Part IB — Geometry
Theorems with proof

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

*Parts of Analysis II will be found useful for this course.*

Groups of rigid motions of Euclidean space. Rotation and reflection groups in two and three dimensions. Lengths of curves. [2]

Spherical geometry: spherical lines, spherical triangles and the Gauss-Bonnet theorem. Stereographic projection and Möbius transformations. [3]

Triangulations of the sphere and the torus, Euler number. [1]


Embedded surfaces in $\mathbb{R}^3$. The first fundamental form. Length and area. Examples. [1]

Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution. [2]

The second fundamental form and Gaussian curvature. For metrics of the form $du^2 + G(u, v)dv^2$, expression of the curvature as $\sqrt{G_{uu}/G}$. Abstract smooth surfaces and isometries. Euler numbers and statement of Gauss-Bonnet theorem, examples and applications. [3]
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0 Introduction
1 Euclidean geometry

1.1 Isometries of the Euclidean plane

Theorem. Every isometry of $f: \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$f(x) = Ax + b.$$ 

for $A$ orthogonal and $b \in \mathbb{R}^n$.

Proof. Let $f$ be an isometry. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let

$$b = f(0), \quad a_i = f(e_i) - b.$$ 

The idea is to construct our matrix $A$ out of these $a_i$. For $A$ to be orthogonal, \{a_i\} must be an orthonormal basis.

Indeed, we can compute

$$\|a_i\| = \|f(e_i) - f(0)\| = d(f(e_i), f(0)) = d(e_i, 0) = \|e_i\| = 1.$$ 

For $i \neq j$, we have

$$(a_i, a_j) = -(a_i, -a_j) \\
= -\frac{1}{2}(\|a_i - a_j\|^2 - \|a_i\|^2 - \|a_j\|^2) \\
= -\frac{1}{2}(\|f(e_i) - f(e_j)\|^2 - 2) \\
= -\frac{1}{2}(\|e_i - e_j\|^2 - 2) \\
= 0$$

So $a_i$ and $a_j$ are orthogonal. In other words, \{a_i\} forms an orthonormal set. It is an easy result that any orthogonal set must be linearly independent. Since we have found $n$ orthonormal vectors, they form an orthonormal basis.

Hence, the matrix $A$ with columns given by the column vectors $a_i$ is an orthogonal matrix. We define a new isometry

$$g(x) = Ax + b.$$ 

We want to show $f = g$. By construction, we know $g(x) = f(x)$ is true for $x = 0, e_1, \ldots, e_n$.

We observe that $g$ is invertible. In particular,

$$g^{-1}(x) = A^{-1}(x - b) = A^T x - A^T b.$$ 

Moreover, it is an isometry, since $A^T$ is orthogonal (or we can appeal to the more general fact that inverses of isometries are isometries).

We define

$$h = g^{-1} \circ f.$$ 

Since it is a composition of isometries, it is also an isometry. Moreover, it fixes $x = 0, e_1, \ldots, e_n$.

It currently suffices to prove that $h$ is the identity.
Let \( x \in \mathbb{R}^n \), and expand it in the basis as
\[
x = \sum_{i=1}^{n} x_i e_i.
\]
Let
\[
y = h(x) = \sum_{i=1}^{n} y_i e_i.
\]
We can compute
\[
d(x, e_i)^2 = (x - e_i, x - e_i) = \|x\|^2 + 1 - 2x_i
d(x, 0)^2 = \|x\|^2.
\]
Similarly, we have
\[
d(y, e_i)^2 = (y - e_i, y - e_i) = \|y\|^2 + 1 - 2y_i
d(y, 0)^2 = \|y\|^2.
\]
Since \( h \) is an isometry and fixes \( 0, e_1, \ldots, e_n \), and by definition \( h(x) = y \), we must have
\[
d(x, 0) = d(y, 0), \quad d(x, e_i) = d(y, e_i).
\]
The first equality gives \( \|x\|^2 = \|y\|^2 \), and the others then imply \( x_i = y_i \) for all \( i \).
In other words, \( x = y = h(x) \). So \( h \) is the identity.

1.2 Curves in \( \mathbb{R}^n \)

**Proposition.** If \( \Gamma \) is continuously differentiable (i.e. \( C^1 \)), then the length of \( \Gamma \) is given by
\[
\text{length}(\Gamma) = \int_a^b \|\Gamma'(t)\| \, dt.
\]

**Proof.** To simplify notation, we assume \( n = 3 \). However, the proof works for all possible dimensions. We write
\[
\Gamma(t) = (f_1(t), f_2(t), f_3(t)).
\]
For every \( s \neq t \in [a, b] \), the mean value theorem tells us
\[
\frac{f_i(t) - f_i(s)}{t - s} = f'_i(\xi_i)
\]
for some \( \xi_i \in (s, t) \), for all \( i = 1, 2, 3 \).

Now note that \( f'_i \) are continuous on a closed, bounded interval, and hence uniformly continuous. For all \( \varepsilon \in 0 \), there is some \( \delta > 0 \) such that \( |t - s| < \delta \) implies
\[
|f'_i(\xi_i) - f'(\xi)| < \frac{\varepsilon}{3}
\]
for all \( \xi \in (s, t) \). Thus, for any \( \xi \in (s, t) \), we have
\[
\left\| \frac{\Gamma(t) - \Gamma(s)}{t - s} - \Gamma'(\xi) \right\| = \left\| \begin{pmatrix} f'_1(\xi_1) \\ f'_2(\xi_2) \\ f'_3(\xi_3) \end{pmatrix} - \begin{pmatrix} f'_1(\xi) \\ f'_2(\xi) \\ f'_3(\xi) \end{pmatrix} \right\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
In other words, 
\[ \| \Gamma(t) - \Gamma(s) - (t - s)\Gamma'(\xi) \| \leq \varepsilon (t - s). \]

We relabel \( t = t_i, \ s = t_{i-1} \) and \( \xi = \frac{t_i + t_{i-1}}{2}. \)

Using the triangle inequality, we have
\[
(t_i - t_{i-1}) \left\| \Gamma' \left( \frac{t_i + t_{i-1}}{2} \right) \right\| - \varepsilon (t_i - t_{i-1}) < \| \Gamma(t_i) - \Gamma(t_{i-1}) \|
\]
\[
< (t_i - t_{i-1}) \left\| \Gamma' \left( \frac{t_i + t_{i-1}}{2} \right) \right\| + \varepsilon (t_i - t_{i-1}).
\]

Summing over all \( i, \) we obtain
\[
\sum_i (t_i - t_{i-1}) \left\| \Gamma' \left( \frac{t_i + t_{i-1}}{2} \right) \right\| - \varepsilon (b - a) < S_D
\]
\[
< \sum_i (t_i - t_{i-1}) \left\| \Gamma' \left( \frac{t_i + t_{i-1}}{2} \right) \right\| + \varepsilon (b - a),
\]

which is valid whenever mesh(\( D \)) < \( \delta \).

Since \( \Gamma' \) is continuous, and hence integrable, we know
\[
\sum_i (t_i - t_{i-1}) \left\| \Gamma' \left( \frac{t_i + t_{i-1}}{2} \right) \right\| \to \int_a^b \|\Gamma'(t)\| \ dt
\]
as mesh(\( D \)) \to 0, and

\[
\text{length}(\Gamma) = \lim_{\text{mesh}(D) \to 0} S_D = \int_a^b \|\Gamma'(t)\| \ dt.
\]
2 Spherical geometry

2.1 Triangles on a sphere

Theorem (Spherical cosine rule).

\[ \sin a \sin b \cos \gamma = \cos c - \cos a \cos b. \]

Proof. We use the fact from IA Vectors and Matrices that

\[ (C \times B) \cdot (A \times C) = (A \cdot C)(B \cdot C) - (C \cdot C)(B \cdot A), \]

which follows easily from the double-epsilon identity

\[ \varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}. \]

In our case, since \( C \cdot C = 1 \), the right hand side is

\[ (A \cdot C)(B \cdot C) - (B \cdot A). \]

Thus we have

\[ -\cos \gamma = n_1 \cdot n_2 = \frac{C \times B \cdot A \times C}{\sin a \sin b} = \frac{(A \cdot C)(B \cdot C) - (B \cdot A)}{\sin a \sin b} = \frac{\cos b \cos a - \cos c}{\sin a \sin b}. \]

Corollary (Pythagoras theorem). If \( \gamma = \frac{\pi}{2} \), then

\[ \cos c = \cos a \cos b. \]

Theorem (Spherical sine rule).

\[ \frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}. \]

Proof. We use the fact that

\[ (A \times C) \times (C \times B) = (C \cdot (B \times A))C, \]

which we again are not bothered to prove again. The left hand side is

\[ -(n_1 \times n_2) \sin a \sin b \]

Since the angle between \( n_1 \) and \( n_2 \) is \( \pi + \gamma \), we know \( n_1 \times n_2 = C \sin \gamma \). Thus the left hand side is

\[ -C \sin a \sin b \sin \gamma. \]

Thus we know

\[ C \cdot (A \times B) = \sin a \sin b \sin \gamma. \]
However, since the scalar triple product is cyclic, we know
\[ C \cdot (A \times B) = A \cdot (B \times C). \]

In other words, we have
\[ \sin a \sin b \sin \gamma = \sin b \sin c \sin \alpha. \]

Thus we have
\[ \frac{\sin \gamma}{\sin \alpha} = \frac{\sin \alpha}{\sin \gamma}. \]

Similarly, we know this is equal to \( \frac{\sin \beta}{\sin \gamma}. \)

**Corollary** (Triangle inequality). For any \( P, Q, R \in S^2 \), we have
\[ d(P, Q) + d(Q, R) \geq d(P, R), \]
with equality if and only if \( Q \) lies in the line segment \( PR \) of shortest length.

**Proof.** The only case left to check is if \( d(P, R) = \pi \), since we do not allow our triangles to have side length \( \pi \). But in this case they are antipodal points, and any \( Q \) lies in a line through \( PR \), and equality holds.

**Proposition.** Given a curve \( \Gamma \) on \( S^2 \subseteq \mathbb{R}^3 \) from \( P \) to \( Q \), we have
\[ \ell = \text{length}(\Gamma) \geq d(P, Q). \]
Moreover, if \( \ell = d(P, Q) \), then the image of \( \Gamma \) is a spherical line segment \( PQ \).

**Proof.** Let \( \Gamma : [0, 1] \to S \) and \( \ell = \text{length}(\Gamma) \). Then for any dissection \( D \) of \( [0, 1] \), say \( 0 = t_0 < \cdots < t_N = 1 \), write \( P_i = \Gamma(t_i) \). We define
\[ \tilde{S}_D = \sum_i d(P_{i-1}, P_i) > S_D = \sum_i |\overrightarrow{P_{i-1}P_i}|, \]
where the length in the right hand expression is the distance in Euclidean 3-space.

Now suppose \( \ell < d(P, Q) \). Then there is some \( \varepsilon > 0 \) such that \( \ell(1 + \varepsilon) < d(P, Q) \).

Recall from basic trigonometric that if \( \theta > 0 \), then \( \sin \theta < \theta \). Also,
\[ \frac{\sin \theta}{\theta} \to 1 \text{ as } \theta \to 0. \]

Thus we have
\[ \theta \leq (1 + \varepsilon) \sin \theta. \]

for small \( \theta \). What we really want is the double of this:
\[ 2\theta \leq (1 + \varepsilon)2\sin \theta. \]

This is useful since these lengths appear in the following diagram:
This means for $P,Q$ sufficiently close, we have $d(P,Q) \leq (1 + \varepsilon)|\overrightarrow{PQ}|$.

From Analysis II, we know $\Gamma$ is uniformly continuous on $[0,1]$. So we can choose $D$ such that
\[ d(P_{i-1},P_i) \leq (1 + \varepsilon)|\overrightarrow{P_{i-1}P_i}| \]
for all $i$. So we know that for sufficiently fine $D$,
\[ \tilde{S}_D \leq (1 + \varepsilon)S_D < d(P,Q), \]
since $S_D \to \ell$. However, by the triangle inequality $\tilde{S}_D \geq d(P,Q)$. This is a contradiction. Hence we must have $\ell \geq d(P,Q)$.

Suppose now $\ell = d(P,Q)$ for some $\Gamma : [0,1] \to S$, $\ell = \text{length}(\Gamma)$. Then for every $t \in [0,1]$, we have
\[
\begin{align*}
d(P,Q) &= \ell = \text{length} \Gamma|_{[0,t]} + \text{length} \Gamma|_{[t,1]} \\
&\geq d(P,\Gamma(t)) + d(\Gamma(t),Q) \\
&\geq d(P,Q).
\end{align*}
\]
Hence we must have equality all along the way, i.e.
\[ d(P,Q) = d(P,\Gamma(t)) + d(\Gamma(t),Q) \]
for all $\Gamma(t)$.

However, this is possible only if $\Gamma(t)$ lies on the shorter spherical line segment $PQ$, as we have previously proved. So done.

**Proposition** (Gauss-Bonnet theorem for $S^2$). If $\Delta$ is a spherical triangle with angles $\alpha, \beta, \gamma$, then
\[ \text{area}(\Delta) = (\alpha + \beta + \gamma) - \pi. \]

**Proof.** We start with the concept of a double lune. A double lune with angle $0 < \alpha < \pi$ is two regions $S$ cut out by two planes through a pair of antipodal points, where $\alpha$ is the angle between the two planes.

![double lune diagram](image)

It is not hard to show that the area of a double lune is $4\alpha$, since the area of the sphere is $4\pi$.

Now note that our triangle $\Delta = ABC$ is the intersection of 3 single lunes, with each of $A,B,C$ as the pole (in fact we only need two, but it is more convenient to talk about 3).
Therefore $\Delta$ together with its antipodal partner $\Delta'$ is a subset of each of the 3 double lunes with areas $4\alpha, 4\beta, 4\gamma$. Also, the union of all the double lunes cover the whole sphere, and overlap at exactly $\Delta$ and $\Delta'$. Thus

$$4(\alpha + \beta + \gamma) = 4\pi + 2(\text{area}(\Delta) + \text{area}(\Delta')) = 4\pi + 4\text{area}(\Delta).$$

\[\square\]

### 2.2 Möbius geometry

**Lemma.** If $\pi': S^2 \to \mathbb{C}_\infty$ denotes the stereographic projection from the South Pole instead, then

$$\pi'(P) = \frac{1}{\pi(P)}.$$

**Proof.** Let $P(x, y, z)$. Then

$$\pi(x, y, z) = \frac{x + iy}{1 - z}.$$

Then we have

$$\pi'(x, y, z) = \frac{x + iy}{1 + z},$$

since we have just flipped the $z$ axis around. So we have

$$\pi(P)\pi'(P) = \frac{x^2 + y^2}{1 - z^2} = 1,$$

noting that we have $x^2 + y^2 + z^2 = 1$ since we are on the unit sphere. \[\square\]

**Theorem.** Via the stereographic projection, every rotation of $S^2$ induces a Möbius map defined by a matrix in $\text{SU}(2) \subseteq \text{GL}(2, \mathbb{C})$, where

$$\text{SU}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

**Proof.**

(i) Consider the $r(\hat{z}, \theta)$, the rotations about the $z$ axis by $\theta$. These corresponds to the Möbius map $\zeta \mapsto e^{i\theta}\zeta$, which is given by the unitary matrix

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$
(ii) Consider the rotation $r(\hat{y}, \frac{\pi}{2})$. This has the matrix
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
z \\
y \\
-x
\end{pmatrix}.
\]
This corresponds to the map
\[
\zeta = \frac{x + iy}{1 - z} \mapsto \zeta' = \frac{z + iy}{1 + x}.
\]
We want to show this is a Möbius map. To do so, we guess what the Möbius map should be, and check it works. We can manually compute that $-1 \mapsto \infty$, $1 \mapsto 0$, $i \mapsto i$.

\[
\begin{pmatrix}
z \\
y \\
-x
\end{pmatrix}
=
\begin{pmatrix}
x \\
y \\
-x
\end{pmatrix}
= \left( \zeta' \right)
= \left( \frac{z + iy}{1 + x} \right)
= \left( \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy) - (z^2 + y^2)} \right)
= \left( \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy) + (x^2 - 1)} \right)
= \left( \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy + x - 1)} \right)
= \left( \frac{z + iy}{x + 1} \right).
\]
So done. We finally have to write this in the form of an SU(2) matrix:
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\]
(iii) We claim that $\text{SO}(3)$ is generated by $r\left(\hat{y}, \frac{\pi}{2}\right)$ and $r(\hat{z}, \theta)$ for $0 \leq \theta < 2\pi$.

To show this, we observe that $r(\hat{x}, \varphi) = r(\hat{y}, \frac{\pi}{2})r(\hat{z}, \varphi)r(\hat{y}, -\frac{\pi}{2})$. Note that we read the composition from right to left. You can convince yourself this is true by taking a physical sphere and try rotating. To prove it formally, we can just multiply the matrices out.

Next, observe that for $v \in S^2 \subseteq \mathbb{R}^3$, there are some angles $\varphi, \psi$ such that $g = r(\hat{z}, \psi)r(\hat{x}, \varphi)$ maps $v$ to $\hat{x}$. We can do so by first picking $r(\hat{x}, \varphi)$ to rotate $v$ into the $(x, y)$-plane. Then we rotate about the $z$-axis to send it to $\hat{x}$.

Then for any $\theta$, we have $r(v, \theta) = g^{-1}r(\hat{x}, \theta)g$, and our claim follows by composition.

(iv) Thus, via the stereographic projection, every rotation of $S^2$ corresponds to products of Möbius transformations of $\mathbb{C}_\infty$ with matrices in $\text{SU}(2)$.

**Theorem.** The group of rotations $\text{SO}(3)$ acting on $S^2$ corresponds precisely with the subgroup $\text{PSU}(2) = \text{SU}(2)/\pm 1$ of Möbius transformations acting on $\mathbb{C}_\infty$.

**Proof.** Let $g \in \text{PSU}(2)$ be a Möbius transformation

$$g(z) = \frac{az + b}{bz + \bar{a}}.$$  

Suppose first that $g(0) = 0$. So $b = 0$. So $a\bar{a} = 1$. Hence $a = e^{i\theta/2}$. Then $g$ corresponds to $r(\hat{z}, \theta)$, as we have previously seen.

In general, let $g(0) = w \in \mathbb{C}_\infty$. Let $Q \in S^2$ be such that $\pi(Q) = w$. Choose a rotation $A \in \text{SO}(3)$ such that $A(Q) = -\hat{z}$. Since $A$ is a rotation, let $\alpha \in \text{PSU}(2)$ be the corresponding Möbius transformation. By construction we have $\alpha(w) = 0$. Then the composition $\alpha \circ g$ fixes zero. So it corresponds to some $B = r(z, \theta)$. We then see that $g$ corresponds to $A^{-1}B \in \text{SO}(3)$. So done. \(\square\)
3 Triangulations and the Euler number

Theorem. The Euler number $e$ is independent of the choice of triangulation.

Proposition. For every geodesic triangulation of $S^2$ (and respectively $T$) has $e = 2$ (respectively, $e = 0$).

Proof. For any triangulation $\tau$, we denote the “faces” of $\Delta_1, \cdots, \Delta_F$, and write $\tau_i = \alpha_i + \beta_i + \gamma_i$ for the sum of the interior angles of the triangles (with $i = 1, \cdots, F$).

Then we have

$$\sum \tau_i = 2\pi V,$$

since the total angle around each vertex is $2\pi$. Also, each triangle has three edges, and each edge is from two triangles. So $3F = 2E$. We write this in a more convenient form:

$$F = 2E - 2F.$$

How we continue depends on whether we are on the sphere or the torus.

- For the sphere, Gauss-Bonnet for the sphere says the area of $\Delta_i$ is $\tau_i - \pi$. Since the area of the sphere is $4\pi$, we know

$$4\pi = \sum \text{area}(\Delta_i) = \sum (\tau_i - \pi) = 2\pi V - F\pi = 2\pi V - (2E - 2F)\pi = 2\pi(F - E + V).$$

So $F - E + V = 2$.

- For the torus, we have $\tau_i = \pi$ for every face in $\hat{Q}$. So

$$2\pi V = \sum \tau_i = \pi F.$$

So

$$2V = F = 2E - 2F.$$

So we get

$$2(F - V + E) = 0,$$

as required. \qed
4 Hyperbolic geometry

4.1 Review of derivatives and chain rule

**Proposition** (Chain rule). Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$. Let $f : U \to \mathbb{R}^m$ and $g : V \to U$ be smooth. Then $f \circ g : V \to \mathbb{R}^m$ is smooth and has a derivative

$$d(f \circ g)_p = (df)_{g(p)} \circ (dg)_p.$$ 

In terms of the Jacobian matrices, we get

$$J(f \circ g)_p = J(f)_{g(p)} J(g)_p.$$ 

4.2 Riemannian metrics

4.3 Two models for the hyperbolic plane

**Proposition.** The elements of $\text{PSL}(2, \mathbb{R})$ are isometries of $H$, and this preserves the lengths of curves.

**Proof.** It is easy to check that $\text{PSL}(2, \mathbb{R})$ is generated by

(i) Translations $z \mapsto z + a$ for $a \in \mathbb{R}$

(ii) Dilations $z \mapsto az$ for $a > 0$

(iii) The single map $z \mapsto -\frac{1}{z}$.

So it suffices to show each of these preserves the metric $\frac{|dz|^2}{|z|^4}$, where $z = x + iy$. The first two are straightforward to see, by plugging it into formula and notice the metric does not change.

We now look at the last one, given by $z \mapsto -\frac{1}{z}$. The derivative at $z$ is

$$f'(z) = \frac{1}{z^2}.$$ 

So we get

$$dz \mapsto d\left(-\frac{1}{z}\right) = \frac{dz}{z^2}.$$ 

So

$$\left|d\left(-\frac{1}{z}\right)\right|^2 = \frac{|dz|^2}{|z|^4}.$$ 

We also have

$$\text{Im} \left(-\frac{1}{z}\right) = -\frac{1}{|z|^2} \text{Im} \bar{z} = \frac{\text{Im} z}{|z|^2}.$$ 

So

$$\frac{|d(-1/z)|^2}{\text{Im}(-1/z)^2} = \left(\frac{|dz|^2}{|z|^4}\right) / \left(\frac{|\text{Im} z|^2}{|z|^4}\right) = \frac{|dz|^2}{|\text{Im} z|^2}.$$ 

So this is an isometry, as required. 

**Lemma.** Given any two distinct points $z_1, z_2 \in H$, there exists a unique hyperbolic line through $z_1$ and $z_2$. 

Proof. This is clear if \( \text{Re} \ z_1 = \text{Re} \ z_2 \) — we just pick the vertical half-line through them, and it is clear this is the only possible choice.

Otherwise, if \( \text{Re} \ z_1 \neq \text{Re} \ z_2 \), then we can find the desired circle as follows:

\[
\text{It is also clear this is the only possible choice.} \quad \square
\]

**Lemma.** \( \text{PSL}(2, \mathbb{R}) \) acts transitively on the set of hyperbolic lines in \( H \).

**Proof.** It suffices to show that for each hyperbolic line \( \ell \), there is some \( g \in \text{PSL}(2, \mathbb{R}) \) such that \( g(\ell) = L^+ \). This is clear when \( \ell \) is a vertical half-line, since we can just apply a horizontal translation.

If it is a semicircle, suppose it has end-points \( s < t \in \mathbb{R} \). Then consider

\[
g(z) = \frac{z - t}{z - s}.
\]

This has determinant \(-s + t > 0\). So \( g \in \text{PSL}(2, \mathbb{R}) \). Then \( g(t) = 0 \) and \( g(s) = \infty \). Then we must have \( g(\ell) = L^+ \), since \( g(\ell) \) is a hyperbolic line, and the only hyperbolic lines passing through \( \infty \) are the vertical half-lines. So done. \( \square \)

**Proposition.** If \( \gamma : [0, 1] \to H \) is a piecewise \( C^1 \)-smooth curve with \( \gamma(0) = z_1, \gamma(1) = z_2 \), then \( \text{length}(\gamma) \geq \rho(z_1, z_2) \), with equality if \( \gamma \) is a monotonic parametrisation of \([z_1, z_2] \subseteq \ell \), where \( \ell \) is the hyperbolic line through \( z_1 \) and \( z_2 \).

**Proof.** We pick an isometry \( g \in \text{PSL}(2, \mathbb{R}) \) so that \( g(\ell) = L^+ \). So without loss of generality, we assume \( z_1 = iu \) and \( z_2 = iv \), with \( u < v \in \mathbb{R} \).

We decompose the path as \( \gamma(t) = x(t) + iy(t) \). Then we have

\[
\text{length}(\gamma) = \int_0^1 \frac{1}{y} \sqrt{x'^2 + y'^2} \, dt
\geq \int_0^1 \frac{|y'|}{y} \, dz
\geq \left| \int_0^1 \frac{y'}{y} \, dt \right|
= \left| \log y(t) \right|_0^1
= \log \left( \frac{v}{u} \right)
\]

This calculation also tells us that \( \rho(z_1, z_2) = \log \left( \frac{v}{u} \right) \) so \( \text{length}(\gamma) \geq \rho(z_1, z_2) \) with equality if and only if \( x(t) = 0 \) (hence \( \gamma \subseteq L^+ \)) and \( y \geq 0 \) (hence monotonic). \( \square \)
Corollary (Triangle inequality). Given three points \( z_1, z_2, z_3 \in H \), we have
\[
\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3),
\]
with equality if and only if \( z_2 \) lies between \( z_1 \) and \( z_2 \).

4.4 Geometry of the hyperbolic disk

Lemma. Let \( G \) be the set of isometries of the hyperbolic disk. Then

(i) Rotations \( z \mapsto e^{i\theta}z \) (for \( \theta \in \mathbb{R} \)) are elements of \( G \).

(ii) If \( a \in D \), then \( g(z) = \frac{z - a}{1 - \bar{a}z} \) is in \( G \).

Proof.

(i) This is clearly an isometry, since this is a linear map, preserves \( |z| \) and \( |dz| \), and hence also the metric
\[
\frac{4|dz|^2}{(1 - |z|^2)^2}.
\]

(ii) First, we need to check this indeed maps \( D \) to itself. To do this, we first make sure it sends \( \{|z| = 1\} \) to itself. If \( |z| = 1 \), then
\[
|1 - \bar{a}z| = \bar{z}(1 - \bar{a}z) = |\bar{z} - \bar{a}| = |z - a|.
\]
So
\[
|g(z)| = 1.
\]
Finally, it is easy to check \( g(a) = 0 \). By continuity, \( G \) must map \( D \) to itself. We can then show it is an isometry by plugging it into the formula.

Proposition. If \( 0 \leq r < 1 \), then
\[
\rho(0, re^{i\theta}) = 2 \tanh^{-1} r.
\]
In general, for \( z_1, z_2 \in D \), we have
\[
g(z_1, z_2) = 2 \tanh^{-1} \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|}.
\]

Proof. By the lemma above, we can rotate the hyperbolic disk so that \( re^{i\theta} \) is rotated to \( r \). So
\[
\rho(0, re^{i\theta}) = \rho(0, r).
\]
We can evaluate this by performing the integral
\[
\rho(0, r) = \int_0^r \frac{2 \, dt}{1 - t^2} = 2 \tanh^{-1} r.
\]
For the general case, we apply the Möbius transformation
\[
g(z) = \frac{z - z_1}{1 - \bar{z}_1 z}
\]
Then we have
\[ g(z_1) = 0, \quad g(z_2) = \frac{z_2 - z_1}{1 - z_1 \bar{z}_2} = \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right| e^{i\theta}. \]
So
\[ \rho(z_1, z_2) = \rho(g(z_1), g(z_2)) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|. \]
Now note that \(g(\ell')\) is also a line meeting \(L^+\) perpendicularly at \(Q\), since \(g\) fixes \(L^+\) and preserves angles. So we must have \(g(\ell') = \ell'\). Then in particular \(g(P) \in \ell'\). So we must have \(g(P) = P\) or \(g(P) = P'\), where \(P'\) is the image under reflection in \(L^+\).

Now it suffices to prove that if \(g(P) = P\) for any one \(P\), then \(g(P)\) must be the identity (if \(g(P) = P'\) for all \(P\), then \(g\) must be given by \(g(z) = -\overline{z}\)).

Now suppose \(g(P) = P\), and let \(A \in H^+\), where \(H^+ = \{z \in H : \text{Re } z > 0\}\).

Now if \(g(A) \neq A\), then \(g(A) = A'\). Then \(\rho(A', P) = \rho(A, P)\). But

\[
\rho(A', P) = \rho(A', B) + \rho(B, P) = \rho(A, B) + \rho(B, P) > \rho(A, P),
\]

by the triangle inequality, noting that \(B \notin (AP)\). This is a contradiction. So \(g\) must fix everything.

4.5 Hyperbolic triangles

**Theorem** (Gauss-Bonnet theorem for hyperbolic triangles). For each hyperbolic triangle \(\Delta\), say, \(ABC\), with angles \(\alpha, \beta, \gamma \geq 0\) (note that zero angle is possible), we have

\[
\text{area}(\Delta) = \pi - (\alpha + \beta + \gamma).
\]

**Proof.** First do the case where \(\gamma = 0\), so \(C\) is “at infinity”. Recall that we like to use the disk model if we have a distinguished point in the hyperbolic plane. If we have a distinguished point at infinity, it is often advantageous to use the upper half plane model, since \(\infty\) is a distinguished point at infinity.

So we use the upper-half plane model, and wlog \(C = \infty\) (apply \(\text{PSL}(2, \mathbb{R})\)) if necessary. Then \(AC\) and \(BC\) are vertical half-lines. So \(AB\) is the arc of a semi-circle. So \(AB\) is an arc of a semicircle.
We use the transformation $z \mapsto z + a$ (with $a \in \mathbb{R}$) to center the semi-circle at 0. We then apply $z \mapsto bz$ (with $b > 0$) to make the circle have radius 1. Thus wlog $AB \subseteq \{x^2 + y^2 = 1\}$.

Now we have

$$\text{area}(T) = \int_{\cos(\pi - \alpha)}^{\cos \beta} \int_{\sqrt{1 - x^2}}^{\infty} \frac{1}{y^2} dy \, dx$$

$$= \int_{\cos(\pi - \alpha)}^{\cos \beta} \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= \left[ - \cos^{-1}(x) \right]_{\cos(\pi - \alpha)}^{\cos \beta}$$

$$= \pi - \alpha - \beta,$$

as required.

In general, we use $H$ again, and we can arrange $AC$ in a vertical half-line. Also, we can move $AB$ to $x^2 + y^2 = 1$, noting that this transformation keeps $AC$ vertical.

We consider $\Delta_1 = AB\infty$ and $\Delta_2 = CB\infty$. Then we can immediately write

$$\text{area}(\Delta_1) = \pi - \alpha - (\beta + \delta)$$

$$\text{area}(\Delta_2) = \pi - \delta - (\pi - \gamma) = \gamma - \delta.$$

So we have

$$\text{area}(T) = \text{area}(\Delta_2) - \text{area}(\Delta_1) = \pi - \alpha - \beta - \gamma,$$

as required.

**Theorem (Hyperbolic cosine rule).** In a triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$, we have

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

**Proof.** See example sheet 2.

4.6 **Hyperboloid model**
5 Smooth embedded surfaces (in $\mathbb{R}^3$)

5.1 Smooth embedded surfaces

**Proposition.** Let $\sigma : V \to U$ and $\tilde{\sigma} : \tilde{V} \to U$ be two $C^\infty$ parametrisations of a surface. Then the homeomorphism

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$$

is in fact a diffeomorphism.

**Proof.** Since differentiability is a local property, it suffices to consider $\varphi$ on some small neighbourhood of a point in $V$. Pick our favorite point $(v_0, u_0) \in \tilde{V}$. We know $\sigma = \sigma(u,v)$ is differentiable. So it has a Jacobian matrix

$$\begin{pmatrix}
x_u & x_v \\
y_u & y_v \\
z_u & z_v
\end{pmatrix}.$$ 

By definition, this matrix has rank two at each point. wlog, we assume the first two rows are linearly independent. So

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0$$

at $(v_0, u_0) \in \tilde{V}$. We define a new function

$$F(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}.$$ 

Now the inverse function theorem applies. So $F$ has a local $C^\infty$ inverse, i.e. there are two open neighbourhoods $(u_0, v_0) \in N$ and $F(u_0, v_0) \in N' \subseteq \mathbb{R}^2$ such that $f : N \to N'$ is a diffeomorphism.

Writing $\pi : \tilde{\sigma} \to N'$ for the projection $\pi(x,y,z) = (x,y)$ we can put these things in a commutative diagram:

$$\begin{array}{ccc}
\sigma(N) & \xrightarrow{\pi} & N' \\
\downarrow{\sigma} & & \\
N & \xrightarrow{F} & N'
\end{array}$$

We now let $\tilde{N} = \tilde{\sigma}^{-1}(\sigma(N))$ and $\tilde{F} = \pi \circ \tilde{\sigma}$, which is yet again smooth. Then we have the following larger commutative diagram:

$$\begin{array}{ccc}
\sigma(N) & \xrightarrow{\pi} & N' \\
\downarrow{\sigma} & & \leftarrow F \\
N & \xleftarrow{F} & \tilde{N}
\end{array}$$

Then we have

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} = \sigma^{-1} \circ \pi^{-1} \circ \pi \circ \tilde{\sigma} = F^{-1} \circ \tilde{F},$$

which is smooth, since $F^{-1}$ and $\tilde{F}$ are. Hence $\varphi$ is smooth everywhere. By symmetry of the argument, $\varphi^{-1}$ is smooth as well. So this is a diffeomorphism. $\square$
Corollary. The tangent plane $T_Q S$ is independent of parametrization.

Proof. We know

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\varphi_1(\tilde{u}, \tilde{v}), \varphi_2(\tilde{u}, \tilde{v})).$$

We can then compute the partial derivatives as

$$\tilde{\sigma}_u = \varphi_1, \tilde{u} \sigma_u + \varphi_2, \tilde{u} \sigma_v$$
$$\tilde{\sigma}_v = \varphi_1, \tilde{v} \sigma_u + \varphi_2, \tilde{v} \sigma_v$$

Here the transformation is related by the Jacobian matrix

$$\begin{pmatrix} \varphi_1, \tilde{u} & \varphi_1, \tilde{v} \\ \varphi_2, \tilde{u} & \varphi_2, \tilde{v} \end{pmatrix} = J(\varphi).$$

This is invertible since $\varphi$ is a diffeomorphism. So $(\sigma_u, \sigma_v)$ and $(\tilde{\sigma}_u, \tilde{\sigma}_v)$ are different basis of the same two-dimensional vector space. So done.

Proposition. If we have two parametrizations related by $\tilde{\sigma} = \sigma \circ \varphi : \tilde{V} \to U$, then $\varphi : V \to \tilde{V}$ is an isometry of Riemannian metrics (on $V$ and $\tilde{V}$).

Proposition. The area of $T$ is independent of the choice of parametrization. So it extends to more general subsets $T \subseteq S$, not necessarily living in the image of a parametrization.

Proof. Exercise!

5.2 Geodesics

Proposition. A smooth curve $\gamma$ satisfies the geodesic ODEs if and only if $\gamma$ is a stationary point of the energy function for all proper variation, i.e. if we define the function

$$E(\tau) = \text{energy}(\gamma_\tau) : (-\epsilon, \epsilon) \to \mathbb{R},$$

then

$$\left. \frac{dE}{d\tau} \right|_{\tau=0} = 0.$$

Proof. We let $\gamma(t) = (u(t), v(t))$. Then we have

$$\text{energy}(\gamma) = \int_a^b (E(u, v)\dot{u}^2 + 2F(u, v)\dot{u}\dot{v} + G(u, v)\dot{v}^2) \, dt = \int_a^b I(u, v, \dot{u}, \dot{v}) \, dt.$$

We consider this as a function of four variables $u, \dot{u}, v, \dot{v}$, which are not necessarily related to one another. From the calculus of variations, we know $\gamma$ is stationary if and only if

$$\frac{d}{dt} \left( \frac{\partial I}{\partial \dot{u}} \right) = \frac{\partial I}{\partial u}, \quad \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{v}} \right) = \frac{\partial I}{\partial v}.$$

The first equation gives us

$$\frac{d}{dt} (2(E\dot{u} + F\dot{v})) = E_u \ddot{u}^2 + 2F_u \ddot{u} \ddot{v} + G_u \ddot{v}^2,$$

which is exactly the geodesic ODE. Similarly, the second equation gives the other geodesic ODE. So done.
5 Smooth embedded surfaces (in $\mathbb{R}^3$) IB Geometry (Theorems with proof)

Corollary. If a curve $\Gamma$ minimizes the energy among all curves from $P = \Gamma(a)$ to $Q = \Gamma(b)$, then $\Gamma$ is a geodesic.

Proof. For any $a_1, a_2$ such that $a \leq a_1 \leq b_1 \leq b$, we let $\Gamma_1 = \Gamma|_{[a_1, b_1]}$. Then $\Gamma_1$ also minimizes the energy between $a_1$ and $b_1$ for all curves between $\Gamma(a_1)$ and $\Gamma(b_1)$.

If we picked $a_1, b_1$ such that $\Gamma([a_1, b_1]) \subseteq U$ for some parametrized neighbourhood $U$, then $\Gamma_1$ is a geodesic by the previous proposition. Since the parametrized neighbourhoods cover $S$, at each point $t_0 \in [a, b]$, we can find $a_1, b_1$ such that $\Gamma([a_1, b_1]) \subseteq U$. So done.

Lemma. Let $V \subseteq \mathbb{R}^2$ be an open set with a Riemannian metric, and let $P, Q \in V$. Consider $C^\infty$ curves $\gamma : [a, b] \to V$ such that $\gamma(0) = P, \gamma(1) = Q$. Then such a $\gamma$ will minimize the energy (and therefore is a geodesic) if and only if $\gamma$ minimizes the length and has constant speed.

Proof. Recall the Cauchy-Schwartz inequality for continuous functions $f, g \in C[0, 1]$, which says

$$
\left( \int_0^1 f(x)g(x) \, dx \right)^2 \leq \left( \int_0^1 f(x)^2 \, dx \right) \left( \int_0^1 g(x)^2 \, dx \right),
$$

with equality if $g = \lambda f$ for some $\lambda \in \mathbb{R}$, or $f = 0$, i.e. $g$ and $f$ are linearly dependent.

We now put $f = 1$ and $g = \|\dot{\gamma}\|$. Then Cauchy-Schwartz says

$$(\text{length } \gamma)^2 \leq \text{energy}(\gamma),$$

with equality if and only if $\dot{\gamma}$ is constant.

From this, we see that a curve of minimal energy must have constant speed. Then it follows that minimizing energy is the same as minimizing length if we move at constant speed. \qed

Proposition. A curve $\Gamma$ is a geodesic iff and only if it minimizes the energy locally, and this happens if it minimizes the length locally and has constant speed.

Here minimizing a quantity locally means for every $t \in [a, b]$, there is some $\varepsilon > 0$ such that $\Gamma|_{[t-\varepsilon, t+\varepsilon]}$ minimizes the quantity.

Proposition. In fact, the geodesic ODEs imply $\|\Gamma'(t)\|$ is constant.

Lemma (Gauss’ lemma). The geodesic circles $\{r = r_0\} \subseteq W$ are orthogonal to their radii, i.e. to $\gamma^\theta$, and the Riemannian metric (first fundamental form) on $W$ is

$$
dr^2 + G(r, \theta) \, d\theta^2.
$$

5.3 Surfaces of revolution

Proposition. We assume $\|\dot{\gamma}\| = 1$, i.e. $\ddot{u}^2 + f^2(u)\dot{v}^2 = 1$.

(i) Every unit speed meridians is a geodesic.
(ii) A (unit speed) parallel will be a geodesic if and only if
\[
\frac{df}{du}(u_0) = 0,
\]
i.e. \(u_0\) is a critical point for \(f\).

**Proof.**

(i) In a meridian, \(v = v_0\) is constant. So the second equation holds. Also, we know \( \|\dot{\gamma}\| = |\dot{u}| = 1\). So \(\ddot{u} = 0\). So the first geodesic equation is satisfied.

(ii) Since \(o = o_u\), we know \(f(u_0)^2\dot{v}^2 = 1\). So
\[
\dot{v} = \pm \frac{1}{f(u_0)}.
\]
So the second equation holds. Since \(\dot{v}\) and \(f\) are non-zero, the first equation is satisfied if and only if \(\frac{df}{du} = 0\).

\[\Box\]

### 5.4 Gaussian curvature

**Proposition.** We let
\[
N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}
\]
be our unit normal for a surface patch. Then at each point, we have
\[
N_u = a\sigma_u + b\sigma_v,
\]
\[
N_v = c\sigma_u + d\sigma_v,
\]
where
\[
- \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.
\]
In particular,
\[
K = ad - bc.
\]

**Proof.** Note that
\[
N \cdot N = 1.
\]
Differentiating gives
\[
N \cdot N_u = 0 = N \cdot N_v.
\]
Since \(\sigma_u, \sigma_v\) and \(N\) for an orthogonal basis, at least there are some \(a, b, c, d\) such that
\[
N_u = a\sigma_u + b\sigma_v,
\]
\[
N_v = c\sigma_u + d\sigma_v.
\]
By definition of \(\sigma_u\), we have
\[
N \cdot \sigma_u = 0.
\]
So differentiating gives
\[
N_u \cdot \sigma_u + N \cdot \sigma_{uu} = 0.
\]
So we know
\[ N_u \cdot \sigma_u = -L. \]
Similarly, we find
\[ N_u = \sigma_v = -M = N_v \cdot \sigma_u, \quad N_v \cdot \sigma_v = -N. \]
We dot our original definition of \( N_u, N_v \) in terms of \( a, b, c, d \) with \( \sigma_u \) and \( \sigma_v \) to obtain
\[ -L = aE + bF \quad -M = aF + bG \]
\[ -M = cE + dF \quad -N = cF + dG. \]
Taking determinants, we get the formula for the curvature.

**Theorem.** Suppose for a parametrization \( \sigma : V \to U \subseteq S \subseteq \mathbb{R}^3 \), the first fundamental form is given by
\[ du^2 + G(u,v) \ dv^2 \]
for some \( G \in C^\infty(V) \). Then the Gaussian curvature is given by
\[ K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}. \]
In particular, we do not need to compute the second fundamental form of the surface.

**Proof.** We set
\[ e = \sigma_u, \quad f = \frac{\sigma_v}{\sqrt{G}}. \]
Then \( e \) and \( f \) are unit and orthogonal. We also let \( N = e \times f \) be a third unit vector orthogonal to \( e \) and \( f \) so that they form a basis of \( \mathbb{R}^3 \).

Using the notation of the previous proposition, we have
\[ N_u \times N_v = (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v) \]
\[ = (ad - bc)\sigma_u \times \sigma_v \]
\[ = K\sigma_u \times \sigma_v \]
\[ = K\sqrt{G}e \times f \]
\[ = K\sqrt{G}N. \]
Thus we know
\[ K\sqrt{G} = (N_u \times N_v) \cdot N \]
\[ = (N_u \times N_v) \cdot (e \times f) \]
\[ = (N_u \cdot e)(N_v \cdot f) - (N_u \cdot f)(N_v \cdot e). \]
Since \( N \cdot e = 0 \), we know
\[ N_u \cdot e + N \cdot e_u = 0. \]
Hence to evaluate the expression above, it suffices to compute \( N \cdot e_u \) instead of \( N_u \cdot e \).
Since $\mathbf{e} \cdot \mathbf{e} = 1$, we know
$$\mathbf{e} \cdot \mathbf{e}_u = 0 = \mathbf{e} \cdot \mathbf{e}_v.$$

So we can write
$$\mathbf{e}_u = \alpha \mathbf{f} + \lambda_1 \mathbf{N}$$
$$\mathbf{e}_v = \beta \mathbf{f} + \lambda_2 \mathbf{N}.$$

Similarly, we have
$$\mathbf{f}_u = -\tilde{\alpha} \mathbf{e} + \mu_1 \mathbf{N}$$
$$\mathbf{f}_v = -\tilde{\beta} \mathbf{e} + \mu_2 \mathbf{N}.$$

Our objective now is to find the coefficients $\mu_i, \lambda_i$, and then
$$K \sqrt{G} = \lambda_1 \mu_2 - \lambda_2 \mu_1.$$

Since we know $\mathbf{e} \cdot \mathbf{f} = 0$, differentiating gives
$$\mathbf{e}_u \cdot \mathbf{f} + \mathbf{e} \cdot \mathbf{f}_u = 0$$
$$\mathbf{e}_v \cdot \mathbf{f} + \mathbf{e} \cdot \mathbf{f}_v = 0.$$

Thus we get
$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta.$$

But we have
$$\alpha = \mathbf{e}_u \cdot \mathbf{f} = \sigma_u \cdot \sqrt{G} \mathbf{e}_u \cdot \mathbf{e}_v = \left( \left( \sigma_u \cdot \sigma_v \right)_u - \frac{1}{2} \left( \sigma_u \cdot \sigma_u \right)_v \right) \frac{1}{\sqrt{G}} = 0,$$

since $\sigma_u \cdot \sigma_v = 0, \sigma_u \cdot \sigma_u = 1$. So $\alpha$ vanishes.

Also, we have
$$\beta = \mathbf{e}_v \cdot \mathbf{f} = \sigma_v \cdot \sqrt{G} = \frac{1}{2} \sqrt{G} = (\sqrt{G})_u.$$

Finally, we can use our equations again to find
$$\lambda_1 \mu_1 - \lambda_2 \mu_1 = \mathbf{e}_u \cdot \mathbf{f}_v - \mathbf{e}_v \cdot \mathbf{f}_u$$
$$= (\mathbf{e} \cdot \mathbf{f}_v)_u - (\mathbf{e} \cdot \mathbf{f}_u)_v$$
$$= -\tilde{\beta}_u - (-\tilde{\alpha})_u$$
$$= -(\sqrt{G})_{uu}.$$

So we have
$$K \sqrt{G} = -(\sqrt{G})_{uu},$$

as required. Phew.

**Corollary** (Theorema Egregium). If $S_1$ and $S_2$ have locally isometric charts, then $K$ is locally the same.
5 Smooth embedded surfaces (in $\mathbb{R}^3$) IB Geometry (Theorems with proof)

Proof. We know that this corollary is valid under the assumption of the previous theorem, i.e. the existence of a parametrization $\sigma$ of the surface $S$ such that the first fundamental form is

$$du^2 + G(u,v)\,dv^2.$$

Suitable $\sigma$ includes, for each point $P \in S$, the geodesic polars $(\rho, \theta)$. However, $P$ itself is not in the chart, i.e. $P \not\in \sigma(U)$, and there is no guarantee that there will be some geodesic polar that covers $P$. To solve this problem, we notice that $K$ is a $C^\infty$ function of $S$, and in particular continuous. So we can determine the curvature at $P$ as

$$K(P) = \lim_{\rho \to 0} K(\rho, \sigma).$$

So done.

Note also that every surface of revolution has such a suitable parametrization, as we have previously explicitly seen. \qed
6 Abstract smooth surfaces

Theorem (Gauss-Bonnet theorem). If the sides of a triangle $ABC \subseteq S$ are geodesic segments, then

$$\int_{ABC} K \, dA = (\alpha + \beta + \gamma) - \pi,$$

where $\alpha, \beta, \gamma$ are the angles of the triangle, and $dA$ is the “area element” given by

$$dA = \sqrt{EG - F^2} \, du \, dv,$$

on each domain $U \subseteq S$ of a chart, with $E, F, G$ as in the respective first fundamental form.

Moreover, if $S$ is a compact surface, then

$$\int_S K \, dA = 2\pi \epsilon(S).$$