1. Let $\mathbb{R}^2$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^2$?

(a) The set $C$ of continuous functions.
(b) The set $\{ f \in C : |f(t)| \leq 1 \text{ for all } t \in [0, 1] \}$.
(c) The set $\{ f \in C : f(t) \to 0 \text{ as } t \to \infty \}$.
(d) The set $\{ f \in C : f(t) \to 1 \text{ as } t \to \infty \}$.
(e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4x(t) = 0$.
(f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4x(t) = \sin t$.
(g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(h) The set of solutions of $(\dot{x}(t))^2 + (x(t))^2 = 0$.

2. Suppose that the vectors $e_1, \ldots, e_n$ form a basis for $V$. Which of the following are also bases?

(a) $e_1 + e_2, e_2 + e_3, \ldots, e_{n-1} + e_n, e_n$;
(b) $e_1 + e_2, e_2 + e_3, \ldots, e_{n-1} + e_n, e_n + e_1$;
(c) $e_1 - e_n, e_2 + e_{n-1}, \ldots, e_n + (-1)^n e_1$.

3. Let $T$, $U$ and $W$ be subspaces of $V$.

(i) Show that $T \cup U$ is a subspace of $V$ only if either $T \subseteq U$ or $U \subseteq T$.
(ii) Give explicit counter-examples to the following statements:

(a) $T + (U \cap W) = (T + U) \cap (T + W)$;
(b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.
(iii) Show that each of the equalities in (ii) cannot be replaced by a valid inclusion of one side in the other.

4. For each of the following pairs of vector spaces $(V, W)$ over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here $P$ denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a,b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.)

(a) $V = \mathbb{R}^4$, $W = \{ x \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0 \}$.
(b) $V = \mathbb{R}^5$, $W = \{ p \in P : \deg p \leq 5 \}$.
(c) $V = C[0,1]$, $W = C[-1,1]$.
(d) $V = C[0,1]$, $W = \{ f \in C[0,1] : f(0) = 0, f \text{ continuously differentiable} \}$.
(e) $V = \mathbb{R}^2$, $W = \{ \text{ solutions of } \ddot{x}(t) + x(t) = 0 \}$.
(f) $V = \mathbb{R}^4$, $W = C[0,1]$.
(g) (Harder:) $V = P$, $W = \mathbb{R}^3$.

5. (i) If $\alpha$ and $\beta$ are linear maps from $U$ to $V$ show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements:

(a) $\text{Im}(\alpha + \beta) = \text{Im}(\alpha) + \text{Im}(\beta)$;
(b) $\text{Ker}(\alpha + \beta) = \text{Ker}(\alpha) \cap \text{Ker}(\beta)$.

Show that in general each of these equalities cannot be replaced by a valid inclusion of one side in the other.
(ii) Let $\alpha$ be a linear map from $V$ to $V$. Show that if $\alpha^2 = \alpha$ then $V = \text{Ker}(\alpha) \oplus \text{Im}(\alpha)$. Does your proof still hold if $V$ is infinite dimensional? Is the result still true?

6. Let $U = \{ x \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0, \ 2x_1 + 2x_2 + x_5 = 0 \}$, $W = \{ x \in \mathbb{R}^5 : x_1 + x_5 = 0, \ x_2 = x_3 = x_4 \}$.

Find bases for $U$ and $W$ containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that $U + W = \{ x \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4 \}$.
7. Let $\alpha: U \to V$ be a linear map between two finite dimensional vector spaces and let $W$ be a vector subspace of $U$. Show that the restriction of $\alpha$ to $W$ is a linear map $\alpha|_W: W \to V$ which satisfies

$$r(\alpha) \geq r(\alpha|_W) \geq r(\alpha) - \dim(U) + \dim(W).$$

Give examples (with $W \neq U$) to show that either of the two inequalities can be an equality.

8. (i) Let $\alpha: V \to V$ be an endomorphism of a finite dimensional vector space $V$. Show that

$$V \geq \text{Im}(\alpha) \geq \text{Im}(\alpha^2) \geq \ldots \quad \text{and} \quad \{0\} \leq \text{Ker}(\alpha) \leq \text{Ker}(\alpha^2) \leq \ldots.$$

If $r_k = r(\alpha^k)$, deduce that $r_k \geq r_{k+1}$ and that $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$. Conclude that if, for some $k \geq 0$, we have $r_k = r_{k+1}$, then $r_k = r_{k+1} = \ldots$ for all $\ell \geq 0$.

(ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $r(\alpha)$ and $r(\alpha^2)$?

9. Let $\alpha: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing $\alpha$ relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range. Write down bases for the domain and range with respect to which the matrix of $\alpha$ is the identity.

10. Let $U_1, \ldots, U_k$ be subspaces of a vector space $V$ and let $B_i$ be a basis for $U_i$. Show that the following statements are equivalent:

   (i) $U = \sum_i U_i$ is a direct sum, i.e. every element of $U$ can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.

   (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all $j$.

   (iii) The $B_i$ are pairwise disjoint and their union is a basis for $\sum_i U_i$.

   Give an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

11. Let $Y$ and $Z$ be subspaces of the finite dimensional vector spaces $V$ and $W$, respectively. Show that $R = \{\alpha \in \mathcal{L}(V,W) : \alpha(Y) \subseteq Z\}$ is a subspace of the space $\mathcal{L}(V,W)$ of all linear maps from $V$ to $W$. What is the dimension of $R$?

12. Recall that $\mathbb{F}^n$ has standard basis $e_1, \ldots, e_n$. Let $U$ be a subspace of $\mathbb{F}^n$. Show that there is a subset $I$ of $\{1, 2, \ldots, n\}$ for which the subspace $W = \langle \{e_i : i \in I\} \rangle$ is a complementary subspace to $U$ in $\mathbb{F}^n$.

13. Suppose $X$ and $Y$ are linearly independent subsets of a vector space $V$; no member of $X$ is expressible as a linear combination of members of $Y$, and no member of $Y$ is expressible as a linear combination of members of $X$. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

14. Show that any two subspaces of the same dimension in a finite dimensional real vector space have a common complementary subspace.

15. Let $T, U, V, W$ be vector spaces over $\mathbb{F}$ and let $\alpha: T \to U$, $\beta: V \to W$ be fixed linear maps. Show that the mapping $\Phi: \mathcal{L}(U, V) \to \mathcal{L}(T, W)$ which sends $\theta$ to $\beta \circ \theta \circ \alpha$ is linear. If the spaces are finite-dimensional and $\alpha$ and $\beta$ have rank $r$ and $s$ respectively, find the rank of $\Phi$. 

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- October 2015
1. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix $A$ is invertible if and only if it can be written as a product of elementary matrices. Use this method to find the inverse of
\[
\begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 1 \\
0 & 3 & -1
\end{pmatrix}.
\]

2. (Another proof of the row rank column rank equality.) Let $A$ be an $m \times n$ matrix of (column) rank $r$. Show that $r$ is the least integer for which $A$ factorises as $A = BC$ with $B \in \text{Mat}_{m,r}(F)$ and $C \in \text{Mat}_{r,n}(F)$. Using the fact that $(BC)^T = C^TB^T$, deduce that the (column) rank of $A^T$ equals $r$.

3. Let $V$ be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of $V^*$ dual to the basis $\{x_1, x_2, x_3, x_4\}$ for $V$. Determine, in terms of the $\xi_i$, the bases dual to each of the following:
   (a) $\{x_2, x_1, x_4, x_3\}$;
   (b) $\{x_1, 2x_2, x_3, x_4\}$;
   (c) $\{x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4\}$;
   (d) $\{x_1, x_2 - x_1, x_3 - x_2 + x_1, x_4 - x_3 + x_2 - x_1\}$.

4. Let $P_n$ be the space of real polynomials of degree at most $n$. For $x \in \mathbb{R}$ define $\epsilon_x \in P_n^*$ by $\epsilon_x(p) = p(x)$. Show that $\epsilon_0, \ldots, \epsilon_n$ form a basis for $P_n^*$, and identify the basis of $P_n$ to which it is dual.

5. (a) Show that if $x \neq y$ are vectors in the finite dimensional vector space $V$, then there is a linear functional $\theta \in V^*$ such that $\theta(x) \neq \theta(y)$.
   (b) Suppose that $V$ is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{0\}$.

6. For $A \in \text{Mat}_{m,m}(F)$ and $B \in \text{Mat}_{n,n}(F)$, let $\tau_A(B)$ denote $\text{tr}AB$. Show that, for each fixed $A$, $\tau_A: \text{Mat}_{m,n}(F) \to F$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\text{Mat}_{m,m}(F) \to \text{Mat}_{m,n}(F)^*$. 

7. (a) Let $V$ be a non-zero finite dimensional real vector space. Show that there are no endomorphisms $\alpha, \beta$ of $V$ with $\alpha \beta - \beta \alpha = \text{id}_V$.
   (b) Let $V$ be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms $\alpha$ and $\beta$ of $V$ such that $\alpha \beta - \beta \alpha = \text{id}_V$.

8. Suppose that $\psi: U \times V \to F$ is a bilinear form of rank $r$ on finite dimensional vector spaces $U$ and $V$ over $F$. Show that there exist bases $e_1, \ldots, e_m$ for $U$ and $f_1, \ldots, f_n$ for $V$ such that
\[
\psi \left( \sum_{i=1}^m x_i e_i, \sum_{j=1}^n y_j f_j \right) = \sum_{k=1}^r x_k y_k
\]
for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in F$. What are the dimensions of the left and right kernels of $\psi$?

9. Let $A$ and $B$ be $n \times n$ matrices over a field $F$. Show that the $2n \times 2n$ matrix
\[
C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix}
\]

by elementary row operations (which you should specify). By considering the determinants of $C$ and $D$, obtain another proof that $\det AB = \det A \det B$. 

S.J.Wadsley@dpmms.cam.ac.uk - 1 - October 2015
10. Let $A, B$ be $n \times n$ matrices, where $n \geq 2$. Show that, if $A$ and $B$ are non-singular, then

(i) $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$,  
(ii) $\det(\text{adj} A) = (\det A)^{n-1}$,  
(iii) $\text{adj}(\text{adj} A) = (\det A)^{n-2}A$.

What happens if $A$ is singular? [Hint: Consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]

Show that the rank of the adjugate matrix is 

$$r(\text{adj} A) = \begin{cases} n & \text{if } r(A) = n \\ 1 & \text{if } r(A) = n - 1 \\ 0 & \text{if } r(A) \leq n - 2. \end{cases}$$

11. Show that the dual of the space $\mathbb{P}^n$ of all sequences of real numbers, via the mapping which sends a linear form $\psi$ to the bases $v_1, v_2, \ldots, v_n$, is given by $\mathbb{P}^n$. Consider a basis $v_1, v_2, \ldots, v_n$ of $\mathbb{P}^n$. Show that the rank of the adjugate matrix is $r(\text{adj} A) = (\det A)^{n-2}A$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:

(a) The map $D$ defined by $D(p)(t) = p'(t)$.
(b) The map $S$ defined by $S(p)(t) = p(t^2)$.
(c) The map $E$ defined by $E(p)(t) = p(t - 1)$.
(d) The composite $DS$.
(e) The composite $SD$.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

12. Suppose that $\psi: V \times V \to \mathbb{F}$ is a bilinear form on a finite dimensional vector space $V$. Take $U$ a subspace of $V$ with $U = W^\perp$ some subspace $W$ of $V$. Suppose that $\psi|_{U \times U}$ is non-singular. Show that $\psi$ is also non-singular.

13. Let $V$ be a vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that $g$ is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$.

14. Let $\alpha: V \to V$ be an endomorphism of a real finite dimensional vector space $V$ with $\text{tr}(\alpha) = 0$.

(i) Show that, if $\alpha \neq 0$, there is a vector $v$ with $\alpha(v)$ linearly independent. Deduce that there is a basis for $V$ relative to which $\alpha$ is represented by a matrix $A$ with all of its diagonal entries equal to 0.

(ii) Show that there are endomorphisms $\beta, \gamma$ of $V$ with $\alpha = \beta \gamma - \gamma \beta$.

The final question is based on non-examinable material.

15. Let $Y$ and $Z$ be subspaces of the finite dimensional vector spaces $V$ and $W$ respectively. Suppose that $\alpha: V \to W$ is a linear map such that $\alpha(Y) \subset Z$. Show that $\alpha$ induces linear maps $\alpha|_Y: Y \to Z$ via $\alpha|_Y(y) = \alpha(y)$ and $\overline{\alpha}: V/Y \to W/Z$ via $\overline{\alpha}(v + Y) = \alpha(v) + Z$.

Consider a basis $(v_1, \ldots, v_n)$ for $V$ containing a basis $(v_1, \ldots, v_k)$ for $Y$ and a basis $(w_1, \ldots, w_m)$ for $W$ containing a basis $(w_1, \ldots, w_l)$ for $Z$. Show that the matrix representing $\alpha$ with respect to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ is a block matrix of the form $
abla A C
\nabla 0 B$. Explain how to determine the matrices representing $\alpha|_Y$ with respect to the bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ and representing $\overline{\alpha}$ with respect to the bases $(v_{k+1} + Y, \ldots, v_n + Y)$ and $(w_{l+1} + Z, \ldots, w_m + Z)$ from this block matrix.
1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 3 & -2 \\
0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & -1 \\
0 & 3 & -2 \\
0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & -1 \\
-1 & 3 & -1 \\
-1 & 1 & 1
\end{pmatrix}.
\]

The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank of a suitable matrix, find the eigenvalues of the \( n \times n \) matrix \( A \) with each diagonal entry equal to \( \lambda \) and all other entries 1. Hence write down the determinant of \( A \).

3. Let \( \alpha \) be an endomorphism of the finite dimensional vector space \( V \) over \( \mathbb{F} \), with characteristic polynomial \( \chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0 \). Show that \( \det(\alpha) = (-1)^nc_0 \) and \( \text{tr}(\alpha) = -c_{n-1} \).

4. Let \( V \) be a vector space, let \( \pi_1, \pi_2, \ldots, \pi_k \) be endomorphisms of \( V \) such that \( \text{id}_V = \pi_1 + \cdots + \pi_k \) and \( \pi_i \pi_j = 0 \) for any \( i \neq j \). Show that \( V = U_1 \oplus \cdots \oplus U_k \), where \( U_j = \text{im}(\pi_j) \).

Let \( \alpha \) be an endomorphism on the vector space \( V \), satisfying the equation \( \alpha^3 = \alpha \). Prove directly that \( V = V_0 \oplus V_1 \oplus V_{-1} \), where \( V_0 \) is the \( \lambda \)-eigenspace of \( \alpha \).

5. Let \( \alpha \) be an endomorphism of a finite dimensional complex vector space. Show that if \( \lambda \) is an eigenvalue for \( \alpha \) then \( \lambda^2 \) is an eigenvalue for \( \alpha^2 \). Show further that every eigenvalue of \( \alpha^2 \) arises in this way. Are the eigenspaces \( \ker(\alpha - \lambda \text{id}) \) and \( \ker(\alpha^2 - \lambda^2 \text{id}) \) necessarily the same?

6. (Another proof of the Diagonalisability Theorem.) Let \( V \) be a vector space of finite dimension. Show that if \( \alpha_1 \) and \( \alpha_2 \) are endomorphisms of \( V \), then the nullity \( n(\alpha_1\alpha_2) \) satisfies \( n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2) \). Deduce that if \( \alpha \) is an endomorphism of \( V \) such that \( p(\alpha) = 0 \) for some polynomial \( p(t) \) which is a product of distinct linear factors, then \( \alpha \) is diagonalisable.

7. Let \( A \) be a square complex matrix of finite order — that is, \( A^m = I \) for some \( m > 0 \). Show that \( A \) can be diagonalised.

8. Show that none of the following matrices are similar:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Is the matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

similar to any of them? If so, which?

9. Find a basis with respect to which \( \begin{pmatrix}
0 & -1 \\
1 & 2
\end{pmatrix} \) is in Jordan normal form. Hence compute \( \begin{pmatrix}
0 & -1 \\
1 & 2
\end{pmatrix}^n \).

10. (a) Recall that the Jordan normal form of a \( 3 \times 3 \) complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for \( 4 \times 4 \) complex matrices.

(b) Let \( A \) be a \( 5 \times 5 \) complex matrix with \( A^4 = A^2 \neq A \). What are the possible minimal and characteristic polynomials? If \( A \) is not diagonalisable, how many possible JNFs are there for \( A \)?

11. Let \( V \) be a vector space of dimension \( n \) and \( \alpha \) an endomorphism of \( V \) with \( \alpha^n = 0 \) but \( \alpha^{n-1} \neq 0 \). Show that there is a vector \( y \) such that \( \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y) \) is a basis for \( V \).

Show that if \( \beta \) is an endomorphism of \( V \) which commutes with \( \alpha \), then \( \beta = p(\alpha) \) for some polynomial \( p \). [Hint: consider \( \beta(y) \).] What is the form of the matrix for \( \beta \) with respect to the above basis?
12. Let $\alpha$ be an endomorphism of the finite-dimensional vector space $V$, and assume that $\alpha$ is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of $\alpha^{-1}$ in terms of those of $\alpha$.

13. Prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary invertible square matrix $A$, describe the Jordan normal form of $A^{-1}$ in terms of that of $A$.

Prove that any square complex matrix is similar to its transpose.

14. Let $C$ be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where $A$ and $B$ are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of $\lambda$, show that if $C$ is invertible then there exists a real number $\lambda$ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices $P$ and $Q$ are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let $C$ be the circulant matrix

$$
\begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_n \\
a_n & a_0 & a_1 & \ldots & a_{n-1} \\
a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_1 & a_2 & a_3 & \ldots & a_0
\end{pmatrix}
$$

Show that the determinant of $C$ is $\det C = \prod_{j=0}^n f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n + 1))$.

16. Let $V$ denote the space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ and let $\alpha$ be the differentiation endomorphism $f \mapsto f'$.

(i) Show that every real number $\lambda$ is an eigenvalue of $\alpha$. Show also that $\ker(\alpha - \lambda \iota)$ has dimension 1.

(ii) Show that $\alpha - \lambda \iota$ is surjective for every real number $\lambda$. 

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1. The square matrices $A$ and $B$ over the field $F$ are congruent if $B = P^TAP$ for some invertible matrix $P$ over $F$. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

$$
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}, \quad \begin{pmatrix}
0 & 2 \\
2 & 0
\end{pmatrix}, \quad \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
4 & 4 \\
4 & 5
\end{pmatrix}.
$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$
x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2xz.
$$

If $A$ is the matrix of the first of these (say), find a non-singular matrix $P$ such that $P^TAP$ is diagonal with entries $\pm 1$.

3. (i) Show that the function $\psi(A, B) = \text{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\text{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\text{tr}(AB^T)| \leq \text{tr}(AA^T)^{1/2}\text{tr}(BB^T)^{1/2}$.

(ii) Show that the map $A \mapsto \text{tr}(A^2)$ is a quadratic form on $\text{Mat}_n(\mathbb{R})$. Find its rank and signature.

4. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space $V$.

(i) Find the rank and signature of $\psi$ in the case $V = \mathbb{C}^3$ and

$$
\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.
$$

(ii) Show in general that if $n > 2$ then $\psi(u, v) = \frac{1}{n} \sum_{k=1}^{n} \zeta^{-k} \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.

5. Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite. Write down an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.

6. Let $W \leq V$ with $V$ an inner product space. An endomorphism $\pi$ of $V$ is called an idempotent if $\pi^2 = \pi$. Show that the orthogonal projection onto $W$ is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

7. Let $S$ be an $n \times n$ real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. An endomorphism $\alpha$ of a finite dimensional inner product space $V$ is positive definite if it is self-adjoint and satisfies $\langle \alpha(x), x \rangle > 0$ for all non-zero $x \in V$.

(i) Prove that a positive definite endomorphism has a unique positive definite square root.

(ii) Let $\alpha$ be an invertible endomorphism of $V$ and $\alpha^*$ its adjoint. By considering $\alpha^*\alpha$, show that $\alpha$ can be factored as $\beta\gamma$ with $\beta$ unitary and $\gamma$ positive definite.

9. Let $V$ be a finite dimensional complex inner product space, and let $\alpha$ be an endomorphism on $V$. Assume that $\alpha$ is normal, that is, $\alpha$ commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that $\alpha$ and $\alpha^*$ have a common eigenvector $v$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle v \rangle^\perp$ is invariant under both $\alpha$ and $\alpha^*$. Deduce that there is an orthonormal basis of eigenvectors of $\alpha$.

10. Find a linear transformation which simultaneously reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 4yz, \quad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6xz$$

to the forms

$$X^2 + Y^2 + Z^2, \quad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2, \quad 2xy$ simultaneously to diagonal forms?
11. Show that if $A$ is an $m \times n$ real matrix of rank $n$ then $A^T A$ is invertible. Find a corresponding result for complex matrices.

12. Let $P_n$ be the $(n+1)$-dimensional space of real polynomials of degree $\leq n$. Define
\[
(f, g) = \int_{-1}^{+1} f(t)g(t)dt.
\]
Show that $(\ , \ )$ is an inner product on $P_n$ and that the endomorphism $\alpha : P_n \rightarrow P_n$ defined by
\[
\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)
\]
is self-adjoint. What are the eigenvalues of $\alpha$?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1-t^2)^k$. Prove the following.
(i) For $i \neq j$, $(s_i, s_j) = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for $P_n$.
(iii) For all $1 \leq k \leq n$, $s_k$ spans the orthogonal complement of $P_{k-1}$ in $P_k$.
(iv) $s_k$ is an eigenvector of $\alpha$. (Give its eigenvalue.)

What is the relation between the $s_k$ and the result of applying Gram-Schmidt to the sequence $1, x, x^2, x^3$ and so on? (Calculate the first few terms?)

13. Let $f_1, \ldots, f_t, f_{t+1}, \ldots, f_{t+u}$ be linear functionals on the finite-dimensional real vector space $V$. Show that $Q(x) = f_1(x)^2 + \cdots + f_t(x)^2 - f_{t+1}(x)^2 - \cdots - f_{t+u}(x)^2$ is a quadratic form on $V$. Suppose $Q$ has rank $p + q$ and signature $p - q$. Show that $p \leq t$ and $q \leq u$.

14. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1$?

15. Suppose that $\alpha$ is an orthogonal endomorphism on the finite-dimensional real inner product space $V$. Prove that $V$ can be decomposed into a direct sum of mutually orthogonal $\alpha$-invariant subspaces of dimension 1 or 2. Determine the possible matrices of $\alpha$ with respect to orthonormal bases in the cases where $V$ has dimension 1 or dimension 2.