

Part IB — Quantum Mechanics

Theorems

Based on lectures by J. M. Evans

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Physical background

Photoelectric effect. Electrons in atoms and line spectra. Particle diffraction. [1]

Schrödinger equation and solutions

De Broglie waves. Schrödinger equation. Superposition principle. Probability interpretation, density and current. [2]

Stationary states. Free particle, Gaussian wave packet. Motion in 1-dimensional potentials, parity. Potential step, square well and barrier. Harmonic oscillator. [4]

Observables and expectation values

Position and momentum operators and expectation values. Canonical commutation relations. Uncertainty principle. [2]

Observables and Hermitian operators. Eigenvalues and eigenfunctions. Formula for expectation value. [2]

Hydrogen atom

Spherically symmetric wave functions for spherical well and hydrogen atom.

Orbital angular momentum operators. General solution of hydrogen atom. [5]

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0 Introduction

0.1 Light quanta

0.2 Bohr model of the atom

0.3 Matter waves

1 Wavefunctions and the Schrödinger equation

1.1 Particle state and probability

1.2 Operators

1.3 Time evolution of wavefunctions

Proposition. The probability density

$$P(x, t) = |\Psi(x, t)|^2$$

obeys a conservation equation

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x},$$

where

$$j(x, t) = -\frac{i\hbar}{2m} \left(\Psi^* \frac{d\Psi}{dx} - \frac{d\Psi^*}{dx} \Psi \right)$$

is the *probability current*.

2 Some examples in one dimension

2.1 Introduction

2.2 Infinite well — particle in a box

2.3 Parity

2.4 Potential well

2.5 The harmonic oscillator

3 Expectation and uncertainty

3.1 Inner products and expectation values

Proposition. The operators \hat{x} , \hat{p} and H are all Hermitian (for real potentials).

Proposition (Cauchy-Schwarz inequality). If ψ and ϕ are any normalizable states, then

$$\|\psi\|\|\phi\| \geq |(\psi, \phi)|.$$

3.2 Ehrenfest's theorem

Theorem (Ehrenfest's theorem).

$$\begin{aligned}\frac{d}{dt}\langle\hat{x}\rangle_{\Psi} &= \frac{1}{m}\langle\hat{p}\rangle_{\Psi} \\ \frac{d}{dt}\langle\hat{p}\rangle_{\Psi} &= -\langle V'(\hat{x})\rangle_{\Psi}.\end{aligned}$$

3.3 Heisenberg's uncertainty principle

Theorem (Heisenberg's uncertainty principle). If ψ is any normalized state (at any fixed time), then

$$(\Delta x)_{\psi}(\Delta p)_{\psi} \geq \frac{\hbar}{2}.$$

4 More results in one dimensions

4.1 Gaussian wavepackets

4.2 Scattering

4.3 Potential step

4.4 Potential barrier

4.5 General features of stationary states

5 Axioms for quantum mechanics

5.1 States and observables

5.2 Measurements

Proposition. Let Q be Hermitian (an observable), i.e. $Q^\dagger = Q$. Then

- (i) Eigenvalues of Q are real.
- (ii) Eigenstates of Q with different eigenvalues are orthogonal (with respect to the complex inner product).
- (iii) Any state can be written as a (possibly infinite) linear combination of eigenstates of Q , i.e. eigenstates of Q provide a basis for V . Alternatively, the set of eigenstates is *complete*.

Proposition.

- (i) The expectation value of Q in state ψ is

$$\langle Q \rangle_\psi = (\psi, Q\psi) = \sum \lambda_n P_n,$$

with notation as in the previous part.

- (ii) The uncertainty $(\delta Q)_\psi$ is given by

$$(\Delta Q)_\psi^2 = \langle (Q - \langle Q \rangle_\psi)^2 \rangle_\psi = \langle Q^2 \rangle_\psi - \langle Q \rangle_\psi^2 = \sum_n (\lambda_n - \langle Q \rangle_\psi)^2 P_n.$$

5.3 Evolution in time

Theorem (Ehrenfest's theorem). If Q is any operator with no explicit time dependence, then

$$i\hbar \frac{d}{dt} \langle Q \rangle_\Psi = \langle [Q, H] \rangle_\Psi,$$

where

$$[Q, H] = QH - HQ$$

is the commutator.

5.4 Discrete and continuous spectra

5.5 Degeneracy and simultaneous measurements

6 Quantum mechanics in three dimensions

6.1 Introduction

6.2 Separable eigenstate solutions

6.3 Angular momentum

Proposition.

(i) $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$.

(ii) $[\mathbf{L}^2, L_i] = 0$.

(iii) $[L_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k$ and $[L_i, \hat{p}_j] = i\hbar\varepsilon_{ijk}\hat{p}_k$

6.4 Joint eigenstates for a spherically symmetric potential

7 The hydrogen atom

7.1 Introduction

7.2 General solution

7.3 Comments