Part III: Riemannian Geometry (Lent 2017)

Example Sheet 1

1. (i) Prove that any connection $\nabla$ on $M$ uniquely determines a covariant derivative on the cotangent bundle $T^*M$ (still to be denoted by $\nabla$), such that $\nabla_X : \Omega^1(M) \to \Omega^1(M)$ satisfies $X(\alpha, Y) = \langle \nabla_X \alpha, Y \rangle + \langle \alpha, \nabla_X Y \rangle$. Here $\alpha \in \Omega^1(M)$, $X, Y$ are vector fields on $M$, and $\langle \cdot , \cdot \rangle$ denotes the evaluation of a 1-form on a tangent vector. In particular, prove that if $\alpha = \sum_j \alpha_j dx^j$ in local coordinates and $\Gamma^i_{jk}$ are the coefficients of $\nabla$ on the tangent bundle then $(\nabla_X \alpha)_j = \sum_{ik} (\frac{\partial \alpha_j}{\partial x^k} - \Gamma^i_{jk} \alpha^k) X^i$.

Show further that if $\nabla$ is the Levi–Civita of some metric $(g_{ij})$ on $M$ then the induced connection is compatible with the dual metric $g$ by writing out an appropriate version of ‘Leibniz formula’ for $\nabla$. Give the expression for $(\nabla_X \alpha)_j$ in local coordinates. Show that if $\nabla$ is the Levi–Civita of a Riemannian metric $g$ on $M$ then $\nabla g = 0$. (Thus a Riemannian metric is covariant constant, or ‘parallel’, with respect to its Levi–Civita connection.)

(ii) Generalize the definition of the induced connection (still denoted by $\nabla$) to the case of $(0,q)$-tensor bundle $T^*M \otimes \mathbb{R}^q$, $q > 1$, by writing out an appropriate version of ‘Leibniz formula’ for $\nabla$. Give the expression for $\nabla$ in local coordinates. Show that if $\nabla$ is the Levi–Civita of a Riemannian metric $g$ on $M$ then $\nabla g = 0$. (Thus a Riemannian metric is covariant constant, or ‘parallel’, with respect to its Levi–Civita connection.)

2. (i) Let $M$ be a Riemannian manifold. Show that the Levi–Civita covariant derivative of $R(X, Y) \in \Gamma(\text{End} \ T M)$ is given by $\nabla_Z R(X, Y) = [\nabla_Z X, Y] + [X, \nabla_Z Y] - R(\nabla_Z X, Y) - R(X, \nabla_Z Y)$. Deduce from this a version of the second Bianchi identity for the Levi–Civita connection

$$\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0.$$

(ii) When $\dim M \geq 3$, show, using (*), that if $\text{Ric} = fg$ for some smooth function $f$, then $f$ is constant ($M$ then is said to be an Einstein manifold).

(You might like to consider a map $\delta : \Gamma(\text{Sym}^2 T^*M) \to \Gamma(T^*M)$ defined by $(\delta h)(X) = -\sum_{i=1}^n (\nabla_{e_i} h)(e_i, X)$, where $\{e_i\}$ is any local orthonormal frame field on $M$, and put $h = \text{Ric}$.)

3. For this question, recall that the Riemann curvature tensor $(R_{ij,kl})$ of $(M, g)$ defines a symmetric bilinear form on the fibres of $\Lambda^2 T^*M$. Show that if $\dim M = 3$ then the Riemann curvature is determined at each point of $M$ by the Ricci curvature $\text{Ric}(g)$.

[Hint: the assignment of $\text{Ric}(g)$ to $R(g)$ is a linear map, at each point of $M$. A special feature of the dimension 3 is that the spaces of 1-forms and 2-forms on $\mathbb{R}^3$ have the same dimension.]

4. Prove that the scalar curvature $s(p)$, $p \in M$ is given by

$$s(p) = \frac{n}{\omega_{n-1}} \int_{S^{n-1}} \text{Ric}_p(x, x) \, dx$$

where $\omega_{n-1}$ is the volume of the unit sphere $S^{n-1}$ in $T_p M$.

5. Let $G$ be a Lie group endowed with a Riemannian metric $g$ which is left and right invariant and let $X, Y, Z$ be left invariant vector fields of $G$.

(i) Show that $g([X, Y], Z) + g(Y, [X, Z]) = 0$. (Consider the flow of $X$.)

(ii) Show that $\nabla_X X = 0$. (Hint: consider $g(Y, \nabla_X X)$.)
(iii) Show that $\nabla_X Y = \frac{1}{2} [X, Y]$. 
(iv) Prove that $R(X, Y)Z = \frac{1}{4} [[X, Y], Z]$. 
(v) Suppose that $X$ and $Y$ are orthonormal, and let $K(\sigma)$ be the sectional curvature of the 2-plane $\sigma$ spanned by $X$ and $Y$. Prove that 
$$K(\sigma) = \frac{1}{4} ||[X, Y]|^2_g$$

6. Let $M$ be a Riemannian manifold. $M$ is said to be \textit{locally symmetric} if $\nabla R = 0$, where $R = (R_{ijkl})$ is the curvature tensor of $M$.

(i) Let $M$ be a locally symmetric space and let $\gamma : [0, \ell] \to M$ be a geodesic on $M$. Let $X, Y, Z$ be parallel vector fields along $\gamma$. Prove that $R(X, Y)Z$ is a parallel vector field along $\gamma$.

(ii) Suppose that $M$ is locally symmetric, connected and 2-dimensional. Prove that $M$ has constant sectional curvature.

(iii) Prove that if $M$ has constant sectional curvature, then it is locally symmetric.

7. Let $N$ be a connected Riemannian manifold and let $f : M \to N$ be a local diffeomorphism. Show that one can put a Riemannian metric on $M$ such that $f$ becomes a local isometry. Show that if $M$ is complete then $N$ is complete. Is the converse true? Is the converse true if $f$ is a covering map?

8. A geodesic $\gamma : [0, \infty) \to M$ is called a \textit{ray} if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in (0, \infty)$. Show that if $M$ is complete and non-compact, there is a ray leaving from every point in $M$.

9. A Riemannian manifold $M$ is said to be \textit{homogeneous} if given $p$ and $q$ in $M$, there exists an isometry of $M$ taking $p$ to $q$. Show that a homogeneous Riemannian manifold is complete.

10. Let $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \}$ and let $h : S^3 \to S^3$ be given by
$$h(z_1, z_2) = (e^{2\pi i/q}z_1, e^{2\pi i r/q}z_2),$$
where $q$ and $r$ are co-prime integers and $q > 1$.

(i) Show that $G = \{ \text{id}, h, \ldots, h^{q-1} \}$ is a group of isometries of $S^3$ (equipped with the standard metric) that acts in such a way that $S^3/G$ is a manifold and the projection $p : S^3 \to S^3/G$ is a local diffeomorphism. (The manifolds $S^3/G$ are called \textit{lens spaces}.)

(ii) Consider on $S^3/G$ the metric induced by $p$. Show that all the geodesics of $S^3/G$ are closed, but they could have different lengths.

11. Let $M$ be a complete Riemannian manifold and let $N \subset M$ be a closed submanifold. Let $p \in M$, $p \not\in N$, and let $d(p, N)$ be the distance from $p$ to $N$. Show that there exists a point $q \in N$ such that $d(p, q) = d(p, N)$. Show that a minimizing geodesic between $p$ and $q$ must be orthogonal to $N$ at $q$.

12. Let $M$ be an orientable Riemannian manifold of even dimension and positive sectional curvature. Show that any closed geodesic in $M$ is homotopic to a closed curve with length strictly smaller than that of $\gamma$.

13. Suppose that for every smooth Riemannian metric on a manifold $M$, $M$ is complete. Show that $M$ is compact (Hint: think about rays as in Problem 8.)

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Part III: Riemannian Geometry (Lent 2017)

Example Sheet 2

1. Give an example of a non-compact complete Riemannian manifold with Ricci curvature (strictly) positive-definite at each point.

2. Let $G$ be a Lie group whose Lie algebra $\mathfrak{g}$ has trivial centre. Suppose that $G$ admits a bi-invariant (i.e. left- and right-invariant) Riemannian metric. Show that $G$ and its universal cover are compact. Deduce that $SL(2, \mathbb{R})$ admits no bi-invariant metric.

3. (i) Show that the Hodge star on $\Lambda^2(\mathbb{R}^4)^*$ determines an orthogonal decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda^+ \oplus \Lambda^-$ into the $\pm 1$ eigenspaces and $\dim \Lambda^+ = \dim \Lambda^- = 3$. Deduce that on every oriented 4-dimensional Riemannian manifold $M$ there is a decomposition of 2-forms $\Omega^2(M) = \Omega^+ \oplus \Omega^-$, so that $\alpha \wedge \alpha = \pm |\alpha|^2 g \omega_g$, for every $\alpha \in \Omega^\pm$, where $\omega_g$ is the volume form. (2-forms in the subspaces $\Omega^\pm$ are called, respectively, the self- and anti-self-dual forms on $M$.)

(ii) Now assume that $M$ is a compact 4-dimensional oriented Riemannian manifold. Show that the expression $\int_M \alpha \wedge \beta$, for closed $\alpha, \beta \in \Omega^2(M)$, induces a well-defined symmetric bilinear form on the de Rham cohomology $H^2_{dR}(M)$. Let $(b^+(M), b^-(M))$ denote the signature of this bilinear form. Show that $b^\pm(M) = \dim \mathcal{H}^\pm$, where $\mathcal{H}^\pm$ denotes the space of harmonic (anti-)self-dual forms on $M$.

4. (i) Derive explicit formulas for $\ast$, $\delta$, and Laplace–Beltrami operator in Euclidean space. In particular, show that if $\alpha = \sum_{i_1 < \ldots < i_p} \alpha_I dx_{i_1} \wedge \ldots \wedge dx_{i_p} \quad (I = i_1, \ldots, i_p)$, then

$$\Delta \alpha = - \sum_{i_1 < \ldots < i_p} \left( \sum_{i=1}^n \frac{\partial^2 \alpha_I}{\partial x_i^2} \right) dx_{i_1} \wedge \ldots \wedge dx_{i_p}.
$$

(ii) For $u, v \in C^\infty(M)$, show that $\Delta(uv) = \Delta(u)v - 2\langle du, dv \rangle_g + u \Delta v \quad (M \text{ is an oriented Riemannian manifold}).$

5. Calculate explicitly the expression of the Laplacian for functions:

(a) on the hyperbolic plane $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, where the metric is $g(x, y) = \frac{dx^2 + dy^2}{y^2}$;

(b) on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, in local coordinates given by stereographic projections. (The metric on $S^n$ is the standard ‘round’ metric induced by the embedding.)

*Express the Laplacian on the Euclidean $\mathbb{R}^{n+1} \setminus \{0\}$ in terms of the Laplacian on the unit sphere $S^n$ (recall that the Euclidean metric can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^{n+1}$, and $dS^2$ is the ‘round’ metric on $S^n$). Deduce a formula for the Laplacian on spherically-symmetric functions $f(r)$.

6. Let $\alpha$ and $\beta$ be $n$-forms on a compact oriented manifold $M^n$ such that $\int_M \alpha = \int_M \beta$. Prove that $\alpha$ and $\beta$ differ by an exact form. (Stokes’ theorem may be assumed.)

http://www.dpmms.cam.ac.uk/~agk22/riemannian2.pdf
7. Show that the partial differential equation $\Delta f = \varphi$ for a function $f \in C^\infty(M)$ on a compact oriented Riemannian manifold $(M, g)$, with a given $\varphi \in C^\infty(M)$, has a solution if and only if $\int_M \varphi \omega_g = 0$. ($\omega_g$ denotes the volume form.) Is the solution unique? *Discuss the solvability of $\Delta(\Delta f) = \varphi$ when $f, \varphi \in C^\infty(M)$ and more generally when $f, \varphi$ are $p$-forms.

8. Let $M$ be a compact oriented Riemannian manifold and $F$ a diffeomorphism of $M$ which preserves the volume form on $M$. We say that a form $\alpha \in \Omega^p(M)$ is invariant under $F$ if $\alpha \circ F = \alpha$, and we say that the Laplacian $\Delta$ is invariant under $F$ if $\Delta \circ F = \Delta(\alpha \circ F)$, for all $\alpha \in \Omega^p(M)$. Suppose that $\Delta$ and $\alpha$ are invariant under $F$ and $\alpha$ is $L^2$-orthogonal to each harmonic form on $M$. Prove that there is an invariant solution $\eta$ of $\Delta \eta = \alpha$.

9. (Holonomy transformations.) Show that the parallel transport defined by the Levi-Civita connection over any closed loop based at $x \in M$ defines an orthogonal linear transformation of $T_xM$ which is in $SO(T_xM)$ when $M$ is oriented.

An orthogonal almost complex structure on a manifold $(M, g)$ is an endomorphism $J$ of its tangent bundle $TM$ such that $J^2 = -1$ and $g(JX, JY) = g(X, Y)$, for all $X, Y \in \text{ Vect}(M)$. If $M$ admits such $J$, show that $M$ is orientable and even-dimensional. Show that $\omega = g(J \cdot, \cdot)$ defines a 2-form on $M$ with $\omega^n \neq 0$ at each point ($\dim M = 2n$).

Show further that the following statements are equivalent:

(a) $\nabla J = 0$,
(b) $\nabla \omega = 0$,
(c) the parallel transport defined by $\nabla$ along closed loops is represented by elements of $U(n) \subset SO(2n)$ (after some natural identifications).

Here $\nabla$ denotes the (induced) Levi-Civita connection on respective vector bundles. (Each of (a),(b),(c) is in fact equivalent to $M$ being a Kähler complex manifold with Kähler form $\omega$ and $J$ corresponding to multiplication by $i$ in local complex coordinates.)

10. (i) For any two bilinear forms $h, k$ on tangent spaces to $M$, define a $(0, 4)$-tensor $(h \cdot k)(X, Y, Z, T) = h(X, Z)k(Y, T) + h(Y, T)k(X, Z) - h(X, T)k(Y, Z) - h(Y, Z)k(X, T)$, where $X, Y, Z, T \in T_xM$. Show that the curvature tensor $R = (R_{ijkl})$ of a Riemannian $n$-dimensional manifold $(M, g)$, $n \geq 4$, has an $SO(n)$-invariant, orthogonal decomposition $R = \frac{s}{2n(n-1)}g \cdot g + \frac{1}{n-2}(\text{Ric} - \frac{s}{n}g) \cdot g + W$, where $W$ satisfies $W(X, Y, Z, T) + W(Z, X, Y, T) + W(Y, Z, X, T) = 0$ (1st Bianchi identity) and $\sum_{i=1}^n W(X, e_i, Y, e_i) = 0$ for all $X, Y, Z, T \in T_xM$ and where $e_i$ is an orthonormal basis of $T_xM$. ($W$ is called the Weyl tensor of $(M, \varphi)$).

(ii) Suppose that $\dim M = 4$ and $M$ is oriented. We consider $R$ as a symmetric bilinear form on the fibres of $\Lambda^2T^*M$. Let a bilinear form, $B : \Lambda^+ \times \Lambda^- \rightarrow \mathbb{R}$ be the restriction of $R$ defined using the decomposition into self- and anti-self-dual forms as in Question 3. Show that $B$ is equivalent to the trace-free part $\text{Ric}_0$ of the Ricci curvature (with respect to the metric $g$), that is, $g^{ik}(\text{Ric}_0)_{kj} = g^{ik}\text{Ric}_0 - \frac{1}{2}s \delta^i_j$ in local coordinates, where $(g^{ij})$ denotes the inverse matrix of $g = (g_{ij})$ (the summation convention is assumed).

Show further that $R$, with respect to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, has the form

\[
\begin{pmatrix}
W^+ + \frac{s}{12}I & B \\
B^T \frac{1}{12}I & W^- + \frac{s}{12}I
\end{pmatrix},
\]

where $W^\pm : \Lambda^\pm \times \Lambda^\pm \rightarrow \mathbb{R}$ are symmetric bilinear forms with $\text{tr} W^- + \text{tr} W^+ = 0$ and $W = W^+ \oplus W^-$ is an $SO(4)$-invariant orthogonal decomposition of the Weyl tensor.

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Example Sheet 3

1. Show that the volume form of a Riemannian manifold is parallel, $\nabla \omega_g = 0$, with respect to the Levi–Civita connection of $g$.

2. (normal frame fields) Show that for each point $p \in M$ of a Riemannian manifold there exists an orthonormal frame field $e_1, \ldots, e_n$ defined on some neighbourhood of $p$ and such that $\nabla e_i$ vanishes at $p$ for each $i$. [Hint: you might like to first verify that the coefficients $\Gamma^i_{jk}$ of the Levi–Civita connection in the geodesic coordinates at $p \in M$, vanish at $p$.]

3. Show that for the Levi–Civita connection, the following diagram commutes

$$\Gamma(\Lambda^p T^* M) \xrightarrow{\nabla} \Gamma(T^* M \otimes \Lambda^p T^* M) \xrightarrow{d} \Gamma(\Lambda^{p+1} T^* M),$$

where $\text{alt}(\xi \otimes \alpha) = \xi \wedge \alpha$ denotes projection to the subspace of anti-symmetric tensors ($p > 0$). Deduce the formula for the exterior derivative of one-forms

$$d\alpha(X, Y) = (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X)$$
as stated in the Lectures. * Show that these results hold for any torsion-free connection $\nabla$ on $M$.

4. (i) The divergence of a $(1,3)$-tensor $R$ may be defined as a $(0,3)$-tensor (i.e. a tri-linear function of tangent vectors $X,Y,Z$)

$$(\text{div } R)(X,Y,Z) = \text{tr}(V \rightarrow (\nabla_V R)(X,Y,Z)).$$

Show that if $R = (R^i_{j,k,l})$ is the $(1,3)$-curvature tensor, then

$$(\text{div } R)(X,Y,Z) = (\nabla_X \text{Ric})(Y,Z) - (\nabla_Y \text{Ric})(X,Z).$$

Deduce that $\text{div } R = 0$ if and only if the Ricci curvature satisfies:

$$(\nabla_X \text{Ric})(Y,Z) = (\nabla_Y \text{Ric})(X,Z) \text{ for all } X,Y,Z.$$  

(ii) (M. Berger) On a closed Riemannian manifold $(M,g)$ show that if $\text{div } R = 0$ and the sectional curvature $K \geq 0$, then $\nabla \text{Ric} = 0$. [Hint: use the relation $2 \text{tr } \nabla \text{Ric} = ds$ to conclude $\text{tr } \nabla \text{Ric} = 0$. Here $(\text{tr } \nabla \text{Ric})(X) = \sum_i \nabla_{e_i} \text{Ric}(e_i, X)$, for orthonormal $e_i$.]

5. Show that the co-differential $\delta$ on $p$-forms may be equivalently defined by

$$\delta\eta(X_2, \ldots, X_p) = -\sum_{i=1}^n (\nabla_{e_i} \eta)(e_i, X_2, \ldots, X_p) = -\sum_{i=1}^n (i(e_i)(\nabla_{e_i} \eta))(X_2, \ldots, X_p)$$

where $e_i$ is some/any local orthonormal frame field. (In particular, $\delta$ and $\Delta$, are independent of the choice of orientation and in fact may be defined on non-orientable manifolds too.)

http://www.dpmms.cam.ac.uk/~agk22/riemannian3.pdf
6. (i) Show that $\text{Hol}^0(M)$ is a normal subgroup of $\text{Hol}(M)$ and that there is a natural, surjective group homomorphism $\pi_1(M) \to \text{Hol}(M)/\text{Hol}^0(M)$.

(ii) Show that $\text{Hol}(\tilde{M}) = \text{Hol}^0(M)$, where $\tilde{M}$ denotes the universal Riemannian cover of $M$.

7. Determine the holonomy of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ (with the round metric).

8. A theorem due to de Rham asserts that if we decompose the tangent bundle of a Riemannian manifold $(M, g)$ into irreducible components according to the holonomy representation, $TM = \tau_1 \oplus \ldots \oplus \tau_k$, then around each point $p \in M$ there is a neighbourhood $U$ that decomposes into a Riemannian product, $(U, g) = (U_1 \times \ldots \times U_k, g_1 + \ldots + g_k)$, so that $TU_i = \tau_i$ for each $i$.

Deduce that if the holonomy group of $(M, g)$ has no invariant subspaces (i.e. the holonomy representation is irreducible) and the Ricci tensor is parallel $\nabla \text{Ric} = 0$, then $M$ is Einstein ($\text{Ric} = \lambda g$, for some $\lambda \in \mathbb{R}$). [Hint: eigenvalues.]

9.* (This question requires Frobenius theorem.) Suppose that $(M, g)$ admits a parallel field of $k$-dimensional tangent subspaces ($k \leq n - 1$), i.e. a rank $k$ subbundle of $TM$ invariant under parallel transport. Show that every such distribution is integrable (involutive).

10. Using the skew-symmetric linear maps

$$X \wedge Y : T_pM \to t_pM, \quad X \wedge Y(V) = g(X, V)Y - g(Y, V)X,$$

show that $\Lambda^2 T_pM \cong so(T_pM)$. (Elements $X \wedge Y$ of $\Lambda^2 T_pM$ are sometimes called bi-vectors.) Now let $\mathcal{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM$ be a linear endomorphism induced by the curvature $(0,4)$-tensor. Deduce that the image of $\mathcal{R}$ is contained in the holonomy algebra $\mathfrak{R}(\Lambda^2 T_pM) \subset \text{hol}_p(M)$.

11. (i) Show that a compact Riemannian manifold with irreducible holonomy representation and $\text{Ric} \geq 0$ has finite fundamental group.

(ii) Let $G$ be a compact Lie group endowed with a bi-invariant metric. Show that $G$ admits a finite cover by $G' \times T^k$, where $G'$ is compact simply connected and $T^k$ is a torus.

* Show that if $G$ has finite fundamental group, then its Lie algebra has trivial centre.

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