Part III — Riemannian Geometry
Definitions

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Lent 2017

This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:

A closer look at geodesics and curvature. Brief review from the Differential Geometry course. Geodesic coordinates and Gauss' lemma. Jacobi fields, completeness and the Hopf–Rinow theorem. Variations of energy, Bonnet–Myers diameter theorem and Synge's theorem.

Hodge theory and Riemannian holonomy. The Hodge star and Laplace–Beltrami operator. The Hodge decomposition theorem (with the 'geometry part' of the proof). Bochner–Weitzenböck formulae. Holonomy groups. Interplays with curvature and de Rham cohomology.

Ricci curvature. Fundamental groups and Ricci curvature. The Cheeger–Gromoll splitting theorem.

Pre-requisites

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course Differential Geometry offered in Michaelmas Term is the ideal pre-requisite.

Contents

1 Basics of Riemannian manifolds

Definition (Riemannian metric)**.** Let M be a smooth manifold. A *Riemannian metric* g on M is an inner product on the tangent bundle TM varying smoothly with the fibers. Formally, this is a global section of $T^*M \otimes T^*M$ that is fiberwise symmetric and positive definite.

The pair (M, g) is called a *Riemannian manifold*.

Definition (Isometry)**.** Let (M, g) and (N, h) be Riemannian manifolds. We say $f : M \to N$ is an *isometry* if it is a diffeomorphism and $f^*h = g$. In other words, for any $p \in M$ and $u, v \in T_pM$, we need

$$h\big((\mathrm{d}f)_p u, (\mathrm{d}f)_p v\big) = g(u, v).$$

Definition (Left-invariant vector field)**.** Let G be a Lie group, and X a vector field. Then X is *left invariant* if for any $x \in G$, we have $\mathrm{d}(L_x)X = X$.

Definition (Levi-Civita connection)**.** Let (M, g) be a Riemannian manifold. The *Levi-Civita connection* is the unique connection $\nabla : \Omega^0_M(TM) \to \Omega^1_M(TM)$ on M satisfying

(i) Compatibility with metric:

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$

(ii) Symmetry/torsion-free:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Definition (Christoffel symbols)**.** In local coordaintes, the *Christoffel symbols* are defined by

$$\nabla_{\partial_j} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

2 Riemann curvature

Definition (Curvature)**.** Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The *curvature 2-form* is the section

$$R = -\nabla \circ \nabla \in \Gamma(\textstyle\bigwedge^2 T^* M \otimes T^* M \otimes TM) \subseteq \Gamma(T^{1,3} M).$$

Definition (Ricci curvature)**.** The *Ricci curvature* of g at $p \in M$ is

$$\operatorname{Ric}_p(X, Y) = \operatorname{tr}(v \mapsto R_p(X, v)Y).$$

In terms of coordinates, we have

$$\operatorname{Ric}_{ij} = R^q_{i,jq} = g^{pq} R_{pi,jq},$$

where g^{pq} denotes the inverse of g.

This Ric is a symmetric bilinear form on $T_p M$. This can be determined by the quadratic form

$$\operatorname{Ric}(X) = \frac{1}{n-1} \operatorname{Ric}_p(X, X).$$

The coefficient $\frac{1}{n-1}$ is just a convention.

Definition (Scalar curvature)**.** The *scalar curvature* of g is the trace of Ric respect to g. Explicitly, this is defined by

$$s = g^{ij} \operatorname{Ric}_{ij} = g^{ij} R^q_{i,jq} = R^{qi}{}_{iq}.$$

3 Geodesics

3.1 Definitions and basic properties

Definition (Lift)**.** Let $\pi : E \to M$ be a vector bundle with typical fiber V. Consider a curve $\gamma : (-\varepsilon, \varepsilon) \to M$. A *lift* of γ is a map $\gamma^E : (-\varepsilon, \varepsilon) \to E$ if $\pi \circ \gamma^E = \gamma$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & & E \\ & \overset{\gamma^E}{\nearrow} & \downarrow \pi \\ (-\varepsilon, \varepsilon) & \xrightarrow{\gamma} & M \end{array}.$$

Definition (Covariant derivative)**.** The uniquely defined operation in the proposition above is called the *covariant derivative.*

Definition (Horizontal lift)**.** Let ∇ be a connection on E with $\Gamma^i_{jk}(x)$ the coefficients in a local trivialization. We say a lift γ^E is *horizontal* if

$$\frac{\nabla \gamma^E}{\mathrm{d}t} = 0.$$

Definition (Parallel transport)**.** Let $\gamma : [0, 1] \to M$ be a curve in M. Given any $a_0 \in E_{\gamma(0)}$, the unique horizontal lift of γ with $\gamma^E(0) = (\gamma(0), a_0)$ is called the *parallel transport* of a_0 along $\gamma(0)$. We sometimes also call $\gamma^E(1)$ the parallel transport.

Definition (Geodesic)**.** A curve $\gamma(t)$ on a Riemannian manifold (M, g) is called a *geodesic curve* if its canonical lift is horizontal with respect to the Levi-Civita connection. In other words, we need

$$\frac{\nabla \dot{\gamma}}{\mathrm{d}t} = 0.$$

Definition (Exponential map)**.** Let (M, g) be a Riemannian manifold, and $p \in M$. We define $\exp_p$ by

$$\exp_p(a) = \gamma(1, a) \in M$$

for $a \in T_pM$ whenever this is defined.

3.2 Jacobi fields

Definition (Jacobi field)**.** Let $\gamma : [0, L] \to M$ be a geodesic. A *Jacobi field* is a vector field J along γ that is a solution of the *Jacobi equation* on $[0, L]$

$$\frac{\nabla^2}{\mathrm{d}t^2} J + R(\dot{\gamma}, J)\dot{\gamma} = 0. \qquad (\dagger)$$

3.3 Further properties of geodesics

Notation. We write $\Omega(p, q)$ for the set of all piecewise C^1 curves from p to q.

Definition (Distance)**.** Suppose M is connected, which is the same as it being path connected. Let $(p, q) \in M$. We define

$$d(p, q) = \inf_{\xi \in \Omega(p,q)} \text{length}(\xi),$$

where

Definition (Minimal geodesic)**.** A *minimal geodesic* is a curve $\gamma : [0, 1] \to M$ such that

$$d(\gamma(0), \gamma(1)) = \ell(\gamma).$$

3.4 Completeness and the Hopf–Rinow theorem

Definition (Geodesically complete)**.** We say a manifold (M, g) is *geodesically complete* if each geodesic extends for all time. In other words, for all $p \in M$, $\exp_p$ is defined on *all* of T_pM.

3.5 Variations of arc length and energy

Definition (Energy)**.** The *energy function* $E : \Omega(p, q) \to \mathbb{R}$ is given by

$$E(\gamma) = \frac{1}{2} \int_0^T |\dot{\gamma}|^2 \, \mathrm{d}t,$$

where $\gamma : [0, T] \to M$.

3.6 Applications

Definition (Conjugate points)**.** Let $\gamma(t)$ be a geodesic. Then

$$p = \gamma(\alpha), \quad q = \gamma(\beta)$$

are *conjugate points* if there exists some non-trivial J such that $J(\alpha) = 0 = J(\beta)$.

Definition (Diameter)**.** The *diameter* of a Riemannian manifold (M, g) is

$$\text{diam}(M, g) = \sup_{p,q \in M} d(p, q).$$

Notation. Let $h, \hat{h}$ be two symmetric bilinear forms on a real vector space. We say $h \geq \hat{h}$ if $h - \hat{h}$ is non-negative definite.

If $h, \hat{h} \in \Gamma(S^2T^*M)$ are fields of symmetric bilinear forms, we write $h \geq \hat{h}$ if $h_p \geq \hat{h}_p$ for all $p \in M$.

Definition (Riemannian covering map)**.** Let (M, g) and $(\tilde{M}, \tilde{g})$ be two Riemannian manifolds, and $f : \tilde{M} \to M$ be a smooth covering map. We say f is a *Riemannian covering map* if it is a local isometry. Alternatively, $f^*g = \tilde{g}$. We say $\tilde{M}$ is a Riemannian cover of M.

Definition (Bonnet–Myers diameter theorem)**.** Let (M, g) be a complete n-dimensional manifold with

$$\text{Ric}(g) \geq \frac{n-1}{r^2} g,$$

where $r > 0$ is some positive number. Then

$$\text{diam}(M, g) \leq \text{diam}\, S^n(r) = \pi r.$$

In particular, M is compact and $\pi_1(M)$ is finite.

4 Hodge theory on Riemannian manifolds

4.1 Hodge star and operators

Definition (Hodge star)**.** The *Hodge star operator* on (M^n, g) is the linear map

$$\star : \textstyle\bigwedge^p(T_x^* M) \to \bigwedge^{n-p}(T_x^* M)$$

satisfying the property that for all $\alpha, \beta \in \bigwedge^p(T_x^* M)$, we have

$$\alpha \wedge \star\beta = \langle \alpha, \beta\rangle_g \, \omega_g.$$

Definition (Co-differential (δ))**.** We define $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ for $0 \leq p \leq \dim M$ by

$$\delta = \begin{cases} (-1)^{n(p+1)+1} \star \mathrm{d}\star & p \neq 0 \\ 0 & p = 0 \end{cases}.$$

This is (sometimes) called the *co-differential*.

Definition (Laplace–Beltrami operator Δ)**.** The *Laplace–Beltrami operator* is

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} : \Omega^p(M) \to \Omega^p(M).$$

This is also known as the (Hodge) *Laplacian*.

Definition (L^2 inner product)**.** For $\xi, \eta \in \Omega^p(M)$, we define the L^2 *inner product* by

$$\langle\langle \xi, \eta\rangle\rangle_g = \int_M \langle \xi, \eta\rangle_g \, \omega_g,$$

where $\xi, \eta \in \Omega^p(M)$.

Definition (Harmonic forms)**.** A *harmonic form* is a p-form ω such that $\Delta\omega = 0$. We write

$$\mathcal{H}^p = \{\alpha \in \Omega^p(M) : \Delta\alpha = 0\}.$$

4.2 Hodge decomposition theorem

Definition (Weak solution)**.** A weak solution to the equation $\Delta\omega = \alpha$ is a linear functional $\ell : \Omega^p(M) \to \mathbb{R}$ such that

(i) $\ell(\Delta\varphi) = \langle \alpha, \varphi\rangle$ for all $\varphi \in \Omega^p(M)$.

(ii) ℓ is *bounded*, i.e. there is some C such that $|\ell(\beta)| < C\|\beta\|$ for all β.

4.3 Divergence

Definition (Divergence)**.** The *divergence* of a vector field $X \in \mathrm{Vect}(M)$ is

$$\mathrm{div} X = \mathrm{tr}(\nabla X).$$

Definition (Interior product)**.** Let $X \in \mathrm{Vect}(M)$. We define the *interior product* $i(X) : \Omega^p(M) \to \Omega^{p-1}(M)$ by

$$(i(X)\psi)(Y_1, \cdots, Y_{p-1}) = \psi(X, Y_1, \cdots, Y_{p-1}).$$

This is sometimes written as $i(X)\psi = X \lrcorner \psi$.

4.4 Introduction to Bochner's method

Definition (Covariant Laplacian)**.** The *covariant Laplacian* is

$$\nabla^* \nabla : \Gamma(E) \to \Gamma(E)$$

Definition (Normal frame field)**.** A local orthonormal frame $\{e_k\}$ field is *normal* at p if further

$$\nabla e_k|_p = 0$$

for all k.

5 Riemannian holonomy groups

Definition (Holonomy transformation)**.** The *holonomy transformation* $P(\gamma)$ sends $X_0 \in T_x M$ to $X(1) \in T_y M$.

Definition (Holonomy group)**.** The *holonomy group* of M at $x \in M$ is

$$\mathrm{Hol}_x(M) = \{P(\gamma) : \gamma \in \Omega(x, x)\} \subseteq \mathrm{O}(T_x M).$$

The group operation is given by composition of linear maps, which corresponds to composition of paths.

Definition (Restricted holonomy group)**.** We define

$$\mathrm{Hol}^0_x(M) = \{P(\gamma) : \gamma \in \Omega(x, x) \text{ nullhomotopic}\}.$$

Definition (Holonomy algebra)**.** The *holonomy algebra* $\mathfrak{hol}(M)$ is the Lie algebra of $\mathrm{Hol}(M)$.

6 The Cheeger–Gromoll splitting theorem

Definition (Ray)**.** Let (M, g) be a Riemannian manifold. A *ray* is a map $r(t) : [0, \infty) \to M$ if r is a geodesic, and minimizes the distance between any two points on the curve.

Definition (Line)**.** A *line* is a map $\ell(t) : \mathbb{R} \to M$ such that $\ell(t)$ is a geodesic, and minimizes the distance between any two points on the curve.

Definition (Connected at infinity)**.** A complete manifold is said to be *connected at infinity* if for all compact set $K \subseteq M$, there exists a compact $C \supseteq K$ such that for every two points $p, q \in M \setminus C$, there exists a path $\gamma \in \Omega(p, q)$ such that $\gamma(t) \in M \setminus K$ for all t.

We say M is *disconnected at infinity* if it is not connected at infinity.