Part IB — Fluid Dynamics

Theorems with proof

Based on lectures by P. F. Linden

Notes taken by Dexter Chua

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Parallel viscous flow

Plane Couette flow, dynamic viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematic viscosity, brief description of viscous boundary layers (skin depth). [3]

Kinematics

Material time derivative. Conservation of mass and the kinematic boundary condition. Incompressibility; streamfunction for two-dimensional flow. Streamlines and path lines. [2]

Dynamics

Statement of Navier-Stokes momentum equation. Reynolds number. Stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line stretching, irrotational flow remains irrotational. [4]

Potential flows

Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flows with potential forces. Oscillations in a manometer and of a bubble. [3]

Geophysical flows

Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation. [4]

Contents

0 Introduction

1 Parallel viscous flow

1.1 Preliminaries

1.2 Stress

Law. For a Newtonian fluid, we have

$$\tau_s \propto \frac{U}{h}.$$

1.3 Steady parallel viscous flow

1.4 Derived properties of a flow

1.5 More examples

2 Kinematics

2.1 Material time derivative

2.2 Conservation of mass

2.3 Kinematic boundary conditions

2.4 Streamfunction for incompressible flow

3 Dynamics

3.1 Navier-Stokes equations

Law (Navier-Stokes equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}.$$

3.2 Pressure

3.3 Reynolds number

3.4 A case study: stagnation point flow ($\mathbf{u} = \mathbf{0}$)

3.5 Momentum equation for inviscid ($\nu = 0$) incompressible fluid

Proposition (Euler momentum equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mathbf{f}.$$

Proposition (Momentum integral for steady flow)**.**

$$\int_{\partial \mathcal{D}} (\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n} + \chi \mathbf{n}) \;\mathrm{d}S = 0.$$

Proposition (Bernoulli's equation)**.**

$$\frac{1}{2}\rho \frac{\partial |\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla \left(\frac{1}{2}\rho |\mathbf{u}|^2 + p + \chi\right).$$

3.6 Linear flows

3.7 Vorticity equation

Proposition (Vorticity equation)**.**

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

4 Inviscid irrotational flow

4.1 Three-dimensional potential flows

4.2 Potential flow in two dimensions

4.3 Time dependent potential flows

5 Water waves

5.1 Dimensional analysis

5.2 Equation and boundary conditions

5.3 Two-dimensional waves (straight crested waves)

5.4 Group velocity

5.5 Rayleigh-Taylor instability

6 Fluid dynamics on a rotating frame

6.1 Equations of motion in a rotating frame

Proposition (Euler's equation in a rotating frame)**.**

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}.$$

6.2 Shallow water equations

6.3 Geostrophic balance