Part IB — Geometry
Theorems

Based on lectures by A. G. Kovalev
Notes taken by Dexter Chua

Lent 2016

Parts of Analysis II will be found useful for this course.

Groups of rigid motions of Euclidean space. Rotation and reflection groups in two and three dimensions. Lengths of curves. [2]

Spherical geometry: spherical lines, spherical triangles and the Gauss-Bonnet theorem. Stereographic projection and Möbius transformations. [3]

Triangulations of the sphere and the torus, Euler number. [1]

Riemannian metrics on open subsets of the plane. The hyperbolic plane. Poincaré models and their metrics. The isometry group. Hyperbolic triangles and the Gauss-Bonnet theorem. The hyperboloid model. [4]

Embedded surfaces in $\mathbb{R}^3$. The first fundamental form. Length and area. Examples. [1]

Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution. [2]

The second fundamental form and Gaussian curvature. For metrics of the form $du^2 + G(u,v)dv^2$, expression of the curvature as $\sqrt{G_{uu}}/\sqrt{G}$. Abstract smooth surfaces and isometries. Euler numbers and statement of Gauss-Bonnet theorem, examples and applications. [3]

Contents

0 Introduction

1 Euclidean geometry

1.1 Isometries of the Euclidean plane

Theorem. Every isometry of $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

for A orthogonal and $\mathbf{b} \in \mathbb{R}^n$.

1.2 Curves in $\mathbb{R}^n$

Proposition. If Γ is continuously differentiable (i.e. C^1), then the length of Γ is given by

$$\text{length}(\Gamma) = \int_a^b \|\Gamma'(t)\| \, \mathrm{d}t.$$

2 Spherical geometry

2.1 Triangles on a sphere

Theorem (Spherical cosine rule)**.**

$$\sin a \sin b \cos \gamma = \cos c - \cos a \cos b.$$

Corollary (Pythagoras theorem)**.** If $\gamma = \frac{\pi}{2}$, then

$$\cos c = \cos a \cos b.$$

Theorem (Spherical sine rule)**.**

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Corollary (Triangle inequality)**.** For any $P, Q, R \in S^2$, we have

$$d(P, Q) + d(Q, R) \geq d(P, R),$$

with equality if and only if Q lies is in the line segment PR of shortest length.

Proposition. Given a curve Γ on $S^2 \subseteq \mathbb{R}^3$ from P to Q, we have $\ell = \text{length}(\Gamma) \geq d(P, Q)$. Moreover, if $\ell = d(P, Q)$, then the image of Γ is a spherical line segment PQ.

Proposition (Gauss-Bonnet theorem for S^2)**.** If Δ is a spherical triangle with angles α, β, γ, then

$$\text{area}(\Delta) = (\alpha + \beta + \gamma) - \pi.$$

2.2 Möbius geometry

Lemma. If $\pi' : S^2 \to \mathbb{C}_\infty$ denotes the stereographic projection from the South Pole instead, then

$$\pi'(P) = \frac{1}{\overline{\pi(P)}}.$$

Theorem. Via the stereographic projection, every rotation of S^2 induces a Möbius map defined by a matrix in $\mathrm{SU}(2) \subseteq \mathrm{GL}(2, \mathbb{C})$, where

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

Theorem. The group of rotations $\mathrm{SO}(3)$ acting on S^2 corresponds precisely with the subgroup $\mathrm{PSU}(2) = \mathrm{SU}(2)/\pm 1$ of Möbius transformations acting on $\mathbb{C}_\infty$.

3 Triangulations and the Euler number

Theorem. The Euler number e is independent of the choice of triangulation.

Proposition. For every geodesic triangulation of S^2 (and respectively T) has $e = 2$ (respectively, $e = 0$).

4 Hyperbolic geometry

4.1 Review of derivatives and chain rule

Proposition (Chain rule)**.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^p$. Let $f : U \to \mathbb{R}^m$ and $g : V \to U$ be smooth. Then $f \circ g : V \to \mathbb{R}^m$ is smooth and has a derivative

$$d(f \circ g)_p = (\mathrm{d}f)_{g(p)} \circ (\mathrm{d}g)_p.$$

In terms of the Jacobian matrices, we get

$$J(f \circ g)_p = J(f)_{g(p)} J(g)_p.$$

4.2 Riemannian metrics

4.3 Two models for the hyperbolic plane

Proposition. The elements of $\mathrm{PSL}(2, \mathbb{R})$ are isometries of H, and this preserves the lengths of curves.

Lemma. Given any two distinct points $z_1, z_2 \in H$, there exists a unique hyperbolic line through z_1 and z_2.

Lemma. $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on the set of hyperbolic lines in H.

Proposition. If $\gamma : [0, 1] \to H$ is a piecewise C^1-smooth curve with $\gamma(0) = z_1, \gamma(1) = z_2$, then $\mathrm{length}(\gamma) \geq \rho(z_1, z_2)$, with equality iff γ is a monotonic parametrisation of $[z_1, z_2] \subseteq \ell$, where ℓ is the hyperbolic line through z_1 and z_2.

Corollary (Triangle inequality)**.** Given three points $z_1, z_2, z_3 \in H$, we have

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3),$$

with equality if and only if z_2 lies between z_1 and z_2.

4.4 Geometry of the hyperbolic disk

Lemma. Let G be the set of isometries of the hyperbolic disk. Then

(i) Rotations $z \mapsto e^{i\theta} z$ (for $\theta \in \mathbb{R}$) are elements of G.

(ii) If $a \in D$, then $g(z) = \frac{z-a}{1-\bar{a}z}$ is in G.

Proposition. If $0 \leq r < 1$, then

$$\rho(0, re^{i\theta}) = 2 \tanh^{-1} r.$$

In general, for $z_1, z_2 \in D$, we have

$$g(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \bar{z_1} z_2} \right|.$$

Proposition. For every point P and hyperbolic line ℓ, with $P \not\in \ell$, there is a unique line ℓ' with $P \in \ell'$ such that ℓ' meets ℓ orthogonally, say $\ell \cap \ell' = Q$, and $\rho(P, Q) \leq \rho(P, \tilde{Q})$ for all $\tilde{Q} \in \ell$.

Lemma (Hyperbolic reflection)**.** Suppose g is an isometry of the hyperbolic half-plane H and g fixes every point in $L^+ = \{iy : y \in \mathbb{R}^+\}$. Then G is either the identity or $g(z) = -\bar{z}$, i.e. it is a reflection in the vertical axis L^+.

4.5 Hyperbolic triangles

Theorem (Gauss-Bonnet theorem for hyperbolic triangles)**.** For each hyperbolic triangle Δ, say, ABC, with angles $\alpha, \beta, \gamma \geq 0$ (note that zero angle is possible), we have
$$\operatorname{area}(\Delta) = \pi - (\alpha + \beta + \gamma).$$

Theorem (Hyperbolic cosine rule)**.** In a triangle with sides a, b, c and angles α, β, γ, we have
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

4.6 Hyperboloid model

5 Smooth embedded surfaces (in $\mathbb{R}^3$)

5.1 Smooth embedded surfaces

Proposition. Let $\sigma : V \to U$ and $\tilde{\sigma} : \tilde{V} \to U$ be two C^∞ parametrisations of a surface. Then the homeomorphism

$$\varphi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$$

is in fact a diffeomorphism.

Corollary. The tangent plane $T_Q S$ is independent of parametrization.

Proposition. If we have two parametrizations related by $\tilde{\sigma} = \sigma \circ \varphi : \tilde{V} \to U$, then $\varphi : \tilde{V} \to V$ is an isometry of Riemannian metrics (on V and $\tilde{V}$).

Proposition. The area of T is independent of the choice of parametrization. So it extends to more general subsets $T \subseteq S$, not necessarily living in the image of a parametrization.

5.2 Geodesics

Proposition. A smooth curve γ satisfies the geodesic ODEs if and only if γ is a stationary point of the energy function for all proper variation, i.e. if we define the function

$$E(\tau) = \text{energy}(\gamma_\tau) : (-\varepsilon, \varepsilon) \to \mathbb{R},$$

then

$$\left.\frac{\mathrm{d}E}{\mathrm{d}\tau}\right|_{\tau=0} = 0.$$

Corollary. If a curve Γ minimizes the energy among all curves from $P = \Gamma(a)$ to $Q = \Gamma(b)$, then Γ is a geodesic.

Lemma. Let $V \subseteq \mathbb{R}^2$ be an open set with a Riemannian metric, and let $P, Q \in V$. Consider C^∞ curves $\gamma : [a, b] \to V$ such that $\gamma(0) = P, \gamma(1) = Q$. Then such a γ will minimize the energy (and therefore is a geodesic) if and only if γ minimizes the length *and* has constant speed.

Proposition. A curve Γ is a geodesic iff and only if it minimizes the energy *locally*, and this happens if it minimizes the length locally and has constant speed.

Here minimizing a quantity locally means for every $t \in [a, b]$, there is some $\varepsilon > 0$ such that $\Gamma|_{[t-\varepsilon, t+\varepsilon]}$ minimizes the quantity.

Proposition. In fact, the geodesic ODEs imply $\|\Gamma'(t)\|$ is constant.

Lemma (Gauss' lemma)**.** The geodesic circles $\{r = r_0\} \subseteq W$ are orthogonal to their radii, i.e. to γ^θ, and the Riemannian metric (first fundamental form) on W is

$$\mathrm{d}r^2 + G(r, \theta)\ \mathrm{d}\theta^2.$$

5.3 Surfaces of revolution

Proposition. We assume $\|\dot{\gamma}\| = 1$, i.e. $\dot{u}^2 + f^2(u)\dot{v}^2 = 1$.

(i) Every unit speed meridians is a geodesic.

(ii) A (unit speed) parallel will be a geodesic if and only if

$$\frac{\mathrm{d}f}{\mathrm{d}u}(u_0) = 0,$$

i.e. u_0 is a critical point for f.

5.4 Gaussian curvature

Proposition. We let

$$\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

be our unit normal for a surface patch. Then at each point, we have

$$\begin{aligned}\mathbf{N}_u &= a\sigma_u + b\sigma_v,\\ \mathbf{N}_v &= c\sigma_u + d\sigma_v,\end{aligned}$$

where

$$-\begin{pmatrix} L & M \\ M & N\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}.$$

In particular,

$$K = ad - bc.$$

Theorem. Suppose for a parametrization $\sigma : V \to U \subseteq S \subseteq \mathbb{R}^3$, the first fundamental form is given by

$$\mathrm{d}u^2 + G(u, v)\,\mathrm{d}v^2$$

for some $G \in C^\infty(V)$. Then the Gaussian curvature is given by

$$K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}.$$

In particular, we do not need to compute the second fundamental form of the surface.

Corollary (Theorema Egregium)**.** If S_1 and S_2 have locally isometric charts, then K is locally the same.

6 Abstract smooth surfaces

Theorem (Gauss-Bonnet theorem)**.** If the sides of a triangle $ABC \subseteq S$ are geodesic segments, then

$$\int_{ABC} K \, \mathrm{d}A = (\alpha + \beta + \gamma) - \pi,$$

where α, β, γ are the angles of the triangle, and $\mathrm{d}A$ is the "area element" given by

$$\mathrm{d}A = \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v,$$

on each domain $\mathcal{U} \subseteq S$ of a chart, with E, F, G as in the respective first fundamental form.

Moreover, if S is a compact surface, then

$$\int_S K \, \mathrm{d}A = 2\pi e(S).$$