ii) There exists an element $\alpha_0 \in (T_xM)^{\otimes p} \otimes (T_x^*M)^{\otimes q}$ such that α_0 is invariant under the action of $\mathrm{Hol}_x(M)$.

Theorem. Let (M, g) be a connected and oriented Riemannian manifold, and consider the decomposition of the bundle of k-forms into irreducible representations of the holonomy group,

$$\bigwedge^k T^*M = \bigoplus_i \Lambda_i^k.$$

In other words, each fiber $(\Lambda_i^k)_x \subseteq \bigwedge^k T_x^*M$ is an irreducible representation of $\mathrm{Hol}_x(g)$. Then

(i) For all $\alpha \in \Omega_i^k(M) \equiv \Gamma(\Lambda_i^l)$, we have $\Delta\alpha \in \Omega_i^k(M)$.

(ii) If M is compact, then we have a decomposition

$$H_{\mathrm{dR}}^k(M) = \bigoplus H_{i,\mathrm{dR}}^k(M),$$

where

$$H_{i,\mathrm{dR}}^k(M) = \{[\alpha] : \alpha \in \Omega_i^k(M), \Delta\alpha = 0\}.$$

The dimensions of these groups are known as the *refined Betti numbers*.

6 The Cheeger–Gromoll splitting theorem

Lemma. If M is disconnected at infinity, then M contains a line.

Theorem (Cheeger–Gromoll line-splitting theorem (1971))**.** If (M, g) is a complete connected Riemannian manifold containing a line, and has $\mathrm{Ric}(g) \geq 0$ at each point, then M is isometric to a Riemannian product $(N \times \mathbb{R}, g_0 + \mathrm{d}t^2)$ for some metric g_0 on N.

Corollary. Let (M, g) be a complete connected Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. Then it is isometric to $X \times \mathbb{R}^q$ for some $q \in \mathbb{N}$ and Riemannian manifold X, where X is complete and does not contain any lines.

Lemma. Let (M, g) be a compact Riemannian manifold, and suppose its universal Riemannian cover $(\tilde{M}, \tilde{g})$ is non-compact. Then $(\tilde{M}, \tilde{g})$ contains a line.

Corollary. Let (M, g) be a compact, connected manifold with $\mathrm{Ric}(g) \geq 0$. Then

- The universal Riemannian cover is isometric to the Riemannian product $X \times \mathbb{R}^N$, with X compact, $\pi_1(X) = 1$ and $\mathrm{Ric}(g_X) \geq 0$.
- If there is some $p \in M$ such that $\mathrm{Ric}(g)_p > 0$, then $\pi_1(M)$ is finite.
- Denote by $I(\tilde{M})$ the group of isometries $\tilde{M} \to \tilde{M}$. Then $I(\tilde{M}) = I(X) \times E(\mathbb{R}^q)$, where $E(\mathbb{R}^q)$ is the group of rigid Euclidean motions,

  $$\mathbf{y} \mapsto A\mathbf{y} + \mathbf{b}$$

  where $\mathbf{b} \in \mathbb{R}^n$ and $A \in \mathrm{O}(q)$.
- If $\tilde{M}$ is homogeneous, then so is X.

Proposition. Consider $S^n \times \mathbb{R}$ for $n = 2$ or 3. Then this does not admit any Ricci-flat metric.