Part IB — Quantum Mechanics
Theorems

Based on lectures by J. M. Evans
Notes taken by Dexter Chua

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Physical background
Photoelectric effect. Electrons in atoms and line spectra. Particle diffraction. [1]

Schrödinger equation and solutions
De Broglie waves. Schrödinger equation. Superposition principle. Probability interpretation, density and current. [2]

Stationary states. Free particle, Gaussian wave packet. Motion in 1-dimensional potentials, parity. Potential step, square well and barrier. Harmonic oscillator. [4]

Observables and expectation values
Position and momentum operators and expectation values. Canonical commutation relations. Uncertainty principle. [2]

Observables and Hermitian operators. Eigenvalues and eigenfunctions. Formula for expectation value. [2]

Hydrogen atom
Spherically symmetric wave functions for spherical well and hydrogen atom.

Orbital angular momentum operators. General solution of hydrogen atom. [5]

Contents

0 Introduction

0.1 Light quanta

0.2 Bohr model of the atom

0.3 Matter waves

1 Wavefunctions and the Schrödinger equation

1.1 Particle state and probability

1.2 Operators

1.3 Time evolution of wavefunctions

Proposition. The probability density

$$P(x,t) = |\Psi(x,t)|^2$$

obeys a conservation equation

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x},$$

where

$$j(x,t) = -\frac{i\hbar}{2m}\left(\Psi^* \frac{\mathrm{d}\Psi}{\mathrm{d}x} - \frac{\mathrm{d}\Psi^*}{\mathrm{d}x}\Psi\right)$$

is the *probability current.*

2 Some examples in one dimension

2.1 Introduction

2.2 Infinite well — particle in a box

2.3 Parity

2.4 Potential well

2.5 The harmonic oscillator

3 Expectation and uncertainty

3.1 Inner products and expectation values

Proposition. The operators $\hat{x}$, $\hat{p}$ and H are all Hermitian (for real potentials).

Proposition (Cauchy-Schwarz inequality)**.** If ψ and ϕ are any normalizable states, then

$$\|\psi\|\|\phi\| \geq |(\psi, \phi)|.$$

3.2 Ehrenfest's theorem

Theorem (Ehrenfest's theorem)**.**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{x}\rangle_\Psi &= \frac{1}{m}\langle \hat{p}\rangle_\Psi \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{p}\rangle_\Psi &= -\langle V'(\hat{x})\rangle_\Psi.
\end{aligned}$$

3.3 Heisenberg's uncertainty principle

Theorem (Heisenberg's uncertainty principle)**.** If ψ is any normalized state (at any fixed time), then

$$(\Delta x)_\psi (\Delta p)_\psi \geq \frac{\hbar}{2}.$$

4 More results in one dimensions

4.1 Gaussian wavepackets

4.2 Scattering

4.3 Potential step

4.4 Potential barrier

4.5 General features of stationary states

5 Axioms for quantum mechanics

5.1 States and observables

5.2 Measurements

Proposition. Let Q be Hermitian (an observable), i.e. $Q^\dagger = Q$. Then

(i) Eigenvalues of Q are real.

(ii) Eigenstates of Q with different eigenvalues are orthogonal (with respect to the complex inner product).

(iii) Any state can be written as a (possibly infinite) linear combination of eigenstates of Q, i.e. eigenstates of Q provide a basis for V. Alternatively, the set of eigenstates is *complete*.

Proposition.

(i) The expectation value of Q in state ψ is

$$\langle Q\rangle_\psi = (\psi, Q\psi) = \sum \lambda_n P_n,$$

with notation as in the previous part.

(ii) The uncertainty $(\delta Q)_\psi$ is given by

$$(\Delta Q)^2_\psi = \langle (Q - \langle Q\rangle_\psi)^2\rangle_\psi = \langle Q^2\rangle_\psi - \langle Q\rangle^2_\psi = \sum_n (\lambda_n - \langle Q\rangle_\psi)^2 P_n.$$

5.3 Evolution in time

Theorem (Ehrenfest's theorem)**.** If Q is any operator with no explicit time dependence, then

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle_\Psi = \langle [Q, H]\rangle_\Psi,$$

where

$$[Q, H] = QH - HQ$$

is the commutator.

5.4 Discrete and continuous spectra

5.5 Degeneracy and simultaneous measurements

6 Quantum mechanics in three dimensions

6.1 Introduction

6.2 Separable eigenstate solutions

6.3 Angular momentum

Proposition.

(i) $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$.

(ii) $[\mathbf{L}^2, L_i] = 0$.

(iii) $[L_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k$ and $[L_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}\hat{p}_k$

6.4 Joint eigenstates for a spherically symmetric potential

7 The hydrogen atom

7.1 Introduction

7.2 General solution

7.3 Comments