Part IB — Markov Chains

Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Discrete-time chains
Definition and basic properties, the transition matrix. Calculation of n-step transition probabilities. Communicating classes, closed classes, absorption, irreducibility. Calculation of hitting probabilities and mean hitting times; survival probability for birth and death chains. Stopping times and statement of the strong Markov property. [5]
Recurrence and transience; equivalence of transience and summability of n-step transition probabilities; equivalence of recurrence and certainty of return. Recurrence as a class property, relation with closed classes. Simple random walks in dimensions one, two and three. [3]
Invariant distributions, statement of existence and uniqueness up to constant multiples. Mean return time, positive recurrence; equivalence of positive recurrence and the existence of an invariant distribution. Convergence to equilibrium for irreducible, positive recurrent, aperiodic chains *and proof by coupling*. Long-run proportion of time spent in a given state. [3]
Time reversal, detailed balance, reversibility, random walk on a graph. [1]
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0 Introduction
1 Markov chains

1.1 The Markov property

**Definition** (Markov chain). Let $X = (X_0, X_1, \cdots)$ be a sequence of random variables taking values in some set $S$, the *state space*. We assume that $S$ is countable (which could be finite).

We say $X$ has the *Markov property* if for all $n \geq 0, i_0, \cdots, i_{n+1} \in S$, we have

$$
P(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = P(X_{n+1} = i_{n+1} \mid X_n = i_n).$$

If $X$ has the Markov property, we call it a *Markov chain*.

We say that a Markov chain $X$ is *homogeneous* if the conditional probabilities $P(X_{n+1} = j \mid X_n = i)$ do not depend on $n$.

1.2 Transition probability

**Definition** ($n$-step transition probability). The $n$-step transition probability from $i$ to $j$ is

$$p_{i,j}(n) = P(X_n = j \mid X_0 = i).$$

**Notation.** Write $P(m) = (p_{i,j}(m))_{i,j \in S}$. 

2 Classification of chains and states

2.1 Communicating classes

Definition (Leading to and communicate). Suppose we have two states $i, j \in S$. We write $i \rightarrow j$ ($i$ leads to $j$) if there is some $n \geq 0$ such that $p_{i,j}(n) > 0$, i.e. it is possible for us to get from $i$ to $j$ (in multiple steps). Note that we allow $n = 0$. So we always have $i \rightarrow i$.

We write $i \leftrightarrow j$ if $i \rightarrow j$ and $j \rightarrow i$. If $i \leftrightarrow j$, we say $i$ and $j$ communicate.

Definition (Communicating classes). The equivalence classes of $\leftrightarrow$ are communicating classes.

Definition (Irreducible chain). A Markov chain is irreducible if there is a unique communication class.

Definition (Closed). A subset $C \subseteq S$ is closed if $p_{i,j} = 0$ for all $i \in C, j \notin C$.

2.2 Recurrence or transience

Notation. For convenience, we will introduce some notations. We write

$$\mathbb{P}_i(A) = \mathbb{P}(A \mid X_0 = i),$$

and

$$\mathbb{E}_i(Z) = \mathbb{E}(Z \mid X_0 = i).$$

Definition (First passage time and probability). The first passage time of $j \in S$ starting from $i$ is

$$T_j = \min\{n \geq 1 : X_n = j\}.$$

Note that this implicitly depends on $i$. Here we require $n \geq 1$. Otherwise $T_i$ would always be 0.

The first passage probability is

$$f_{ij}(n) = \mathbb{P}_i(T_j = n).$$

Definition (Recurrent state). A state $i \in S$ is recurrent (or persistent) if

$$\mathbb{P}_i(T_i < \infty) = 1,$$

i.e. we will eventually get back to the state. Otherwise, we call the state transient.

2.3 Hitting probabilities

Definition (Hitting time). The hitting time of $A \subseteq S$ is the random variable $H^A = \min\{n \geq 0 : X_n \in A\}$. In particular, if we start in $A$, then $H^A = 0$. We also have

$$h^A_i = \mathbb{P}_i(H^A < \infty) = \mathbb{P}_i(\text{ever reach } A).$$

2.4 The strong Markov property and applications

Definition (Stopping time). Let $X$ be a Markov chain. A random variable $T$ (which is a function $\Omega \rightarrow \mathbb{N} \cup \{\infty\}$) is a stopping time for the chain $X = (X_n)$ if for $n \geq 0$, the event $\{T = n\}$ is given in terms of $X_0, \cdots, X_n$. 

2.5 Further classification of states

**Definition** (Mean recurrence time). Let \( T_i \) be the returning time to a state \( i \). Then the *mean recurrence time* of \( i \) is

\[
\mu_i = E_i(T_i) = \begin{cases} 
\infty & \text{if } i \text{ transient} \\
\sum_{n=1}^{\infty} nf_{i,i}(n) & \text{if } i \text{ recurrent}
\end{cases}
\]

**Definition** (Null and positive state). If \( i \) is recurrent, we call \( i \) a *null state* if \( \mu_i = \infty \). Otherwise \( i \) is *non-null* or *positive*.

**Definition** (Period). The *period* of a state \( i \) is \( d_i = \gcd\{n \geq 1 : p_{i,i}(n) > 0\} \).

A state is *aperiodic* if \( d_i = 1 \).

**Definition** (Ergodic state). A state \( i \) is *ergodic* if it is aperiodic and positive recurrent.
3 Long-run behaviour

3.1 Invariant distributions

Definition (Invariant distribution). Let $X_j$ be a Markov chain with transition probabilities $P$. The distribution $\pi = (\pi_k : k \in S)$ is an invariant distribution if

(i) $\pi_k \geq 0$, $\sum_k \pi_k = 1$.

(ii) $\pi = \pi P$.

The first condition just ensures that this is a genuine distribution.

An invariant distribution is also known as an invariant measure, equilibrium distribution or steady-state distribution.

3.2 Convergence to equilibrium
4 Time reversal

Definition (Reversible chain). An irreducible Markov chain $X = (X_0, \cdots, X_N)$ in its invariant distribution $\pi$ is reversible if its reversal has the same transition probabilities as does $X$, i.e.

$$\pi_i p_{i,j} = \pi_j p_{j,i}$$

for all $i, j \in S$.

This equation is known as the detailed balance equation. In general, if $\lambda$ is a distribution that satisfies

$$\lambda_i p_{i,j} = \lambda_j p_{j,i},$$

we say $(P, \lambda)$ is in detailed balance.