Part III — Riemannian Geometry
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:


Pre-requisites

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course Differential Geometry offered in Michaelmas Term is the ideal pre-requisite.
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1 Basics of Riemannian manifolds

**Definition** (Riemannian metric). Let $M$ be a smooth manifold. A *Riemannian metric* $g$ on $M$ is an inner product on the tangent bundle $TM$ varying smoothly with the fibers. Formally, this is a global section of $T^*M \otimes T^*M$ that is fiberwise symmetric and positive definite.

The pair $(M,g)$ is called a *Riemannian manifold*.

**Definition** (Isometry). Let $(M,g)$ and $(N,h)$ be Riemannian manifolds. We say $f : M \to N$ is an *isometry* if it is a diffeomorphism and $f^*h = g$. In other words, for any $p \in M$ and $u, v \in T_pM$, we need

$$h((df)_pu, (df)_pv) = g(u, v).$$

**Definition** (Left-invariant vector field). Let $G$ be a Lie group, and $X$ a vector field. Then $X$ is *left invariant* if for any $x \in G$, we have $d(L_x)X = X$.

**Definition** (Levi-Civita connection). Let $(M,g)$ be a Riemannian manifold. The *Levi-Civita connection* is the unique connection $\nabla : \Omega^0(M)(TM) \to \Omega^1(M)(TM)$ on $M$ satisfying

(i) Compatibility with metric:

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$

(ii) Symmetry/torsion-free:

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

**Definition** (Christoffel symbols). In local coordaintes, the *Christoffel symbols* are defined by

$$\nabla_{\partial_j} \frac{\partial}{\partial x^i} = \Gamma^i_{jk} \frac{\partial}{\partial x^j}. $$


2 Riemann curvature

Definition (Curvature). Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The curvature 2-form is the section

$$R = -\nabla \circ \nabla \in \Gamma(\Lambda^2 T^* M \otimes T^* M \otimes TM) \subseteq \Gamma(T^{1,3}M).$$

Definition (Ricci curvature). The Ricci curvature of $g$ at $p \in M$ is

$$\text{Ric}_p(X, Y) = \text{tr}(v \mapsto R_p(X, v)Y).$$

In terms of coordinates, we have

$$\text{Ric}_{ij} = R^q_{i,jq} = g^{pq} R_{pi,jq},$$

where $g^{pq}$ denotes the inverse of $g$.

This Ric is a symmetric bilinear form on $T_pM$. This can be determined by the quadratic form

$$\text{Ric}(X) = \frac{1}{n-1} \text{Ric}_p(X, X).$$

The coefficient $\frac{1}{n-1}$ is just a convention.

Definition (Scalar curvature). The scalar curvature of $g$ is the trace of Ric respect to $g$. Explicitly, this is defined by

$$s = g^{ij} \text{Ric}_{ij} = g^{ij} R^q_{i,jq} = R^q_{q}. $$
3 Geodesics

3.1 Definitions and basic properties

Definition (Lift). Let \( \pi : E \to M \) be a vector bundle with typical fiber \( V \). Consider a curve \( \gamma : (-\varepsilon, \varepsilon) \to M \). A lift of \( \gamma \) is a map \( \gamma^E : (-\varepsilon, \varepsilon) \to E \) if \( \pi \circ \gamma^E = \gamma \), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\gamma^E & \to & E \\
\downarrow & & \downarrow \pi \\
(-\varepsilon, \varepsilon) & \to & M
\end{array}
\]

Definition (Covariant derivative). The uniquely defined operation in the proposition above is called the covariant derivative.

Definition (Horizontal lift). Let \( \nabla \) be a connection on \( E \) with \( \Gamma^i_{jk}(x) \) the coefficients in a local trivialization. We say a lift \( \gamma^E \) is horizontal if

\[
\nabla \gamma^E \frac{dt}{dt} = 0.
\]

Definition (Parallel transport). Let \( \gamma : [0, 1] \to M \) be a curve in \( M \). Given any \( a_0 \in E_{\gamma(0)} \), the unique horizontal lift of \( \gamma \) with \( \gamma^E(0) = (\gamma(0), a_0) \) is called the parallel transport of \( a_0 \) along \( \gamma(0) \). We sometimes also call \( \gamma^E(1) \) the parallel transport.

Definition (Geodesic). A curve \( \gamma(t) \) on a Riemannian manifold \( (M, g) \) is called a geodesic curve if its canonical lift is horizontal with respect to the Levi-Civita connection. In other words, we need

\[
\nabla \dot{\gamma} \frac{dt}{dt} = 0.
\]

Definition (Exponential map). Let \( (M, g) \) be a Riemannian manifold, and \( p \in M \). We define \( \exp_p \) by

\[
\exp_p(a) = \gamma(1, a) \in M
\]

for \( a \in T_pM \) whenever this is defined.

3.2 Jacobi fields

Definition (Jacobi field). Let \( \gamma : [0, L] \to M \) be a geodesic. A Jacobi field is a vector field \( J \) along \( \gamma \) that is a solution of the Jacobi equation on \( [0, L] \)

\[
\frac{\nabla^2}{dt^2} J + R(\dot{\gamma}, J) \dot{\gamma} = 0.
\]

3.3 Further properties of geodesics

Notation. We write \( \Omega(p, q) \) for the set of all piecewise \( C^1 \) curves from \( p \) to \( q \).
**Definition (Distance).** Suppose \( M \) is connected, which is the same as it being path connected. Let \((p, q) \in M\). We define
\[
d(p, q) = \inf_{\xi \in \Omega(p, q)} \text{length}(\xi),
\]
where

**Definition (Minimal geodesic).** A minimal geodesic is a curve \( \gamma : [0, 1] \to M \) such that
\[
d(\gamma(0), \gamma(1)) = \ell(\gamma).
\]

### 3.4 Completeness and the Hopf–Rinow theorem

**Definition (Geodesically complete).** We say a manifold \((M, g)\) is geodesically complete if each geodesic extends for all time. In other words, for all \( p \in M \), \( \exp_p \) is defined on all of \( T_p M \).

### 3.5 Variations of arc length and energy

**Definition (Energy).** The energy function \( E : \Omega(p, q) \to \mathbb{R} \) is given by
\[
E(\gamma) = \frac{1}{2} \int_0^T |\dot{\gamma}|^2 \, dt,
\]
where \( \gamma : [0, T] \to M \).

### 3.6 Applications

**Definition (Conjugate points).** Let \( \gamma(t) \) be a geodesic. Then
\[
p = \gamma(\alpha), \quad q = \gamma(\beta)
\]
are conjugate points if there exists some non-trivial \( J \) such that \( J(\alpha) = 0 = J(\beta) \).

**Definition (Diameter).** The diameter of a Riemannian manifold \((M, g)\) is
\[
diam(M, g) = \sup_{p, q \in M} d(p, q).
\]

**Notation.** Let \( h, \hat{h} \) be two symmetric bilinear forms on a real vector space. We say \( h \geq \hat{h} \) if \( h - \hat{h} \) is non-negative definite.

If \( h, \hat{h} \in \Gamma(S^2 T^* M) \) are fields of symmetric bilinear forms, we write \( h \geq \hat{h} \) if \( h_p \geq \hat{h}_p \) for all \( p \in M \).

**Definition (Riemannian covering map).** Let \((M, g)\) and \((\tilde{M}, \tilde{g})\) be two Riemannian manifolds, and \( f : \tilde{M} \to M \) be a smooth covering map. We say \( f \) is a Riemannian covering map if it is a local isometry. Alternatively, \( f^* g = \tilde{g} \). We say \( \tilde{M} \) is a Riemannian cover of \( M \).

**Definition (Bonnet–Myers diameter theorem).** Let \((M, g)\) be a complete \( n \)-dimensional manifold with
\[
\text{Ric}(g) \geq \frac{n - 1}{r^2} g,
\]
where \( r > 0 \) is some positive number. Then
\[
diam(M, g) \leq diam S^n(r) = \pi r.
\]
In particular, \( M \) is compact and \( \pi_1(M) \) is finite.
4 Hodge theory on Riemannian manifolds

4.1 Hodge star and operators

**Definition (Hodge star).** The *Hodge star operator* on \((M^n, g)\) is the linear map

\[
\ast : \bigwedge^p (T^*_x M) \rightarrow \bigwedge^{n-p} (T^*_x M)
\]

satisfying the property that for all \(\alpha, \beta \in \bigwedge^p (T^*_x M)\), we have

\[
\alpha \wedge \ast \beta = \langle \alpha, \beta \rangle_g \omega_g.
\]

**Definition (Co-differential (\(\delta\))).** We define \(\delta : \Omega^p (M) \rightarrow \Omega^{p-1} (M)\) for \(0 \leq p \leq \dim M\) by

\[
\delta = \begin{cases} (-1)^{n(p+1)+1} \ast d & p \neq 0 \\ 0 & p = 0 \end{cases}
\]

This is (sometimes) called the *co-differential*.

**Definition (Laplace–Beltrami operator \(\Delta\)).** The *Laplace–Beltrami operator* is

\[
\Delta = d \delta + \delta d : \Omega^p (M) \rightarrow \Omega^p (M).
\]

This is also known as the (Hodge) *Laplacian*.

**Definition (**\(L^2\) inner product**).** For \(\xi, \eta \in \Omega^p (M)\), we define the *\(L^2\) inner product* by

\[
\langle \langle \xi, \eta \rangle \rangle_g = \int_M \langle \xi, \eta \rangle_g \omega_g,
\]

where \(\xi, \eta \in \Omega^p (M)\).

**Definition (Harmonic forms).** A *harmonic form* is a \(p\)-form \(\omega\) such that \(\Delta \omega = 0\). We write

\[
\mathcal{H}^p = \{ \alpha \in \Omega^p (M) : \Delta \alpha = 0 \}.
\]

4.2 Hodge decomposition theorem

**Definition (Weak solution).** A weak solution to the equation \(\Delta \omega = \alpha\) is a linear functional \(\ell : \Omega^p (M) \rightarrow \mathbb{R}\) such that

(i) \(\ell (\Delta \varphi) = \langle \alpha, \varphi \rangle\) for all \(\varphi \in \Omega^p (M)\).

(ii) \(\ell\) is *bounded*, i.e. there is some \(C\) such that \(|\ell(\beta)| < C \|\beta\|\) for all \(\beta\).

4.3 Divergence

**Definition (Divergence).** The *divergence* of a vector field \(X \in \text{Vect}(M)\) is

\[
\text{div} X = \text{tr}(\nabla X).
\]

**Definition (Interior product).** Let \(X \in \text{Vect}(M)\). We define the *interior product* \(i(X) : \Omega^p (M) \rightarrow \Omega^{p-1} (M)\) by

\[
(i(X) \psi)(Y_1, \cdots, Y_{p-1}) = \psi(X, Y_1, \cdots, Y_{p-1}).
\]

This is sometimes written as \(i(X) \psi = X \lrcorner \psi\).
4.4 Introduction to Bochner’s method

**Definition** (Covariant Laplacian). The *covariant Laplacian* is

\[
\nabla^*\nabla : \Gamma(E) \to \Gamma(E)
\]

**Definition** (Normal frame field). A local orthonormal frame \( \{e_k\} \) field is *normal* at \( p \) if further

\[
\nabla e_k|_p = 0
\]

for all \( k \).
5 Riemannian holonomy groups

**Definition** (Holonomy transformation). The *holonomy transformation* $P(\gamma)$ sends $X_0 \in T_x M$ to $X(1) \in T_y M$.

**Definition** (Holonomy group). The *holonomy group* of $M$ at $x \in M$ is

$$\text{Hol}_x(M) = \{ P(\gamma) : \gamma \in \Omega(x, x) \} \subseteq O(T_x M).$$

The group operation is given by composition of linear maps, which corresponds to composition of paths.

**Definition** (Restricted holonomy group). We define

$$\text{Hol}_x^0(M) = \{ P(\gamma) : \gamma \in \Omega(x, x) \text{ nullhomotopic} \}.$$

**Definition** (Holonomy algebra). The *holonomy algebra* $\mathfrak{hol}(M)$ is the Lie algebra of $\text{Hol}(M)$. 
6 The Cheeger–Gromoll splitting theorem

Definition (Ray). Let \((M, g)\) be a Riemannian manifold. A ray is a map \(r(t) : [0, \infty) \to M\) if \(r\) is a geodesic, and minimizes the distance between any two points on the curve.

Definition (Line). A line is a map \(\ell(t) : \mathbb{R} \to M\) such that \(\ell(t)\) is a geodesic, and minimizes the distance between any two points on the curve.

Definition (Connected at infinity). A complete manifold is said to be connected at infinity if for all compact set \(K \subseteq M\), there exists a compact \(C \supseteq K\) such that for every two points \(p, q \in M \setminus C\), there exists a path \(\gamma \in \Omega(p, q)\) such that \(\gamma(t) \in M \setminus K\) for all \(t\).

We say \(M\) is disconnected at infinity if it is not connected at infinity.