Part IB — Geometry

Theorems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Parts of Analysis II will be found useful for this course.

Groups of rigid motions of Euclidean space. Rotation and reflection groups in two and three dimensions. Lengths of curves. [2]

Spherical geometry: spherical lines, spherical triangles and the Gauss-Bonnet theorem. Stereographic projection and Möbius transformations. [3]

Triangulations of the sphere and the torus, Euler number. [1]


Embedded surfaces in $\mathbb{R}^3$. The first fundamental form. Length and area. Examples. [1]

Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution. [2]

The second fundamental form and Gaussian curvature. For metrics of the form $du^2 + G(u, v)dv^2$, expression of the curvature as $\sqrt{G_{uu}}/\sqrt{G}$. Abstract smooth surfaces and isometries. Euler numbers and statement of Gauss-Bonnet theorem, examples and applications. [3]
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0 Introduction
1 Euclidean geometry

1.1 Isometries of the Euclidean plane

Theorem. Every isometry of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is of the form

$$f(x) = Ax + b.$$ 

for $A$ orthogonal and $b \in \mathbb{R}^n$.

1.2 Curves in $\mathbb{R}^n$

Proposition. If $\Gamma$ is continuously differentiable (i.e. $C^1$), then the length of $\Gamma$ is given by

$$\text{length}(\Gamma) = \int_a^b \| \Gamma'(t) \| \, dt.$$ 

2 Spherical geometry

2.1 Triangles on a sphere

Theorem (Spherical cosine rule).
\[ \sin a \sin b \cos \gamma = \cos c - \cos a \cos b. \]

Corollary (Pythagoras theorem). If \( \gamma = \frac{\pi}{2} \), then
\[ \cos c = \cos a \cos b. \]

Theorem (Spherical sine rule).
\[ \frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}. \]

Corollary (Triangle inequality). For any \( P, Q, R \in S^2 \), we have
\[ d(P, Q) + d(Q, R) \geq d(P, R), \]
with equality if and only if \( Q \) lies in the line segment \( PR \) of shortest length.

Proposition. Given a curve \( \Gamma \) on \( S^2 \subseteq \mathbb{R}^3 \) from \( P \) to \( Q \), we have \( \ell = \text{length}(\Gamma) \geq d(P, Q) \). Moreover, if \( \ell = d(P, Q) \), then the image of \( \Gamma \) is a spherical line segment \( PQ \).

Proposition (Gauss-Bonnet theorem for \( S^2 \)). If \( \Delta \) is a spherical triangle with angles \( \alpha, \beta, \gamma \), then
\[ \text{area}(\Delta) = (\alpha + \beta + \gamma) - \pi. \]

2.2 Möbius geometry

Lemma. If \( \pi' : S^2 \to \mathbb{C}_\infty \) denotes the stereographic projection from the South Pole instead, then
\[ \pi'(P) = \frac{1}{\pi(P)}. \]

Theorem. Via the stereographic projection, every rotation of \( S^2 \) induces a Möbius map defined by a matrix in \( \text{SU}(2) \subseteq \text{GL}(2, \mathbb{C}) \), where
\[ \text{SU}(2) = \left\{ \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}. \]

Theorem. The group of rotations \( \text{SO}(3) \) acting on \( S^2 \) corresponds precisely with the subgroup \( \text{PSU}(2) = \text{SU}(2)/\pm 1 \) of Möbius transformations acting on \( \mathbb{C}_\infty \).
3 Triangulations and the Euler number

**Theorem.** The Euler number \( e \) is independent of the choice of triangulation.

**Proposition.** For every geodesic triangulation of \( S^2 \) (and respectively \( T \)) has \( e = 2 \) (respectively, \( e = 0 \)).
4 Hyperbolic geometry

4.1 Review of derivatives and chain rule

**Proposition** (Chain rule). Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$. Let $f : U \to \mathbb{R}^m$ and $g : V \to U$ be smooth. Then $f \circ g : V \to \mathbb{R}^m$ is smooth and has a derivative

$$d(f \circ g)_p = (df)_{g(p)} \circ (dg)_p.$$ 

In terms of the Jacobian matrices, we get

$$J(f \circ g)_p = J(f)_{g(p)} J(g)_p.$$

4.2 Riemannian metrics

4.3 Two models for the hyperbolic plane

**Proposition.** The elements of $\text{PSL}(2, \mathbb{R})$ are isometries of $H$, and this preserves the lengths of curves.

**Lemma.** Given any two distinct points $z_1, z_2 \in H$, there exists a unique hyperbolic line through $z_1$ and $z_2$.

**Lemma.** $\text{PSL}(2, \mathbb{R})$ acts transitively on the set of hyperbolic lines in $H$.

**Proposition.** If $\gamma : [0, 1] \to H$ is a piecewise $C^1$-smooth curve with $\gamma(0) = z_1, \gamma(1) = z_2$, then $\text{length}(\gamma) \geq \rho(z_1, z_2)$, with equality iff $\gamma$ is a monotonic parametrisation of $[z_1, z_2] \subseteq \ell$, where $\ell$ is the hyperbolic line through $z_1$ and $z_2$.

**Corollary (Triangle inequality).** Given three points $z_1, z_2, z_3 \in H$, we have

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3),$$

with equality if and only if $z_2$ lies between $z_1$ and $z_2$.

4.4 Geometry of the hyperbolic disk

**Lemma.** Let $G$ be the set of isometries of the hyperbolic disk. Then

(i) Rotations $z \mapsto e^{i\theta} z$ (for $\theta \in \mathbb{R}$) are elements of $G$.

(ii) If $a \in D$, then $g(z) = \frac{z-a}{1-\bar{a}z}$ is in $G$.

**Proposition.** If $0 \leq r < 1$, then

$$\rho(0, re^{i\theta}) = 2 \tanh^{-1} r.$$ 

In general, for $z_1, z_2 \in D$, we have

$$g(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|.$$ 

**Proposition.** For every point $P$ and hyperbolic line $\ell$, with $P \not\in \ell$, there is a unique line $\ell'$ with $P \in \ell'$ such that $\ell'$ meets $\ell$ orthogonally, say $\ell \cap \ell' = Q$, and $\rho(P, Q) \leq \rho(P, Q')$ for all $Q' \in \ell$.

**Lemma (Hyperbolic reflection).** Suppose $g$ is an isometry of the hyperbolic half-plane $H$ and $g$ fixes every point in $L^+ = \{iy : y \in \mathbb{R}^+\}$. Then $G$ is either the identity or $g(z) = -\bar{z}$, i.e. it is a reflection in the vertical axis $L^+$. 

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4.5 Hyperbolic triangles

**Theorem** (Gauss-Bonnet theorem for hyperbolic triangles). For each hyperbolic triangle \( \Delta \), say, \( ABC \), with angles \( \alpha, \beta, \gamma \geq 0 \) (note that zero angle is possible), we have

\[
\text{area}(\Delta) = \pi - (\alpha + \beta + \gamma).
\]

**Theorem** (Hyperbolic cosine rule). In a triangle with sides \( a, b, c \) and angles \( \alpha, \beta, \gamma \), we have

\[
\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.
\]

4.6 Hyperboloid model
5 Smooth embedded surfaces (in $\mathbb{R}^3$)

5.1 Smooth embedded surfaces

**Proposition.** Let $\sigma : V \to U$ and $\tilde{\sigma} : \tilde{V} \to U$ be two $C^\infty$ parametrisations of a surface. Then the homeomorphism

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$$

is in fact a diffeomorphism.

**Corollary.** The tangent plane $T_Q S$ is independent of parametrization.

**Proposition.** If we have two parametrisations related by $\tilde{\sigma} = \sigma \circ \varphi : \tilde{V} \to U$, then $\varphi : \tilde{V} \to V$ is an isometry of Riemannian metrics (on $V$ and $\tilde{V}$).

**Proposition.** The area of $T$ is independent of the choice of parametrization. So it extends to more general subsets $T \subseteq S$, not necessarily living in the image of a parametrization.

5.2 Geodesics

**Proposition.** A smooth curve $\gamma$ satisfies the geodesic ODEs if and only if $\gamma$ is a stationary point of the energy function for all proper variation, i.e. if we define the function

$$E(\tau) = \text{energy}(\gamma_\tau) : (-\varepsilon, \varepsilon) \to \mathbb{R},$$

then

$$\left. \frac{dE}{d\tau} \right|_{\tau=0} = 0.$$

**Corollary.** If a curve $\Gamma$ minimizes the energy among all curves from $P = \Gamma(a)$ to $Q = \Gamma(b)$, then $\Gamma$ is a geodesic.

**Lemma.** Let $V \subseteq \mathbb{R}^2$ be an open set with a Riemannian metric, and let $P, Q \in V$. Consider $C^\infty$ curves $\gamma : [a, b] \to V$ such that $\gamma(0) = P, \gamma(1) = Q$. Then such a $\gamma$ will minimize the energy (and therefore is a geodesic) if and only if $\gamma$ minimizes the length and has constant speed.

**Proposition.** A curve $\Gamma$ is a geodesic iff and only if it minimizes the energy **locally**, and this happens if it minimizes the length locally and has constant speed.

Here minimizing a quantity locally means for every $t \in [a, b]$, there is some $\varepsilon > 0$ such that $\Gamma|_{[t-\varepsilon, t+\varepsilon]}$ minimizes the quantity.

**Proposition.** In fact, the geodesic ODEs imply $\|\Gamma'(t)\|$ is constant.

**Lemma** (Gauss’ lemma). The geodesic circles $\{r = r_0\} \subseteq W$ are orthogonal to their radii, i.e. to $\gamma^0$, and the Riemannian metric (first fundamental form) on $W$ is

$$dr^2 + G(r, \theta) \, d\theta^2.$$
5.3 Surfaces of revolution

**Proposition.** We assume $\|\dot{\gamma}\| = 1$, i.e. $\dot{u}^2 + f^2(u)\dot{v}^2 = 1$.

(i) Every unit speed meridian is a geodesic.

(ii) A (unit speed) parallel will be a geodesic if and only if

$$\frac{df}{du}(u_0) = 0,$$

i.e. $u_0$ is a critical point for $f$.

5.4 Gaussian curvature

**Proposition.** We let

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

be our unit normal for a surface patch. Then at each point, we have

$$N_u = a\sigma_u + b\sigma_v,$$

$$N_v = c\sigma_u + d\sigma_v,$$

where

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$ 

In particular,

$$K = ad - bc.$$

**Theorem.** Suppose for a parametrization $\sigma : V \to U \subseteq S \subseteq \mathbb{R}^3$, the first fundamental form is given by

$$du^2 + G(u,v) dv^2$$

for some $G \in C^\infty(V)$. Then the Gaussian curvature is given by

$$K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}.$$

In particular, we do not need to compute the second fundamental form of the surface.

**Corollary** (Theorema Egregium). If $S_1$ and $S_2$ have locally isometric charts, then $K$ is locally the same.
6 Abstract smooth surfaces

Theorem (Gauss-Bonnet theorem). If the sides of a triangle $ABC \subseteq S$ are geodesic segments, then

$$\int_{ABC} K \, dA = (\alpha + \beta + \gamma) - \pi,$$

where $\alpha, \beta, \gamma$ are the angles of the triangle, and $dA$ is the “area element” given by

$$dA = \sqrt{EG - F^2} \, du \, dv,$$

on each domain $U \subseteq S$ of a chart, with $E, F, G$ as in the respective first fundamental form.

Moreover, if $S$ is a compact surface, then

$$\int_S K \, dA = 2\pi e(S).$$