Part IB — Geometry
Theorems with proof

Based on lectures by A. G. Kovalev
Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Parts of Analysis II will be found useful for this course.

Groups of rigid motions of Euclidean space. Rotation and reflection groups in two and three dimensions. Lengths of curves. [2]

Spherical geometry: spherical lines, spherical triangles and the Gauss-Bonnet theorem. Stereographic projection and Möbius transformations. [3]

Triangulations of the sphere and the torus, Euler number. [1]

Riemannian metrics on open subsets of the plane. The hyperbolic plane. Poincaré models and their metrics. The isometry group. Hyperbolic triangles and the Gauss-Bonnet theorem. The hyperboloid model. [4]

Embedded surfaces in $\mathbb{R}^3$. The first fundamental form. Length and area. Examples. [1]

Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution. [2]

The second fundamental form and Gaussian curvature. For metrics of the form $du^2 + G(u, v)dv^2$, expression of the curvature as $\sqrt{G_{uu}}/\sqrt{G}$. Abstract smooth surfaces and isometries. Euler numbers and statement of Gauss-Bonnet theorem, examples and applications. [3]

Contents

0 Introduction

1 Euclidean geometry

1.1 Isometries of the Euclidean plane

Theorem. Every isometry of $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

for A orthogonal and $\mathbf{b} \in \mathbb{R}^n$.

Proof. Let f be an isometry. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. Let

$$\mathbf{b} = f(\mathbf{0}), \quad \mathbf{a}_i = f(\mathbf{e}_i) - \mathbf{b}.$$

The idea is to construct our matrix A out of these $\mathbf{a}_i$. For A to be orthogonal, $\{\mathbf{a}_i\}$ must be an orthonormal basis.

Indeed, we can compute

$$\|\mathbf{a}_i\| = \|\mathbf{f}(\mathbf{e}_i) - f(\mathbf{0})\| = d(f(\mathbf{e}_i), f(\mathbf{0})) = d(\mathbf{e}_i, \mathbf{0}) = \|\mathbf{e}_i\| = 1.$$

For $i \neq j$, we have

$$\begin{aligned}(\mathbf{a}_i, \mathbf{a}_j) &= -(\mathbf{a}_i, -\mathbf{a}_j)\\ &= -\frac{1}{2}(\|\mathbf{a}_i - \mathbf{a}_j\|^2 - \|\mathbf{a}_i\|^2 - \|\mathbf{a}_j\|^2)\\ &= -\frac{1}{2}(\|f(\mathbf{e}_i) - f(\mathbf{e}_j)\|^2 - 2)\\ &= -\frac{1}{2}(\|\mathbf{e}_i - \mathbf{e}_j\|^2 - 2)\\ &= 0\end{aligned}$$

So $\mathbf{a}_i$ and $\mathbf{a}_j$ are orthogonal. In other words, $\{\mathbf{a}_i\}$ forms an orthonormal set. It is an easy result that any orthogonal set must be linearly independent. Since we have found n orthonormal vectors, they form an orthonormal basis.

Hence, the matrix A with columns given by the column vectors $\mathbf{a}_i$ is an orthogonal matrix. We define a new isometry

$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

We want to show $f = g$. By construction, we know $g(\mathbf{x}) = f(\mathbf{x})$ is true for $\mathbf{x} = \mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$.

We observe that g is invertible. In particular,

$$g^{-1}(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{b}) = A^T\mathbf{x} - A^T\mathbf{b}.$$

Moreover, it is an isometry, since A^T is orthogonal (or we can appeal to the more general fact that inverses of isometries are isometries).

We define

$$h = g^{-1} \circ f.$$

Since it is a composition of isometries, it is also an isometry. Moreover, it fixes $\mathbf{x} = \mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$.

It currently suffices to prove that h is the identity.

Let $\mathbf{x} \in \mathbb{R}^n$, and expand it in the basis as

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

Let

$$\mathbf{y} = h(\mathbf{x}) = \sum_{i=1}^{n} y_i \mathbf{e}_i.$$

We can compute

$$\begin{aligned} d(\mathbf{x}, \mathbf{e}_i)^2 &= (\mathbf{x} - \mathbf{e}_i, \mathbf{x} - \mathbf{e}_i) = \|\mathbf{x}\|^2 + 1 - 2x_i \\ d(\mathbf{x}, \mathbf{0})^2 &= \|\mathbf{x}\|^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(\mathbf{y}, \mathbf{e}_i)^2 &= (\mathbf{y} - \mathbf{e}_i, \mathbf{y} - \mathbf{e}_i) = \|\mathbf{y}\|^2 + 1 - 2y_i \\ d(\mathbf{y}, \mathbf{0})^2 &= \|\mathbf{y}\|^2. \end{aligned}$$

Since h is an isometry and fixes $\mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$, and by definition $h(\mathbf{x}) = \mathbf{y}$, we must have

$$d(\mathbf{x}, \mathbf{0}) = d(\mathbf{y}, \mathbf{0}), \quad d(\mathbf{x}, \mathbf{e}_i) = d(\mathbf{y}, \mathbf{e}_i).$$

The first equality gives $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$, and the others then imply $x_i = y_i$ for all i. In other words, $\mathbf{x} = \mathbf{y} = h(\mathbf{x})$. So h is the identity. □

1.2 Curves in $\mathbb{R}^n$

Proposition. If Γ is continuously differentiable (i.e. C^1), then the length of Γ is given by

$$\text{length}(\Gamma) = \int_a^b \|\Gamma'(t)\| \;\mathrm{d}t.$$

Proof. To simplify notation, we assume $n = 3$. However, the proof works for all possible dimensions. We write

$$\Gamma(t) = (f_1(t), f_2(t), f_3(t)).$$

For every $s \neq t \in [a, b]$, the mean value theorem tells us

$$\frac{f_i(t) - f_i(s)}{t - s} = f_i'(\xi_i)$$

for some $\xi_i \in (s, t)$, for all $i = 1, 2, 3$.

Now note that f_i' are continuous on a closed, bounded interval, and hence uniformly continuous. For all $\varepsilon \in 0$, there is some $\delta > 0$ such that $|t - s| < \delta$ implies

$$|f_i'(\xi_i) - f'(\xi)| < \frac{\varepsilon}{3}$$

for all $\xi \in (s, t)$. Thus, for any $\xi \in (s, t)$, we have

$$\left\|\frac{\Gamma(t) - \Gamma(s)}{t - s} - \Gamma'(\xi)\right\| = \left\|\begin{pmatrix} f_1'(\xi_1) \\ f_2'(\xi_2) \\ f_3'(\xi_3)\end{pmatrix} - \begin{pmatrix} f_1'(\xi) \\ f_2'(\xi) \\ f_3'(\xi)\end{pmatrix}\right\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

In other words,

$$\|\Gamma(t) - \Gamma(s) - (t-s)\Gamma'(\xi)\| \leq \varepsilon(t-s).$$

We relabel $t = t_i$, $s = t_{i-1}$ and $\xi = \frac{s+t}{2}$.

Using the triangle inequality, we have

$$\begin{aligned}(t_i - t_{i-1})\left\|\Gamma'\left(\frac{t_i + t_{i-1}}{2}\right)\right\| - \varepsilon(t_i - t_{i-1}) &< \|\Gamma(t_i) - \Gamma(t_{i-1})\| \\ &< (t_i - t_{i-1})\left\|\Gamma'\left(\frac{t_i + t_{i-1}}{2}\right)\right\| + \varepsilon(t_i - t_{i-1}).\end{aligned}$$

Summing over all i, we obtain

$$\begin{aligned}\sum_i (t_i - t_{i-1})\left\|\Gamma'\left(\frac{t_i + t_{i-1}}{2}\right)\right\| - \varepsilon(b-a) &< S_{\mathcal{D}} \\ &< \sum_i (t_i - t_{i-1})\left\|\Gamma'\left(\frac{t_i + t_{i-1}}{2}\right)\right\| + \varepsilon(b-a),\end{aligned}$$

which is valid whenever $\text{mesh}(\mathcal{D}) < \delta$.

Since Γ' is continuous, and hence integrable, we know

$$\sum_i (t_i - t_{i-1})\left\|\Gamma'\left(\frac{t_i + t_{i-1}}{2}\right)\right\| \to \int_a^b \|\Gamma'(t)\|\;\mathrm{d}t$$

as $\text{mesh}(\mathcal{D}) \to 0$, and

$$\text{length}(\Gamma) = \lim_{\text{mesh}(\mathcal{D})\to 0} S_{\mathcal{D}} = \int_a^b \|\Gamma'(t)\|\;\mathrm{d}t. \qquad \square$$

2 Spherical geometry

2.1 Triangles on a sphere

Theorem (Spherical cosine rule)**.**

$$\sin a \sin b \cos \gamma = \cos c - \cos a \cos b.$$

Proof. We use the fact from IA Vectors and Matrices that

$$(\mathbf{C} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{C} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{A}),$$

which follows easily from the double-epsilon identity

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

In our case, since $\mathbf{C} \cdot \mathbf{C} = 1$, the right hand side is

$$(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{A}).$$

Thus we have

$$\begin{aligned} -\cos \gamma &= \mathbf{n}_1 \cdot \mathbf{n}_2 \\ &= \frac{\mathbf{C} \times \mathbf{B}}{\sin a} \cdot \frac{\mathbf{A} \times \mathbf{C}}{\sin b} \\ &= \frac{(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{A})}{\sin a \sin b} \\ &= \frac{\cos b \cos a - \cos c}{\sin a \sin b}. \end{aligned}$$

□

Corollary (Pythagoras theorem)**.** If $\gamma = \frac{\pi}{2}$, then

$$\cos c = \cos a \cos b.$$

Theorem (Spherical sine rule)**.**

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Proof. We use the fact that

$$(\mathbf{A} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{B}) = (\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}))\mathbf{C},$$

which we again are not bothered to prove again. The left hand side is

$$-(\mathbf{n}_1 \times \mathbf{n}_2) \sin a \sin b$$

Since the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ is $\pi + \gamma$, we know $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{C} \sin \gamma$. Thus the left hand side is

$$-\mathbf{C} \sin a \sin b \sin \gamma.$$

Thus we know

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \sin a \sin b \sin \gamma.$$

However, since the scalar triple product is cyclic, we know

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}).$$

In other words, we have

$$\sin a \sin b \sin \gamma = \sin b \sin c \sin \alpha.$$

Thus we have

$$\frac{\sin \gamma}{\sin c} = \frac{\sin \alpha}{\sin a}.$$

Similarly, we know this is equal to $\frac{\sin \beta}{\sin b}$. □

Corollary (Triangle inequality)**.** For any $P, Q, R \in S^2$, we have

$$d(P, Q) + d(Q, R) \geq d(P, R),$$

with equality if and only if Q lies is in the line segment PR of shortest length.

Proof. The only case left to check is if $d(P, R) = \pi$, since we do not allow our triangles to have side length π. But in this case they are antipodal points, and any Q lies in a line through PR, and equality holds. □

Proposition. Given a curve Γ on $S^2 \subseteq \mathbb{R}^3$ from P to Q, we have $\ell = \text{length}(\Gamma) \geq d(P, Q)$. Moreover, if $\ell = d(P, Q)$, then the image of Γ is a spherical line segment PQ.

Proof. Let $\Gamma : [0, 1] \to S$ and $\ell = \text{length}(\Gamma)$. Then for any dissection $\mathcal{D}$ of $[0, 1]$, say $0 = t_0 < \cdots < t_N = 1$, write $P_i = \Gamma(t_i)$. We define

$$\tilde{S}_{\mathcal{D}} = \sum_i d(P_{i-1}, P_i) > S_{\mathcal{D}} = \sum_i |\overrightarrow{P_{i-1}P_i}|,$$

where the length in the right hand expression is the distance in Euclidean 3-space.

Now suppose $\ell < d(P, Q)$. Then there is some $\varepsilon > 0$ such that $\ell(1 + \varepsilon) < d(P, Q)$.

Recall from basic trigonometric that if $\theta > 0$, then $\sin \theta < \theta$. Also,

$$\frac{\sin \theta}{\theta} \to 1 \text{ as } \theta \to 0.$$

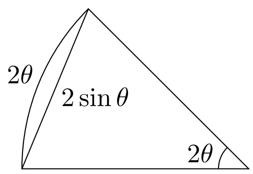
Thus we have

$$\theta \leq (1 + \varepsilon) \sin \theta.$$

for small θ. What we really want is the double of this:

$$2\theta \leq (1 + \varepsilon) 2 \sin \theta.$$

This is useful since these lengths appear in the following diagram:



This means for P, Q sufficiently close, we have $d(P, Q) \leq (1+\varepsilon)|\overrightarrow{PQ}|$.

From Analysis II, we know Γ is uniformly continuous on $[0, 1]$. So we can choose $\mathcal{D}$ such that

$$d(P_{i-1}, P_i) \leq (1+\varepsilon)|\overrightarrow{P_{i-1}P_i}|$$

for all i. So we know that for sufficiently fine $\mathcal{D}$,

$$\tilde{S}_{\mathcal{D}} \leq (1+\varepsilon)S_{\mathcal{D}} < d(P, Q),$$

since $S_{\mathcal{D}} \to \ell$. However, by the triangle inequality $\tilde{S}_{\mathcal{D}} \geq d(P, Q)$. This is a contradiction. Hence we must have $\ell \geq d(P, Q)$.

Suppose now $\ell = d(P, Q)$ for some $\Gamma : [0, 1] \to S$, $\ell = \text{length}(\Gamma)$. Then for every $t \in [0, 1]$, we have

$$\begin{aligned} d(P, Q) = \ell &= \text{length}\,\Gamma|_{[0,t]} + \text{length}\,\Gamma|_{[t,1]} \\ &\geq d(P, \Gamma(t)) + d(\Gamma(t), Q) \\ &\geq d(P, Q). \end{aligned}$$

Hence we must have equality all along the way, i.e.

$$d(P, Q) = d(P, \Gamma(t)) + d(\Gamma(t), Q)$$

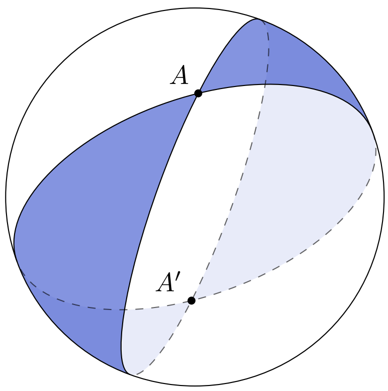
for all $\Gamma(t)$.

However, this is possible only if $\Gamma(t)$ lies on the shorter spherical line segment PQ, as we have previously proved. So done. □

Proposition (Gauss-Bonnet theorem for S^2)**.** If Δ is a spherical triangle with angles α, β, γ, then

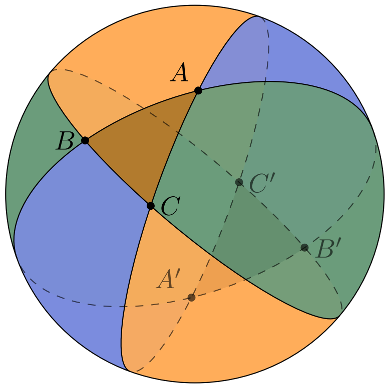
$$\text{area}(\Delta) = (\alpha + \beta + \gamma) - \pi.$$

Proof. We start with the concept of a double lune. A *double lune* with angle $0 < \alpha < \pi$ is two regions S cut out by two planes through a pair of antipodal points, where α is the angle between the two planes.



It is not hard to show that the area of a double lune is 4α, since the area of the sphere is 4π.

Now note that our triangle $\Delta = ABC$ is the intersection of 3 *single* lunes, with each of A, B, C as the pole (in fact we only need two, but it is more convenient to talk about 3).



Therefore Δ together with its antipodal partner Δ' is a subset of each of the 3 double lunes with areas $4\alpha, 4\beta, 4\gamma$. Also, the union of all the double lunes cover the whole sphere, and overlap at exactly Δ and Δ'. Thus

$$4(\alpha + \beta + \gamma) = 4\pi + 2(\text{area}(\Delta) + \text{area}(\Delta')) = 4\pi + 4\,\text{area}(\Delta). \qquad \square$$

2.2 Möbius geometry

Lemma. If $\pi' : S^2 \to \mathbb{C}_\infty$ denotes the stereographic projection from the South Pole instead, then

$$\pi'(P) = \frac{1}{\overline{\pi(P)}}.$$

Proof. Let $P(x, y, z)$. Then

$$\pi(x, y, z) = \frac{x + iy}{1 - z}.$$

Then we have

$$\pi'(x, y, z) = \frac{x + iy}{1 + z},$$

since we have just flipped the z axis around. So we have

$$\overline{\pi(P)}\pi'(P) = \frac{x^2 + y^2}{1 - z^2} = 1,$$

noting that we have $x^2 + y^2 + z^2 = 1$ since we are on the unit sphere. $\square$

Theorem. Via the stereographic projection, every rotation of S^2 induces a Möbius map defined by a matrix in $\mathrm{SU}(2) \subseteq \mathrm{GL}(2, \mathbb{C})$, where

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

Proof.

(i) Consider the $r(\hat{\mathbf{z}}, \theta)$, the rotations about the z axis by θ. These corresponds to the Möbius map $\zeta \mapsto e^{i\theta}\zeta$, which is given by the unitary matrix

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}. \end{pmatrix}$$

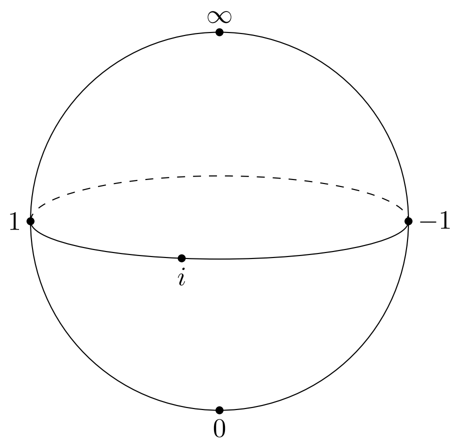
(ii) Consider the rotation $r(\hat{\mathbf{y}}, \frac{\pi}{2})$. This has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ -x \end{pmatrix}.$$

This corresponds to the map

$$\zeta = \frac{x + iy}{1 - z} \mapsto \zeta' = \frac{z + iy}{1 + x}$$

We want to show this is a Möbius map. To do so, we guess what the Möbius map should be, and check it works. We can manually compute that $-1 \mapsto \infty$, $1 \mapsto 0$, $i \mapsto i$.



The only Möbius map that does this is

$$\zeta' = \frac{\zeta - 1}{\zeta + 1}.$$

We now check:

$$\begin{aligned}
\frac{\zeta - 1}{\zeta + 1} &= \frac{x + iy - 1 + z}{x + iy + 1 - z} \\
&= \frac{x - 1 + z + iy}{x + 1 - (z - iy)} \\
&= \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy) - (z^2 + y^2)} \\
&= \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy) + (x^2 - 1)} \\
&= \frac{(z + iy)(x - 1 + z + iy)}{(x + 1)(z + iy + x - 1)} \\
&= \frac{z + iy}{x + 1}.
\end{aligned}$$

So done. We finally have to write this in the form of an SU(2) matrix:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(iii) We claim that SO(3) is generated by $r\left(\hat{\mathbf{y}}, \frac{\pi}{2}\right)$ and $r(\hat{\mathbf{z}}, \theta)$ for $0 \leq \theta < 2\pi$.

To show this, we observe that $r(\hat{\mathbf{x}}, \varphi) = r(\hat{\mathbf{y}}, \frac{\pi}{2}) r(\hat{\mathbf{z}}, \varphi) r(\hat{\mathbf{y}}, -\frac{\pi}{2})$. Note that we read the composition from right to left. You can convince yourself this is true by taking a physical sphere and try rotating. To prove it formally, we can just multiply the matrices out.

Next, observe that for $\mathbf{v} \in S^2 \subseteq \mathbb{R}^3$, there are some angles φ, ψ such that $g = r(\hat{\mathbf{z}}, \psi) r(\hat{\mathbf{x}}, \varphi)$ maps $\mathbf{v}$ to $\hat{\mathbf{x}}$. We can do so by first picking $r(\hat{\mathbf{x}}, \varphi)$ to rotate $\mathbf{v}$ into the (x, y)-plane. Then we rotate about the z-axis to send it to $\hat{\mathbf{x}}$.

Then for any θ, we have $r(\mathbf{v}, \theta) = g^{-1} r(\hat{\mathbf{x}}, \theta) g$, and our claim follows by composition.

(iv) Thus, via the stereographic projection, every rotation of S^2 corresponds to products of Möbius transformations of $\mathbb{C}_\infty$ with matrices in SU(2). □

Theorem. The group of rotations SO(3) acting on S^2 corresponds precisely with the subgroup $\mathrm{PSU}(2) = \mathrm{SU}(2)/\pm 1$ of Möbius transformations acting on $\mathbb{C}_\infty$.

Proof. Let $g \in \mathrm{PSU}(2)$ be a Möbius transformation

$$g(z) = \frac{az + b}{\bar{b}z + \bar{a}}.$$

Suppose first that $g(0) = 0$. So $b = 0$. So $a\bar{a} = 1$. Hence $a = e^{i\theta/2}$. Then g corresponds to $r(\hat{\mathbf{z}}, \theta)$, as we have previously seen.

In general, let $g(0) = w \in \mathbb{C}_\infty$. Let $Q \in S^2$ be such that $\pi(Q) = w$. Choose a rotation $A \in \mathrm{SO}(3)$ such that $A(Q) = -\hat{\mathbf{z}}$. Since A is a rotation, let $\alpha \in \mathrm{PSU}(2)$ be the corresponding Möbius transformation. By construction we have $\alpha(w) = 0$. Then the composition $\alpha \circ g$ fixes zero. So it corresponds to some $B = r(z, \theta)$. We then see that g corresponds to $A^{-1}B \in \mathrm{SO}(3)$. So done. □

3 Triangulations and the Euler number

Theorem. The Euler number e is independent of the choice of triangulation.

Proposition. For every geodesic triangulation of S^2 (and respectively T) has $e = 2$ (respectively, $e = 0$).

Proof. For any triangulation τ, we denote the "faces" of $\Delta_1, \cdots, \Delta_F$, and write $\tau_i = \alpha_i + \beta_i + \gamma_i$ for the sum of the interior angles of the triangles (with $i = 1, \cdots, F$).

Then we have

$$\sum \tau_i = 2\pi V,$$

since the total angle around each vertex is 2π. Also, each triangle has three edges, and each edge is from two triangles. So $3F = 2E$. We write this in a more convenient form:

$$F = 2E - 2F.$$

How we continue depends on whether we are on the sphere or the torus.

- For the sphere, Gauss-Bonnet for the sphere says the area of Δ_i is $\tau_i - \pi$. Since the area of the sphere is 4π, we know

$$\begin{aligned} 4\pi &= \sum \text{area}(\Delta_i) \\ &= \sum (\tau_i - \pi) \\ &= 2\pi V - F\pi \\ &= 2\pi V - (2E - 2F)\pi \\ &= 2\pi(F - E + V). \end{aligned}$$

  So $F - E + V = 2$.

- For the torus, we have $\tau_i = \pi$ for every face in $\mathring{Q}$. So

$$2\pi V = \sum \tau_i = \pi F.$$

  So

$$2V = F = 2E - 2F.$$

  So we get

$$2(F - V + E) = 0,$$

  as required. □

4 Hyperbolic geometry

4.1 Review of derivatives and chain rule

Proposition (Chain rule). Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$. Let $f : U \to \mathbb{R}^m$ and $g : V \to U$ be smooth. Then $f \circ g : V \to \mathbb{R}^m$ is smooth and has a derivative

$$d(f \circ g)_p = (\mathrm{d}f)_{g(p)} \circ (\mathrm{d}g)_p.$$

In terms of the Jacobian matrices, we get

$$J(f \circ g)_p = J(f)_{g(p)} J(g)_p.$$

4.2 Riemannian metrics

4.3 Two models for the hyperbolic plane

Proposition. The elements of $\mathrm{PSL}(2, \mathbb{R})$ are isometries of H, and this preserves the lengths of curves.

Proof. It is easy to check that $\mathrm{PSL}(2, \mathbb{R})$ is generated by

(i) Translations $z \mapsto z + a$ for $a \in \mathbb{R}$

(ii) Dilations $z \mapsto az$ for $a > 0$

(iii) The single map $z \mapsto -\frac{1}{z}$.

So it suffices to show each of these preserves the metric $\frac{|\mathrm{d}z|^2}{y^2}$, where $z = x + iy$. The first two are straightforward to see, by plugging it into formula and notice the metric does not change.

We now look at the last one, given by $z \mapsto -\frac{1}{z}$. The derivative at z is

$$f'(z) = \frac{1}{z^2}.$$

So we get

$$\mathrm{d}z \mapsto \mathrm{d}\left(-\frac{1}{z}\right) = \frac{\mathrm{d}z}{z^2}.$$

So

$$\left|\mathrm{d}\left(-\frac{1}{z}\right)\right|^2 = \frac{|\mathrm{d}z|^2}{|z|^4}.$$

We also have

$$\mathrm{Im}\left(-\frac{1}{z}\right) = -\frac{1}{|z|^2}\,\mathrm{Im}\,\bar{z} = \frac{\mathrm{Im}\, z}{|z|^2}.$$

So

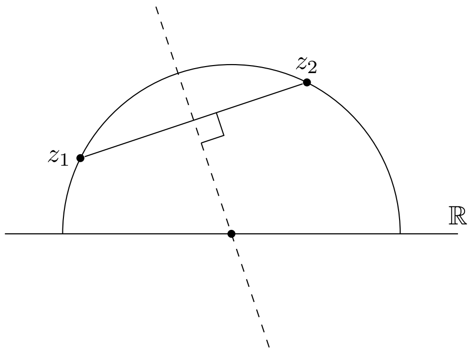
$$\frac{|\mathrm{d}(-1/z)|^2}{\mathrm{Im}(-1/z)^2} = \left(\frac{|\mathrm{d}z|^2}{|z^4|}\right) \Big/ \left(\frac{(\mathrm{Im}\, z)^2}{|z|^4}\right) = \frac{|\mathrm{d}z|^2}{(\mathrm{Im}\, z)^2}.$$

So this is an isometry, as required. □

Lemma. Given any two distinct points $z_1, z_2 \in H$, there exists a unique hyperbolic line through z_1 and z_2.

Proof. This is clear if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ — we just pick the vertical half-line through them, and it is clear this is the only possible choice.

Otherwise, if $\operatorname{Re} z_1 \neq \operatorname{Re} z_2$, then we can find the desired circle as follows:



It is also clear this is the only possible choice. □

Lemma. $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on the set of hyperbolic lines in H.

Proof. It suffices to show that for each hyperbolic line ℓ, there is some $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $g(\ell) = L^+$. This is clear when ℓ is a vertical half-line, since we can just apply a horizontal translation.

If it is a semicircle, suppose it has end-points $s < t \in \mathbb{R}$. Then consider

$$g(z) = \frac{z - t}{z - s}.$$

This has determinant $-s+t > 0$. So $g \in \mathrm{PSL}(2, \mathbb{R})$. Then $g(t) = 0$ and $g(s) = \infty$. Then we must have $g(\ell) = L^+$, since $g(\ell)$ is a hyperbolic line, and the only hyperbolic lines passing through ∞ are the vertical half-lines. So done. □

Proposition. If $\gamma : [0, 1] \to H$ is a piecewise C^1-smooth curve with $\gamma(0) = z_1, \gamma(1) = z_2$, then $\mathrm{length}(\gamma) \geq \rho(z_1, z_2)$, with equality iff γ is a monotonic parametrisation of $[z_1, z_2] \subseteq \ell$, where ℓ is the hyperbolic line through z_1 and z_2.

Proof. We pick an isometry $g \in \mathrm{PSL}(2, \mathbb{R})$ so that $g(\ell) = L^+$. So without loss of generality, we assume $z_1 = iu$ and $z_2 = iv$, with $u < v \in \mathbb{R}$.

We decompose the path as $\gamma(t) = x(t) + iy(t)$. Then we have

$$\begin{aligned}
\mathrm{length}(\gamma) &= \int_0^1 \frac{1}{y}\sqrt{\dot{x}^2 + \dot{y}^2}\;\mathrm{d}t\\
&\geq \int_0^1 \frac{|\dot{y}|}{y}\;\mathrm{d}z\\
&\geq \left|\int_0^1 \frac{\dot{y}}{y}\;\mathrm{d}t\right|\\
&= [\log y(t)]_0^1\\
&= \log\left(\frac{v}{u}\right)
\end{aligned}$$

This calculation also tells us that $\rho(z_1, z_2) = \log\left(\frac{v}{u}\right)$. so $\mathrm{length}(\gamma) \geq \rho(z_1, z_2)$ with equality if and only if $x(t) = 0$ (hence $\gamma \subseteq L^+$) and $\dot{y} \geq 0$ (hence monotonic). □

Corollary (Triangle inequality)**.** Given three points $z_1, z_2, z_3 \in H$, we have

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3),$$

with equality if and only if z_2 lies between z_1 and z_2.

4.4 Geometry of the hyperbolic disk

Lemma. Let G be the set of isometries of the hyperbolic disk. Then

(i) Rotations $z \mapsto e^{i\theta} z$ (for $\theta \in \mathbb{R}$) are elements of G.

(ii) If $a \in D$, then $g(z) = \frac{z-a}{1-\bar{a}z}$ is in G.

Proof.

(i) This is clearly an isometry, since this is a linear map, preserves $|z|$ and $|\mathrm{d}z|$, and hence also the metric

$$\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}.$$

(ii) First, we need to check this indeed maps D to itself. To do this, we first make sure it sends $\{|z| = 1\}$ to itself. If $|z| = 1$, then

$$|1 - \bar{a}z| = |\bar{z}(1 - \bar{a}z)| = |\bar{z} - \bar{a}| = |z - a|.$$

So

$$|g(z)| = 1.$$

Finally, it is easy to check $g(a) = 0$. By continuity, G must map D to itself. We can then show it is an isometry by plugging it into the formula. □

Proposition. If $0 \leq r < 1$, then

$$\rho(0, re^{i\theta}) = 2 \tanh^{-1} r.$$

In general, for $z_1, z_2 \in D$, we have

$$g(z_1, z_2) = 2 \tanh^{-1} \left|\frac{z_1 - z_2}{1 - \bar{z_1} z_2}\right|.$$

Proof. By the lemma above, we can rotate the hyperbolic disk so that $re^{i\theta}$ is rotated to r. So

$$\rho(0, re^{i\theta}) = \rho(0, r).$$

We can evaluate this by performing the integral

$$\rho(0, r) = \int_0^r \frac{2\;\mathrm{d}t}{1 - t^2} = 2 \tanh^{-1} r.$$

For the general case, we apply the Möbius transformation

$$g(z) = \frac{z - z_1}{1 - \bar{z}_1 z}.$$

Then we have

$$g(z_1) = 0, \quad g(z_2) = \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} = \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| e^{i\theta}.$$

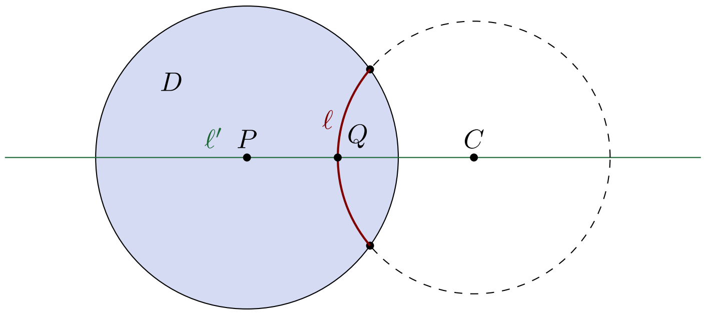
So

$$\rho(z_1, z_2) = \rho(g(z_1), g(z_2)) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|. \qquad \square$$

Proposition. For every point P and hyperbolic line ℓ, with $P \notin \ell$, there is a unique line ℓ' with $P \in \ell'$ such that ℓ' meets ℓ orthogonally, say $\ell \cap \ell' = Q$, and $\rho(P, Q) \leq \rho(P, \tilde{Q})$ for all $\tilde{Q} \in \ell$.

Proof. wlog, assume $P = 0 \in D$. Note that a line in D (that is not a diameter) is a Euclidean circle. So it has a center, say C.

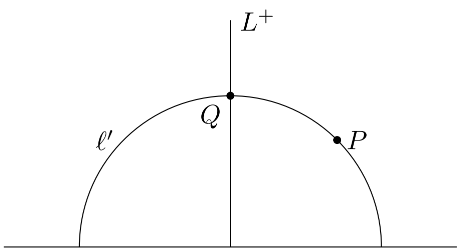
Since any line through P is a diameter, there is clearly only one line that intersects ℓ perpendicularly (recall angles in D is the same as the Euclidean angle).



It is also clear that PQ minimizes the *Euclidean* distance between P and ℓ. While this is not the same as the hyperbolic distance, since hyperbolic lines through P are diameters, having a larger hyperbolic distance is equivalent to having a higher Euclidean distance. So this indeed minimizes the distance. $\square$

Lemma (Hyperbolic reflection)**.** Suppose g is an isometry of the hyperbolic half-plane H and g fixes every point in $L^+ = \{iy : y \in \mathbb{R}^+\}$. Then G is either the identity or $g(z) = -\bar{z}$, i.e. it is a reflection in the vertical axis L^+.

Proof. For every $P \in H \setminus L^+$, there is a unique line ℓ' containing P such that $\ell' \perp L^+$. Let $Q = L^+ \cap \ell'$.



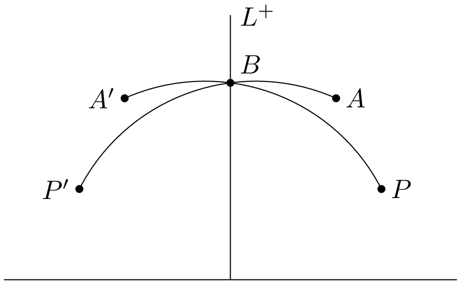
We see ℓ' is a semicircle, and by definition of isometry, we must have

$$\rho(P, Q) = \rho(g(P), Q).$$

Now note that $g(\ell')$ is also a line meeting L^+ perpendicularly at Q, since g fixes L^+ and preserves angles. So we must have $g(\ell') = \ell'$. Then in particular $g(P) \in \ell'$. So we must have $g(P) = P$ or $g(P) = P'$, where P' is the image under reflection in L^+.

Now it suffices to prove that if $g(P) = P$ for any one P, then $g(P)$ must be the identity (if $g(P) = P'$ for all P, then g must be given by $g(z) = -\bar{z}$).

Now suppose $g(P) = P$, and let $A \in H^+$, where $H^+ = \{z \in H : \operatorname{Re} z > 0\}$.



Now if $g(A) \neq A$, then $g(A) = A'$. Then $\rho(A', P) = \rho(A, P)$. But

$$\rho(A', P) = \rho(A', B) + \rho(B, P) = \rho(A, B) + \rho(B, P) > \rho(A, P),$$

by the triangle inequality, noting that $B \notin (AP)$. This is a contradiction. So g must fix everything. □

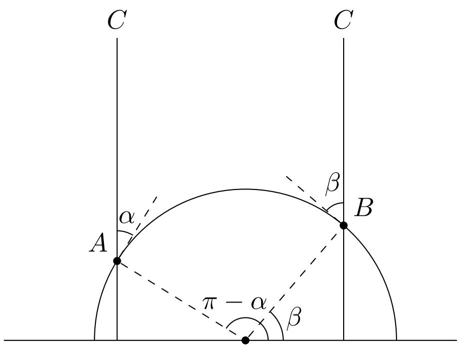
4.5 Hyperbolic triangles

Theorem (Gauss-Bonnet theorem for hyperbolic triangles)**.** For each hyperbolic triangle Δ, say, ABC, with angles $\alpha, \beta, \gamma \geq 0$ (note that zero angle is possible), we have

$$\text{area}(\Delta) = \pi - (\alpha + \beta + \gamma).$$

Proof. First do the case where $\gamma = 0$, so C is "at infinity". Recall that we like to use the disk model if we have a distinguished point in the hyperbolic plane. If we have a distinguished point at *infinity*, it is often advantageous to use the upper half plane model, since ∞ is a distinguished point at infinity.

So we use the upper-half plane model, and wlog $C = \infty$ (apply $\mathrm{PSL}(2, \mathbb{R})$) if necessary. Then AC and BC are vertical half-lines. So AB is the arc of a semi-circle. So AB is an arc of a semicircle.



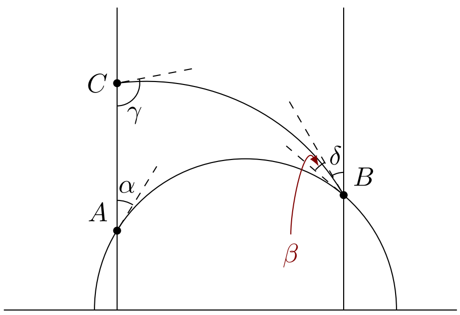
We use the transformation $z \mapsto z + a$ (with $a \in \mathbb{R}$) to center the semi-circle at 0. We then apply $z \mapsto bz$ (with $b > 0$) to make the circle have radius 1. Thus wlog $AB \subseteq \{x^2 + y^2 = 1\}$.

Now we have

$$\begin{aligned}\text{area}(T) &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\;\mathrm{d}y\;\mathrm{d}x\\ &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \frac{1}{\sqrt{1-x^2}}\;\mathrm{d}x\\ &= [-\cos^{-1}(x)]_{\cos(\pi-\alpha)}^{\cos\beta}\\ &= \pi - \alpha - \beta,\end{aligned}$$

as required.

In general, we use H again, and we can arrange AC in a vertical half-line. Also, we can move AB to $x^2 + y^2 = 1$, noting that this transformation keeps AC vertical.



We consider $\Delta_1 = AB\infty$ and $\Delta_2 = CB\infty$. Then we can immediately write

$$\begin{aligned}\text{area}(\Delta_1) &= \pi - \alpha - (\beta + \delta)\\ \text{area}(\Delta_2) &= \pi - \delta - (\pi - \gamma) = \gamma - \delta.\end{aligned}$$

So we have

$$\text{area}(T) = \text{area}(\Delta_2) - \text{area}(\Delta_1) = \pi - \alpha - \beta - \gamma,$$

as required. □

Theorem (Hyperbolic cosine rule)**.** In a triangle with sides a, b, c and angles α, β, γ, we have

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

Proof. See example sheet 2. □

4.6 Hyperboloid model

5 Smooth embedded surfaces (in $\mathbb{R}^3$)

5.1 Smooth embedded surfaces

Proposition. Let $\sigma : V \to U$ and $\tilde{\sigma} : \tilde{V} \to U$ be two C^∞ parametrisations of a surface. Then the homeomorphism

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$$

is in fact a diffeomorphism.

Proof. Since differentiability is a local property, it suffices to consider φ on some small neighbourhood of a point in V. Pick our favorite point $(v_0, u_0) \in \tilde{V}$. We know $\sigma = \sigma(u, v)$ is differentiable. So it has a Jacobian matrix

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_y & z_v \end{pmatrix}.$$

By definition, this matrix has rank two at each point. wlog, we assume the first two rows are linearly independent. So

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0$$

at $(v_0, u_0) \in \tilde{V}$. We define a new function

$$F(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}.$$

Now the inverse function theorem applies. So F has a local C^∞ inverse, i.e. there are two open neighbourhoods $(u_0, v_0) \in N$ and $F(u_0, v_0) \in N' \subseteq \mathbb{R}^2$ such that $f : N \to N'$ is a diffeomorphism.

Writing $\pi : \tilde{\sigma} \to N'$ for the projection $\pi(x, y, z) = (x, y)$ we can put these things in a commutative diagram:

$$\begin{array}{ccc} & & \sigma(N) \\ & \overset{\sigma}{\nearrow} & \downarrow \pi \\ N & \xrightarrow[F]{} & N' \end{array}.$$

We now let $\tilde{N} = \tilde{\sigma}^{-1}(\sigma(N))$ and $\tilde{F} = \pi \circ \tilde{\sigma}$, which is yet again smooth. Then we have the following larger commutative diagram.

$$\begin{array}{ccccc} & & \sigma(N) & & \\ & \overset{\sigma}{\nearrow} & \downarrow \pi & \overset{\tilde{\sigma}}{\nwarrow} & \\ N & \xrightarrow[F]{} & N' & \xleftarrow[\tilde{F}]{} & \tilde{N} \end{array}.$$

Then we have

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} = \sigma^{-1} \circ \pi^{-1} \circ \pi \circ \tilde{\sigma} = F^{-1} \circ \tilde{F},$$

which is smooth, since F^{-1} and $\tilde{F}$ are. Hence φ is smooth everywhere. By symmetry of the argument, φ^{-1} is smooth as well. So this is a diffeomorphism. □

Corollary. The tangent plane $T_Q S$ is independent of parametrization.

Proof. We know

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\varphi_1(\tilde{u}, \tilde{v}), \varphi_2(\tilde{u}, \tilde{v})).$$

We can then compute the partial derivatives as

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} &= \varphi_{1,\tilde{u}}\sigma_u + \varphi_{2,\tilde{u}}\sigma_v \\ \tilde{\sigma}_{\tilde{v}} &= \varphi_{1,\tilde{v}}\sigma_u + \varphi_{2,\tilde{v}}\sigma_v \end{aligned}$$

Here the transformation is related by the Jacobian matrix

$$\begin{pmatrix} \varphi_{1,\tilde{u}} & \varphi_{1,\tilde{v}} \\ \varphi_{2,\tilde{u}} & \varphi_{2,\tilde{v}} \end{pmatrix} = J(\varphi).$$

This is invertible since φ is a diffeomorphism. So $(\sigma_{\tilde{u}}, \sigma_{\tilde{v}})$ and (σ_u, σ_v) are different basis of the same two-dimensional vector space. So done. □

Proposition. If we have two parametrizations related by $\tilde{\sigma} = \sigma \circ \varphi : \tilde{V} \to U$, then $\varphi : \tilde{V} \to V$ is an isometry of Riemannian metrics (on V and $\tilde{V}$).

Proposition. The area of T is independent of the choice of parametrization. So it extends to more general subsets $T \subseteq S$, not necessarily living in the image of a parametrization.

Proof. Exercise! □

5.2 Geodesics

Proposition. A smooth curve γ satisfies the geodesic ODEs if and only if γ is a stationary point of the energy function for all proper variation, i.e. if we define the function

$$E(\tau) = \text{energy}(\gamma_\tau) : (-\varepsilon, \varepsilon) \to \mathbb{R},$$

then

$$\left.\frac{\mathrm{d}E}{\mathrm{d}\tau}\right|_{\tau=0} = 0.$$

Proof. We let $\gamma(t) = (u(t), v(t))$. Then we have

$$\text{energy}(\gamma) = \int_a^b (E(u,v)\dot{u}^2 + 2F(u,v)\dot{u}\dot{v} + G(u,v)\dot{v}^2)\;\mathrm{d}t = \int_a^b I(u, v, \dot{u}, \dot{v})\;\mathrm{d}t.$$

We consider this as a function of four variables $u, \dot{u}, v, \dot{v}$, which are not necessarily related to one another. From the calculus of variations, we know γ is stationary if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial I}{\partial \dot{u}}\right) = \frac{\partial I}{\partial u}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial I}{\partial \dot{v}}\right) = \frac{\partial I}{\partial v}.$$

The first equation gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}(2(E\dot{u} + F\dot{v})) = E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2,$$

which is exactly the geodesic ODE. Similarly, the second equation gives the other geodesic ODE. So done. □

Corollary. If a curve Γ minimizes the energy among all curves from $P = \Gamma(a)$ to $Q = \Gamma(b)$, then Γ is a geodesic.

Proof. For any a_1, a_2 such that $a \leq a_1 \leq b_1 \leq b$, we let $\Gamma_1 = \Gamma|_{[a_1,b_1]}$. Then Γ_1 also minimizes the energy between a_1 and b_1 for all curves between $\Gamma(a_1)$ and $\Gamma(b_1)$.

If we picked a_1, b_1 such that $\Gamma([a_1, b_1]) \subseteq U$ for some parametrized neighbourhood U, then Γ_1 is a geodesic by the previous proposition. Since the parametrized neighbourhoods cover S, at each point $t_0 \in [a, b]$, we can find a_1, b_1 such that $\Gamma([a_1, b_1]) \subseteq U$. So done. □

Lemma. Let $V \subseteq \mathbb{R}^2$ be an open set with a Riemannian metric, and let $P, Q \in V$. Consider C^∞ curves $\gamma : [a, b] \to V$ such that $\gamma(0) = P, \gamma(1) = Q$. Then such a γ will minimize the energy (and therefore is a geodesic) if and only if γ minimizes the length *and* has constant speed.

Proof. Recall the Cauchy-Schwartz inequality for continuous functions $f, g \in C[0, 1]$, which says

$$\left(\int_0^1 f(x)g(x)\;\mathrm{d}x\right)^2 \leq \left(\int_0^1 f(x)^2\;\mathrm{d}x\right)\left(\int_0^1 g(x)^2\;\mathrm{d}x\right),$$

with equality iff $g = \lambda f$ for some $\lambda \in \mathbb{R}$, or $f = 0$, i.e. g and f are linearly dependent.

We now put $f = 1$ and $g = \|\dot\gamma\|$. Then Cauchy-Schwartz says

$$(\text{length}\,\gamma)^2 \leq \text{energy}(\gamma),$$

with equality if and only if $\dot\gamma$ is constant.

From this, we see that a curve of minimal energy must have constant speed. Then it follows that minimizing energy is the same as minimizing length if we move at constant speed. □

Proposition. A curve Γ is a geodesic iff and only if it minimizes the energy *locally*, and this happens if it minimizes the length locally and has constant speed.

Here minimizing a quantity locally means for every $t \in [a, b]$, there is some $\varepsilon > 0$ such that $\Gamma|_{[t-\varepsilon,t+\varepsilon]}$ minimizes the quantity.

Proposition. In fact, the geodesic ODEs imply $\|\Gamma'(t)\|$ is constant.

Lemma (Gauss' lemma)**.** The geodesic circles $\{r = r_0\} \subseteq W$ are orthogonal to their radii, i.e. to γ^θ, and the Riemannian metric (first fundamental form) on W is

$$\mathrm{d}r^2 + G(r, \theta)\;\mathrm{d}\theta^2.$$

5.3 Surfaces of revolution

Proposition. We assume $\|\dot\gamma\| = 1$, i.e. $\dot u^2 + f^2(u)\dot v^2 = 1$.

(i) Every unit speed meridians is a geodesic.

(ii) A (unit speed) parallel will be a geodesic if and only if

$$\frac{\mathrm{d}f}{\mathrm{d}u}(u_0) = 0,$$

i.e. u_0 is a critical point for f.

Proof.

(i) In a meridian, $v = v_0$ is constant. So the second equation holds. Also, we know $\|\dot{\gamma}\| = |\dot{u}| = 1$. So $\ddot{u} = 0$. So the first geodesic equation is satisfied.

(ii) Since $o = o_u$, we know $f(u_0)^2\dot{v}^2 = 1$. So

$$\dot{v} = \pm\frac{1}{f(u_0)}.$$

So the second equation holds. Since $\dot{v}$ and f are non-zero, the first equation is satisfied if and only if $\frac{\mathrm{d}f}{\mathrm{d}u} = 0$. □

5.4 Gaussian curvature

Proposition. We let

$$\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

be our unit normal for a surface patch. Then at each point, we have

$$\begin{aligned} \mathbf{N}_u &= a\sigma_u + b\sigma_v,\\ \mathbf{N}_v &= c\sigma_u + d\sigma_v, \end{aligned}$$

where

$$-\begin{pmatrix} L & M\\ M & N \end{pmatrix} = \begin{pmatrix} a & b\\ c & d \end{pmatrix}\begin{pmatrix} E & F\\ F & G \end{pmatrix}.$$

In particular,

$$K = ad - bc.$$

Proof. Note that

$$\mathbf{N}\cdot\mathbf{N} = 1.$$

Differentiating gives

$$\mathbf{N}\cdot\mathbf{N}_u = 0 = \mathbf{N}\cdot\mathbf{N}_v.$$

Since σ_u, σ_v and $\mathbf{N}$ for an orthogonal basis, at least there are some a, b, c, d such that

$$\begin{aligned} \mathbf{N}_u &= a\sigma_u + b\sigma_v\\ \mathbf{N}_v &= c\sigma_u + d\sigma_v. \end{aligned}$$

By definition of σ_u, we have

$$\mathbf{N}\cdot\sigma_u = 0.$$

So differentiating gives

$$\mathbf{N}_u\cdot\sigma_u + \mathbf{N}\cdot\sigma_{uu} = 0.$$

So we know

$$\mathbf{N}_u \cdot \sigma_u = -L.$$

Similarly, we find

$$\mathbf{N}_u = \sigma_v = -M = N_v \cdot \sigma_u, \quad \mathbf{N}_v \cdot \sigma_v = -N.$$

We dot our original definition of $\mathbf{N}_u, \mathbf{N}_v$ in terms of a, b, c, d with σ_u and σ_v to obtain

$$\begin{aligned} -L &= aE + bF & -M &= aF + bG \\ -M &= cE + dF & -N &= cF + dG. \end{aligned}$$

Taking determinants, we get the formula for the curvature. □

Theorem. Suppose for a parametrization $\sigma : V \to U \subseteq S \subseteq \mathbb{R}^3$, the first fundamental form is given by

$$\mathrm{d}u^2 + G(u, v)\,\mathrm{d}v^2$$

for some $G \in C^\infty(V)$. Then the Gaussian curvature is given by

$$K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}.$$

In particular, we do not need to compute the second fundamental form of the surface.

Proof. We set

$$\mathbf{e} = \sigma_u, \quad \mathbf{f} = \frac{\sigma_v}{\sqrt{G}}.$$

Then $\mathbf{e}$ and $\mathbf{f}$ are unit and orthogonal. We also let $\mathbf{N} = \mathbf{e} \times \mathbf{f}$ be a third unit vector orthogonal to $\mathbf{e}$ and $\mathbf{f}$ so that they form a basis of $\mathbb{R}^3$.

Using the notation of the previous proposition, we have

$$\begin{aligned} \mathbf{N}_u \times \mathbf{N}_v &= (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v) \\ &= (ad - bc)\sigma_u \times \sigma_v \\ &= K\sigma_u \times \sigma_v \\ &= K\sqrt{G}\mathbf{e} \times \mathbf{f} \\ &= K\sqrt{G}\mathbf{N}. \end{aligned}$$

Thus we know

$$\begin{aligned} K\sqrt{G} &= (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} \\ &= (\mathbf{N}_u \times \mathbf{N}_v) \cdot (\mathbf{e} \times \mathbf{f}) \\ &= (\mathbf{N}_u \cdot \mathbf{e})(\mathbf{N}_v \cdot \mathbf{f}) - (\mathbf{N}_u \cdot \mathbf{f})(\mathbf{N}_v \cdot \mathbf{e}). \end{aligned}$$

Since $\mathbf{N} \cdot \mathbf{e} = 0$, we know

$$N_u \cdot \mathbf{e} + \mathbf{N} \cdot \mathbf{e}_u = 0.$$

Hence to evaluate the expression above, it suffices to compute $\mathbf{N} \cdot \mathbf{e}_u$ instead of $\mathbf{N}_u \cdot \mathbf{e}$.

Since $\mathbf{e}\cdot\mathbf{e} = 1$, we know

$$\mathbf{e}\cdot\mathbf{e}_u = 0 = \mathbf{e}\cdot\mathbf{e}_v.$$

So we can write

$$\begin{aligned}\mathbf{e}_u &= \alpha\mathbf{f} + \lambda_1\mathbf{N}\\ \mathbf{e}_v &= \beta\mathbf{f} + \lambda_2\mathbf{N}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbf{f}_u &= -\tilde{\alpha}\mathbf{e} + \mu_1\mathbf{N}\\ \mathbf{f}_v &= -\tilde{\beta}\mathbf{e} + \mu_2\mathbf{N}.\end{aligned}$$

Our objective now is to find the coefficients μ_i, λ_i, and then

$$K\sqrt{G} = \lambda_1\mu_2 - \lambda_2\mu_1.$$

Since we know $\mathbf{e}\cdot\mathbf{f} = 0$, differentiating gives

$$\begin{aligned}\mathbf{e}_u\cdot\mathbf{f} + \mathbf{e}\cdot\mathbf{f}_u &= 0\\ \mathbf{e}_v\cdot\mathbf{f} + \mathbf{e}\cdot\mathbf{f}_v &= 0.\end{aligned}$$

Thus we get

$$\tilde{\alpha} = \alpha,\quad \tilde{\beta} = \beta.$$

But we have

$$\alpha = \mathbf{e}_u\cdot\mathbf{f} = \sigma_{uu}\cdot\frac{\sigma_v}{\sqrt{G}} = \left((\sigma_u\cdot\sigma_v)_u - \frac{1}{2}(\sigma_u\cdot\sigma_u)_v\right)\frac{1}{\sqrt{G}} = 0,$$

since $\sigma_u\cdot\sigma_v = 0, \sigma_u\cdot\sigma_u = 1$. So α vanishes.

Also, we have

$$\beta = \mathbf{e}_v\cdot\mathbf{f} = \sigma_{uv}\cdot\frac{\sigma_v}{\sqrt{G}} = \frac{1}{2}\frac{G_u}{\sqrt{G}} = (\sqrt{G})_u.$$

Finally, we can use our equations again to find

$$\begin{aligned}\lambda_1\mu_1 - \lambda_2\mu_1 &= \mathbf{e}_u\cdot\mathbf{f}_v - \mathbf{e}_v\cdot\mathbf{f}_u\\ &= (\mathbf{e}\cdot\mathbf{f}_v)_u - (\mathbf{e}\cdot\mathbf{f}_u)_v\\ &= -\tilde{\beta}_u - (-\tilde{\alpha})_u\\ &= -(\sqrt{G})_{uu}.\end{aligned}$$

So we have

$$K\sqrt{G} = -(\sqrt{G})_{uu},$$

as required. Phew. □

Corollary (Theorema Egregium)**.** If S_1 and S_2 have locally isometric charts, then K is locally the same.

Proof. We know that this corollary is valid under the assumption of the previous theorem, i.e. the existence of a parametrization σ of the surface S such that the first fundamental form is

$$\mathrm{d}u^2 + G(u, v)\ \mathrm{d}v^2.$$

Suitable σ includes, for each point $P \in S$, the geodesic polars (ρ, θ). However, P itself is not in the chart, i.e. $P \notin \sigma(U)$, and there is no guarantee that there will be some geodesic polar that covers P. To solve this problem, we notice that K is a C^∞ function of S, and in particular continuous. So we can determine the curvature at P as

$$K(P) = \lim_{\rho \to 0} K(\rho, \sigma).$$

So done.

Note also that every surface of revolution has such a suitable parametrization, as we have previously explicitly seen. □

6 Abstract smooth surfaces

Theorem (Gauss-Bonnet theorem)**.** If the sides of a triangle $ABC \subseteq S$ are geodesic segments, then

$$\int_{ABC} K \, \mathrm{d}A = (\alpha + \beta + \gamma) - \pi,$$

where α, β, γ are the angles of the triangle, and $\mathrm{d}A$ is the "area element" given by

$$\mathrm{d}A = \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v,$$

on each domain $\mathcal{U} \subseteq S$ of a chart, with E, F, G as in the respective first fundamental form.

Moreover, if S is a compact surface, then

$$\int_S K \, \mathrm{d}A = 2\pi e(S).$$