Part III — Algebras
Definitions

Based on lectures by C. J. B. Brookes
Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The aim of the course is to give an introduction to algebras. The emphasis will be on non-commutative examples that arise in representation theory (of groups and Lie algebras) and the theory of algebraic D-modules, though you will learn something about commutative algebras in passing.

Topics we discuss include:

- Artinian algebras. Examples, group algebras of finite groups, crossed products. Structure theory. Artin–Wedderburn theorem. Projective modules. Blocks. K_0.
- Noetherian algebras. Examples, quantum plane and quantum torus, differential operator algebras, enveloping algebras of finite dimensional Lie algebras. Structure theory. Injective hulls, uniform dimension and Goldie's theorem.
- Hochschild chain and cochain complexes. Hochschild homology and cohomology. Gerstenhaber algebras.
- Deformation of algebras.
- Coalgebras, bialgebras and Hopf algebras.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields, or III Lie algebras will be helpful but not necessary.

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0 Introduction

Definition (k-algebra)**.** A (unital) associative k-algebra is a k-vector space A together with a linear map $m : A \otimes A \to A$, called the product map, and linear map $u : k \to A$, called the unit map, such that

- The product induced by m is associative.
- $u(1)$ is the identity of the multiplication.

Definition (Ideal)**.** A *left ideal* of A is a k-subspace of A such that if $x \in A$ and $y \in I$, then $xy \in I$. A *right ideal* is one where we require $yx \in I$ instead. An *ideal* is something that is both a left ideal and a right ideal.

Definition (Artinian algebra)**.** An algebra A is *left Artinian* if it satisfies the *descending chain condition* (*DCC*) on left ideals, i.e. if we have a descending chain of left ideals

$$I_1 \geq I_2 \geq I_3 \geq \cdots ,$$

then there is some N such that $I_{N+m} = I_N$ for all $m \geq 0$.

We say an algebra is *Artinian* if it is both left and right Artinian.

Definition (Noetherian algebra)**.** An algebra is *left Noetherian* if it satisfies the *ascending chain condition* (*ACC*) on left ideals, i.e. if

$$I_1 \leq I_2 \leq I_3 \leq \cdots$$

is an ascending chain of left ideals, then there is some N such that $I_{N+m} = I_N$ for all $m \geq 0$.

Similarly, we can define right Noetherian algebras, and say an algebra is *Noetherian* if it is both left and right Noetherian.

1 Artinian algebras

1.1 Artinian algebras

Definition (Module)**.** Let A be an algebra. A *left A-module* is a k-vector space M and a bilinear map

$$\begin{aligned} A \otimes M &\longrightarrow M \\ a \otimes m &\longmapsto xm \end{aligned}$$

such that $(ab)m = a(bm)$ for all $a, b \in A$ and $m \in M$. Right A-modules are defined similarly.

An *A-A-bimodule* is a vector space M that is both a left A-module and a right A-module, such that the two actions commute — for $a, b \in A$ and $x \in M$, we have

$$a(xb) = (ax)b.$$

Definition (Opposite algebra)**.** Let A be a k-algebra. We define the *opposite algebra* A^{op} to be the algebra with the same underlying vector space, but with multiplication given by

$$x \cdot y = yx.$$

Here on the left we have the multiplication in A^{op} and on the right we have the multiplication in A.

Definition (Prime ideal)**.** An ideal P is *prime* if it is a proper ideal, and if I and J are ideals with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

Definition (Annihilator)**.** Let M be a left A-module and $m \in M$. We define the *annihilators* to be

$$\begin{aligned} \mathrm{Ann}(m) &= \{a \in A : am = 0\} \\ \mathrm{Ann}(M) &= \{a \in A : am = 0 \text{ for all } m \in M\} = \bigcap_{m \in M} \mathrm{Ann}(m). \end{aligned}$$

Definition (Simple module)**.** A non-zero module M is *simple* or *irreducible* if the only submodules of M are 0 and M.

Definition (Jacobson radical)**.** The $J(A)$ of A is the intersection of all maximal left ideals.

Definition (Semisimple algebra)**.** An algebra is *semisimple* if $J(A) = 0$.

Definition (Simple algebra)**.** An algebra is *simple* if the only ideals are 0 and A.

Definition (Completely reducible)**.** A module M of A is *completely reducible* iff it is a sum of simple modules.

Definition (Radical)**.** For a module M, we write $\mathrm{Rad}(M)$ for the intersection of maximal submodules of M, and call it the *radical* of M.

1.2 Artin–Wedderburn theorem

Definition (Group algebra)**.** Let G be a group and k a field. The *group algebra* of G over k is

$$kG = \left\{\sum \lambda_g g : g \in G, \lambda_g \in k\right\}.$$

This has a bilinear multiplication given by the obvious formula

$$(\lambda_g g)(\mu_h h) = \lambda_g \mu_h (gh).$$

1.3 Crossed products

Definition (Crossed product)**.** The *crossed product* of a k-algebra B and a group G is specified by the following data:

- A group homomorphism $\phi : G \to \mathrm{Aut}_k(B)$, written
$$\phi_g(\lambda) = \lambda^g;$$
- A function
$$\Psi(g, h) : G \times G \to B.$$

The crossed product algebra has underlying set

$$\sum \lambda_g g : \lambda_g \in B.$$

with operation defined by

$$\lambda g \cdot \mu h = \lambda \mu^g \Psi(g, h)(gh).$$

The function Ψ is required to be such that the resulting product is associative.

Definition (Central simple algebra)**.** A *central simple k-algebra* is a finite-dimensional k-algebra which is a simple algebra, and with a center $Z(A) = k$.

1.4 Projectives and blocks

Definition (Projective module)**.** An A-module is *projective* P if given modules M and M' and maps

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \alpha} & & \\ M' & \xrightarrow{\theta} & M & \longrightarrow & 0 \end{array},$$

then there exists a map $\beta : P \to M'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \beta}\swarrow & \downarrow{\scriptstyle \alpha} & & \\ M' & \xrightarrow{\theta} & M & \longrightarrow & 0 \end{array}.$$

Equivalently, if we have a short exact sequence

$$0 \longrightarrow N \hookrightarrow M' \twoheadrightarrow M \longrightarrow 0,$$

then the sequence

$$0 \longrightarrow \mathrm{Hom}(P, N) \hookrightarrow \mathrm{Hom}(P, M') \twoheadrightarrow \mathrm{Hom}(P, M) \longrightarrow 0$$

is exact.

Definition (Indecomposable)**.** A non-zero module M is indecomposable if M cannot be expressed as the direct sum of two non-zero submodules.

Definition (Block)**.** The *blocks* are the direct summands of A that are indecomposable as ideals.

Definition (Local algebra)**.** An algebra is *local* if it has a unique maximal left ideal, which is $J(A)$, which is the unique maximal right ideal.

Definition (Unique decomposition property)**.** A module M has the *unique decomposition property* if M is a finite direct sum of indecomposable modules, and if

$$M = \bigoplus_{i=1}^{m} M_i = \bigoplus_{i=1}^{n} M_i',$$

then $n = m$, and, after reordering, $M_i = M_i'$.

Definition (Orthogonal idempotent)**.** A collection of idempotents $\{e_i\}$ is *orthogonal* if $e_i e_j = 0$ for $i \neq j$.

Definition (Primitive idempotent)**.** An idempotent is *primitive* if it cannot be expressed as a sum

$$e = e' + e'',$$

where e', e'' are orthogonal idempotents, both non-zero.

Definition (Composition factor)**.** A simple module S is a composition factor of a module M if there are submodules $M_1 \leq M_2$ with

$$M_2/M_1 \cong S.$$

1.5 K_0

Definition (K_0)**.** For any associative k-algebra A, consider the free abelian group with basis labelled by the isomorphism classes $[P]$ of finitely-generated projective A-modules. Then introduce relations

$$[P_1] + [P_2] = [P_1 \oplus P_2],$$

This yields an abelian group which is the quotient of the free abelian group by the subgroup generated by

$$[P_1] + [P_2] - [P_1 \oplus P_2].$$

The abelian group is $K_0(A)$.

Definition (Hattori-Stallings trace map)**.** The map $K_0(A) \to A/[A, A]$ induced by the trace is the *Hattori–Stallings trace map*.

2 Noetherian algebras

2.1 Noetherian algebras

Definition (Noetherian algebra). An algebra is *left Noetherian* if it satisfies the *ascending chain condition* (*ACC*) on left ideals, i.e. if

$$I_1 \leq I_2 \leq I_3 \leq \cdots$$

is an ascending chain of left ideals, then there is some N such that $I_{N+m} = I_N$ for all $m \geq 0$.

Similarly, we say an algebra is *Noetherian* if it is both left and right Noetherian.

Definition (nth Weyl algebra). The *nth Weyl algebra* $A_n(k)$ is the algebra generated by $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ with relations

$$Y_i X_i - X_i Y_i = 1,$$

for all i, and everything else commutes.

Definition (Universal enveloping algebra). Let $\mathfrak{g}$ be a Lie algebra over k, and take a k-vector space basis $x_1, \cdots, x_n$. We form an associative algebra with generators $x_1, \cdots, x_n$ with relations

$$x_i x_j - x_j x_i = [x_i, x_j],$$

and this is the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$.

Definition (Filtered algebra). A *($\mathbb{Z}$-)filtered algebra* A is a collection of k-vector spaces

$$\cdots \leq A_{-1} \leq A_0 \leq A_1 \leq A_2 \leq \cdots$$

such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$, and $1 \in A_0$.

Definition (Exhaustive filtration). A filtration is *exhaustive* if $\bigcup A_i = A$.

Definition (Separated filtration). A filtration is *separated* if $\bigcap A_i = \{0\}$.

Definition (Positive filtration). A filtration is *positive* if $A_i = 0$ for $i < 0$.

Definition (Associated graded algebra). Given a filtration of A, the *associated graded algebra* is the vector space direct sum

$$\operatorname{gr} A = \bigoplus \frac{A_i}{A_{i-1}}.$$

This is given the structure of an algebra by defining multiplication by

$$(a + A_{i-1})(b + A_{j-1}) = ab + A_{i+j-1} \in \frac{A_{i+j}}{A_{i+j-1}}.$$

Definition (Graded algebra). A ($\mathbb{Z}$-)*graded algebra* is an algebra B that is of the form

$$B = \bigoplus_{i \in \mathbb{Z}} B_i,$$

where B_i are k-subspaces, and $B_iB_j \subseteq B_{i+j}$. The B_i's are called the *homogeneous components*.

A *graded ideal* is an ideal of the form

$$\bigoplus J_i,$$

where J_i is a subspace of B_i, and similarly for left and right ideals.

Definition (Rees algebra)**.** Let A be a filtered algebra with filtration $\{A_i\}$. Then the *Rees algebra* $\mathrm{Rees}(A)$ is the subalgebra $\bigoplus A_iT^i$ of the Laurent polynomial algebra $A[T, T^{-1}]$ (where T commutes with A).

Definition (Poisson algebra)**.** An associative algebra B is a *Poisson algebra* if there is a k-bilinear bracket $\{\,\cdot\,,\,\cdot\,\} : B \times B \to B$ such that

– B is a Lie algebra under $\{\,\cdot\,,\,\cdot\,\}$, i.e.

$$\{r, s\} = -\{s, r\}$$

and

$$\{\{r, s\}, t\} + \{\{s, t\}, r\} + \{\{t, r\}, s\} = 0.$$

– We have the *Leibnitz rule*

$$\{r, st\} = s\{r, t\} + \{r, s\}t.$$

2.2 More on $A_n(k)$ and $\mathcal{U}(\mathfrak{g})$

Definition (Poincaré series)**.** Let V be a graded module over a graded algebra S, say

$$V = \bigoplus_{i=0}^{\infty} V_i.$$

Then the *Poincaré series* is

$$P(V, t) = \sum_{i=0}^{\infty} (\dim V_i) t^i.$$

Definition (Gelfand-Kirillov dimension)**.** Let $A = A_n(k)$ or $\mathcal{U}(\mathfrak{g})$ and M a finitely-generated A-module, filtered as before. Then the *Gelfand-Kirillov dimension* $d(M)$ of M is the degree of the Samuel polynomial of $\operatorname{gr} M$ as a $\operatorname{gr} A$-module.

Definition (Gelfand-Kirillov dimension)**.** Let A be a finitely-generated k-algebra, which is filtered as before, and a finitely-generated A-module M, filtered as before. Then the GK-dimension of M is

$$d(M) = \limsup_{n \to \infty} \frac{\log(\dim M_n)}{\log n}.$$

Definition (Multiplicity)**.** Let A be a commutative algebra, and M an A-module. The *multiplicity* of M with $d(M) = d$ is

$$d! \times \text{leading coefficient of } \chi(t).$$

Definition (Holonomic module)**.** An $A_n(k)$ module M is *holonomic* iff $d(M) = n$.

2.3 Injective modules and Goldie's theorem

Definition (Injective module)**.** An A-module E is *injective* if for every diagram of A-module maps

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \overset{\theta}{\rightarrowtail} & N \\ & & \downarrow{\scriptstyle \phi} & \swarrow{\scriptstyle \psi} & \\ & & E & & \end{array},$$

such that θ is injective, there exists a map ψ that makes the diagram commute. Equivalently, $\mathrm{Hom}(\,\cdot\,, E)$ is an exact functor.

Definition (Essential submodule)**.** An *essential submodule* M of an A-module N is one where $M \cap V \neq \{0\}$ for every non-zero submodule V of N. We say N is an *essential extension* of M.

Definition (Injective hull)**.** A maximal essential extension of M is the *injective hull* (or *injective envelope*) of M, written $E(M)$.

Definition (Uniform module)**.** A non-zero module V is *uniform* if given non-zero submodules V_1, V_2, then $V_1 \cap V_2 \neq \{0\}$.

Definition (Domain)**.** An algebra is a *domain* if $xy = 0$ implies $x = 0$ or $y = 0$.

Definition (Uniform dimension)**.** The *uniform dimension*, or *Goldie rank* of M is the number of indecomposable direct summands of $E(M)$.

3 Hochschild homology and cohomology

3.1 Introduction

Definition (Free resolution)**.** Let A be an algebra and M an A-A-bimodule. A *free resolution* of M is an exact sequence of the form

$$\cdots \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \longrightarrow M \ ,$$

where each F_n is a free A-A-bimodule.

Definition (Hochschild chain complex)**.** Let A be a k-algebra with multiplication map $\mu : A \otimes A$. The *Hochschild chain complex* is

$$\cdots \xrightarrow{d_1} A \otimes_k A \otimes_k A \xrightarrow{d_0} A \otimes_k A \xrightarrow{\mu} A \longrightarrow 0.$$

We refer to $A^{\otimes_k (n+2)}$ as the *degree* n term. The differential $d : A^{\otimes_k (n+3)} \to A^{\otimes_k (n+2)}$ is given by

$$d(a_0 \otimes_k \cdots \otimes_k a_{n+1}) = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes_k \cdots \otimes_k a_i a_{i+1} \otimes_k \cdots \otimes_k a_{n+2}.$$

3.2 Cohomology

Definition (Hochschild cochain complex)**.** The *Hochschild cochain complex* of an A-A-bimodule M is what we obtain by applying $\mathrm{Hom}_{A\text{-}A}(\,\cdot\,, M)$ to the Hochschild chain complex of A. Explicitly, we can write it as

$$\mathrm{Hom}_k(k, M) \xrightarrow{\delta_0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta_1} \mathrm{Hom}_k(A \otimes A, M) \longrightarrow \cdots ,$$

where

$$\begin{aligned}
(\delta_0 f)(a) &= af(1) - f(1)a\\
(\delta_1 f)(a_1 \otimes a_2) &= a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2\\
(\delta_2 f)(a_1 \otimes a_2 \otimes a_3) &= a_1 f(a_2 \otimes a_3) - f(a_1 a_2 \otimes a_3)\\
&\quad + f(a_1 \otimes a_2 a_3) - f(a_1 \otimes a_2) a_3\\
(\delta_{n-1} f)(a_1 \otimes \cdots \otimes a_n) &= a_1 f(a_2 \otimes \cdots \otimes a_n)\\
&\quad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)\\
&\quad + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_{n-1}) a_n
\end{aligned}$$

The reason the end ones look different is that we replaced $\mathrm{Hom}_{A\text{-}A}(A^{\otimes (n+2)}, M)$ with $\mathrm{Hom}_k(A^{\otimes n}, M)$.

Definition (Cocycles)**.** The *cocycles* are the elements in $\ker \delta_n$.

Definition (Coboundaries)**.** The *coboundaries* are the elements in $\operatorname{im} \delta_n$.

Definition (Hochschild cohomology groups)**.** We define

$$HH^0(A, M) = \ker \delta_0$$
$$HH^n(A, N) = \frac{\ker \delta_n}{\operatorname{im} \delta_{n-1}}$$

These are k-vector spaces.

Definition (Dimension)**.** The *dimension* of an algebra A is

$$\operatorname{Dim} A = \sup\{n : HH^n(A, M) \neq 0 \text{ for some } A\text{-}A\text{-bimodule } M\}.$$

This can be infinite if such a sup does not exist.

Definition (k-separable)**.** An algebra A is *k-separable* if ${}_A A_A$ embeds as a direct summand of $A \otimes A$.

Definition (Semi-direct product)**.** Let M be an A-A-bimodule. We can form the semi-direct product of A and M, written $A \ltimes M$, which is an algebra with elements $(a, m) \in A \times M$, and multiplication given by

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

Addition is given by the obvious one.

Alternatively, we can write

$$A \ltimes M \cong A + M\varepsilon,$$

where ε commutes with everything and $\varepsilon^2 = 0$. Then $M\varepsilon$ forms an ideal with $(M\varepsilon)^2 = 0$.

Definition (Extension)**.** Let A be an algebra and M and A-A-bimodule. An extension of A by M. An *extension* of A by M is a k-algebra B containing a 2-sided ideal I such that

- $I^2 = 0$;
- $B/I \cong A$; and
- $M \cong I$ as an A-A-bimodule.

Note that since $I^2 = 0$, the left and right multiplication in B induces an A-A-bimodule structure on I, rather than just a B-B-bimodule.

3.3 Star products

Definition (Star product)**.** Let A be a k-algebra, and let V be the underlying vector space. A *star product* is an associative $k[[t]]$-bilinear product on $V[[t]]$ that reduces to the multiplication on A when we set $t = 0$.

Definition (Integrable 2-cocycle)**.** Let $f : A \otimes A \to A$ be a 2-cocycle. Then it is *integrable* if it is the F_1 of a star product.

Definition (Equivalence of star proeducts)**.** Two star products f and g are *equivalent* on $V \otimes k[[t]]$ if there is a $k[[t]]$-linear automorphism Φ of $V[[t]]$ of the form

$$\Phi(a) = a + t\phi_1(a) + t^2\phi_2(a) + \cdots$$

sch that

$$f(a, b) = \Phi^{-1}g(\Phi(a), \Phi(b)).$$

Equivalently, the following diagram has to commute:

$$\begin{CD} V[[t]] \otimes V[[t]] @>{f}>> V[[t]] \\ @V{\Phi\otimes\Phi}VV @VV{\Phi}V \\ V[[t]] \otimes V[[t]] @>{g}>> V[[t]] \end{CD}$$

Star products equivalent to the usual product on $A \otimes k[[t]]$ are called *trivial*.

Definition (Integrable derivation)**.** We say a derivation is *integrable* if there is an automorphism of $A \otimes k[[t]]$ that gives the derivation when we mod t^2.

3.4 Gerstenhaber algebra

Definition (Cup product)**.** The *cup product*

$$\smile : S^m(A, A) \otimes S^n(A, A) \to S^{m+n}(A, A)$$

is defined by

$$(f \smile g)(a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n) = f(a_1 \otimes \cdots \otimes a_m) \cdot g(b_1 \otimes \cdots \otimes b_n),$$

where $a_i, b_j \in A$.

Definition (Gerstenhaber bracket)**.** The *Gerstenhaber bracket* is

$$[f, g] = f \circ g - (-1)^{(n+1)(m+1)} g \circ f$$

Definition (Graded Lie algebra)**.** A *graded Lie algebra* is a vector space

$$L = \bigoplus L_i$$

with a bilinear bracket $[\,\cdot\,,\,\cdot\,] : L \times L \to L$ such that

- $[L_i, L_j] \subseteq L_{i+j}$;
- $[f, g] - (-1)^{mn}[g, f]$; and
- The *graded Jacobi identity* holds:

$$(-1)^{mp}[[f, g], h] + (-1)^{mn}[[g, h], f] + (-1)^{np}[[h, f], g] = 0$$

where $f \in L_m$, $g \in L_n$, $h \in L_p$.

Definition (Gerstenhaber algebra)**.** A *Gerstenhaber algebra* is a graded vector space

$$H = \bigoplus H^i$$

with $H^{\bullet+1}$ a graded Lie algebra with respect to a bracket $[\,\cdot\,,\,\cdot\,] : H^m \times H^n \to H^{m+n-1}$, and an associative product $\smile: H^m \times H^n \to H^{m+n}$ which is graded commutative, such that if $f \in H^m$, then $[f,\,\cdot\,]$ acts as a degree $m-1$ graded derivation of $\smile$:

$$[f, g \smile h] = [f, g] \smile h + (-1)^{(m-1)}ng \smile [f, h]$$

if $g \in H^n$.

Definition (Maurer–Cartan equation)**.** The *Maurer–Cartan equation* is

$$\delta f + \frac{1}{2}[f, f]_{\mathrm{Gerst}} = 0$$

for the element

$$f = \sum t^\lambda F_\lambda,$$

where $F_0(a, b) = ab$.

3.5 Hochschild homology

Definition (Hochschild homology)**.** The *Hochschild homology* groups are

$$HH_0(A, M) = \frac{M}{\operatorname{im} b_0}$$
$$HH_i(A, M) = \frac{\ker b_{i-1}}{\operatorname{im} b_i}$$

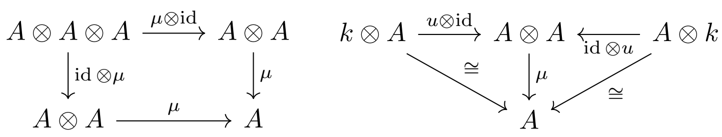
for $i > 0$.

4 Coalgebras, bialgebras and Hopf algebras

Definition (Algebra)**.** A *k-algebra* is a k-vector space A and k-linear maps

$$\mu : A \otimes A \to A \qquad u : k \to A$$
$$x \otimes y \mapsto xy \qquad \lambda \mapsto \lambda I$$

called the *multiplication/product* and *unit* such that the following two diagrams commute:

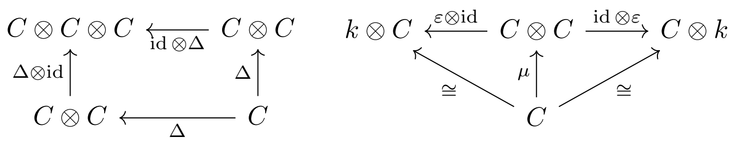


These encode *associativity* and *identity* respectively.

Definition (Coalgebra)**.** A *coalgebra* is a k-vector space C and k-linear maps

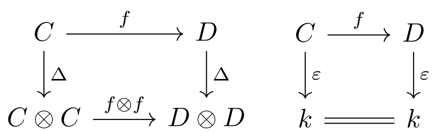
$$\Delta : C \to C \otimes C \qquad \varepsilon : C \to k$$

called *comultiplication/coproduct* and *counit* respectively, such that the following diagrams commute:



These encode *coassociativity* and *coidentity*

A *morphism* of coalgebras $f : C \to D$ is a k-linear map such that the following diagrams commute:



A subspace I of C is a *co-ideal* if $\Delta(I) \leq C \otimes I + I \otimes C$, and $\varepsilon(I) = 0$. In this case, C/I inherits a coproduct and counit.

A *cocommutative coalgebra* is one for which $\tau \circ \Delta = \Delta$, where $\tau : V \otimes W \to W \otimes V$ given by the $v \otimes w \mapsto w \otimes v$ is the "twist map".

Definition (Bialgebra)**.** A *bialgebra* is a k-vector space B and maps $\mu, \upsilon, \Delta, \varepsilon$ such that

(i) (B, μ, u) is an algebra.

(ii) (B, Δ, ε) is a coalgebra.

(iii) Δ and ε are algebra morphisms.

(iv) μ and u are coalgebra morphisms.

Definition (Hopf algebra)**.** A bialgebra $(H, \mu, u, \Delta, \varepsilon)$ is a *Hopf algebra* if there is an *antipode* $S : H \to H$ that is a k-linear map such that

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = u \circ \varepsilon.$$

Definition (Drinfeld double)**.** Let G be a finite group. We define

$$D(G) = (kG)^* \otimes_k kG$$

as a vector space, and the algebra structure is given by the crossed product $(kG)^* \rtimes G$, where G acts on $(kG)^*$ by

$$f^g(x) = f(gxg^{-1}).$$

Then the product is given by

$$(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 f_2^{g_1^{-1}} \otimes g_1 g_2.$$

The coalgebra structure is the tensor of the two coalgebras $(kG)^*$ and kG, with

$$\Delta(\phi_g \otimes h) = \sum_{g_1 g_2 = g} \phi_{g_1} \otimes h \otimes \phi_{g_2} \otimes h.$$

$D(G)$ is *quasitriangular*, i.e. there is an invertible element R of $D(G) \otimes D(G)$ such that

$$R\Delta(x)R^{-1} = \tau(\Delta(x)),$$

where τ is the twist map. This is given by

$$R = \sum_g (\phi_g \otimes 1) \otimes (1 \otimes g)$$

$$R^1 = \sum_g (\phi_g \otimes 1) \otimes (1 \otimes g^{-1}).$$

The equation $R\Delta R^{-1} = \tau\Delta$ results in an isomorphism between $U \otimes V$ and $V \otimes U$ for $D(G)$-bimodules U and V, given by flip follows by the action of R.

Definition (Yang–Baxter equation)**.** R satisfies the *quantum Yang–Baxter equation* (*QYBE*) if

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and the braided form of QYBE (*braid equation*) if

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

Definition (R-symmetric algebra)**.** Given the tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

we form the R-symmetric algebra

$$S_R(V) = \frac{T(V)}{\langle z - R(z) : z \in V \otimes V \rangle}.$$

Definition (Coordinate algebra of quantum matrices)**.** The coordinate algebra of quantum matrices associated with R is

$$\frac{T(E)}{\langle R_{13}(z) - R_{24}^*(z) : z \in E \otimes E \rangle} = S_R(E),$$

where

$$T = R_{24}^* R_{13}^{-1}.$$

The coalgebra structure remains the same as $\mathcal{O}(M_n(k))$, and for the antipode, we write E_1 for the image of e_1 in $S_R(V)$, and similarly F_j for f_j. Then we map

$$\begin{aligned} E_1 &\mapsto \sum_{j=1}^n X_{ij} \otimes E_j \\ F_j &\mapsto \sum_{i=1}^n F_i \otimes X_{ij}. \end{aligned}$$