Part IB — Electromagnetism
Definitions

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Electromagnetism and Relativity
Review of Special Relativity; tensors and index notation. Lorentz force law. Electromagnetic tensor. Lorentz transformations of electric and magnetic fields. Currents and the conservation of charge. Maxwell equations in relativistic and non-relativistic forms. [5]

Electrostatics
Gauss's law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distributions, dipoles. Electrostatic energy. Conductors. [3]

Magnetostatics
Magnetic fields due to steady currents. Ampre's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. Lorentz force on current distributions and force between current-carrying wires. Ohm's law. [3]

Electrodynamics
Faraday's law of induction for fixed and moving circuits. Electromagnetic energy and Poynting vector. 4-vector potential, gauge transformations. Plane electromagnetic waves in vacuum, polarization. [5]

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0 Introduction

1 Preliminaries

1.1 Charge and Current

Definition (Charge density)**.** The *charge density* is the charge per unit volume. The total charge in a region V is

$$Q = \int_V \rho(\mathbf{x}, t)\ \mathrm{d}V$$

Definition (Current and current density)**.** For any surface S, the integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

counts the charge per unit time passing through S. I is the *current*, and $\mathbf{J}$ is the *current density*, "current per unit area".

1.2 Forces and Fields

2 Electrostatics

2.1 Gauss' Law

Definition (Flux through surface)**.** The *flux* of $\mathbf{E}$ through the surface S is defined to be

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}.$$

2.2 Electrostatic potential

Definition (Electrostatic potential)**.** If $\mathbf{E} = -\nabla \phi$, then ϕ is the *electrostatic potential.*

2.2.1 Point charge

2.2.2 Dipole

Definition (Dipole)**.** A *dipole* consists of two point charges, $+Q$ and $-Q$ at $\mathbf{r} = 0$ and $\mathbf{r} = -\mathbf{d}$ respectively.

Definition (Electric dipole moment)**.** We define the *electric dipole moment* to be

$$\mathbf{p} = Q\mathbf{d}.$$

By convention, it points from -ve to +ve.

2.2.3 General charge distribution

2.2.4 Field lines and equipotentials

Definition (Field line)**.** A *field line* is a continuous line tangent to the electric field $\mathbf{E}$. The density of lines is proportional to $|\mathbf{E}|$.

Definition (Equipotentials)**.** *Equipotentials* are surfaces of constant ϕ. Because $\mathbf{E} = -\nabla \phi$, they are always perpendicular to field lines.

2.3 Electrostatic energy

2.4 Conductors

Definition (Conductor)**.** A *conductor* is a region of space which contains lots of charges that are free to move.

3 Magnetostatics

3.1 Ampere's Law

3.2 Vector potential

Definition (Vector potential)**.** If $\mathbf{B} = \nabla \times \mathbf{A}$, then $\mathbf{A}$ is the *vector potential*.

Definition ((Coulomb) gauge)**.** Each choice of $\mathbf{A}$ is called a *gauge*. An $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$ is called a *Coulomb gauge*.

3.3 Magnetic dipoles

Definition (Magnetic dipole moment)**.** The *magnetic dipole moment* is

$$\mathbf{m} = I\mathbf{S}.$$

Definition (Magnetic dipole moment)**.**

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\,\mathrm{d}V'.$$

3.4 Magnetic forces

4 Electrodynamics

4.1 Induction

Definition (Electromotive force (emf)). The *electromotive force* (emf) is

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r}.$$

Despite the name, this is not a force! We can think of it as the work done on a unit charge moving around the curve, or the "voltage" of the system.

Definition (Magnetic flux). The *magnetic flux* is

$$\Phi = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

4.2 Magnetostatic energy

Definition (Inductance). The *inductance* of a curve C, defined as

$$L = \frac{\Phi}{I},$$

is the amount of flux it generates per unit current passing through C. This is a property only of the curve C.

4.3 Resistance

Definition (Resistance). The *resistance* is the R in Ohm's law.

Definition (Resistivity and conductivity). For the wire of length L and cross-sectional area A, we define the *resistivity* to be

$$\rho = \frac{AR}{L},$$

and the conductivity is

$$\sigma = \frac{1}{\rho}.$$

Definition (Joule heating). *Joule heating* is the energy lost in a circuit due to friction. It is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = I^2 R.$$

4.4 Displacement currents

4.5 Electromagnetic waves

Definition (Amplitude, wave number and frequency).

(i) E_0 is the *amplitude*

(ii) k is the *wave number*.

(iii) ω is the *(angular) frequency.*

The wave number is related to the wavelength by

$$\lambda = \frac{2\pi}{k}.$$

Since the wave has to travel at speed c, we must have

$$\omega^2 = c^2 k^2$$

So the value of k determines the value of ω, vice versa.

Definition (Wave vector)**.** $\mathbf{k}$ is the *wave vector*, which is real.

Definition (Linearly polarized wave)**.** A solution with real $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$ is said to be *linearly polarized.*

Definition (Elliptically polarized wave)**.** If $\mathbf{E}_0$ and $\mathbf{B}_0$ are complex, then it is said to be *elliptically polarized.* In the special case where $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$, this is *circular polarization.*

4.6 Poynting vector

Definition (Poynting vector)**.** The *Poynting vector* is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

5 Electromagnetism and relativity

5.1 A review of special relativity

5.1.1 A geometric interlude on (co)vectors

5.1.2 Transformation rules

5.1.3 Vectors and covectors in SR

Definition (Orthonormal basis)**.** An *orthonormal basis* of spacetime is a basis where the metric takes the form $(*)$. An (orthonormal) coordinate system is a choice of orthonormal basis.

Definition (Lorentz transformations)**.** A *Lorentz transformation* is a change-of-basis matrix that preserves the inner product, i.e. orthogonal matrices under the Minkowski metric.

Definition (Vectors and covectors)**.** A vector is an assignment of 4 numbers $V^\mu, \mu = 0, 1, 2, 3$ to each coordinate system such that under a change of basis by Λ, the coordinates V^μ transform as $V^\mu \mapsto \Lambda^\mu{}_\nu V^\nu$.

A covector is an assignment of 4 numbers $V_\mu, \mu = 0, 1, 2, 3$ to each coordinate system such that under a change of basis by Λ, the coordinates V_μ transform as $V_\mu \mapsto \Lambda_\mu{}^\nu V_\nu$.

Definition (Tensor)**.** A *tensor* of type (m, n) is a quantity

$$T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}$$

which transforms as

$$T'^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n} = \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\mu_m}{}_{\rho_m} \Lambda_{\nu_1}{}^{\sigma_1} \cdots \Lambda_{\nu_n}{}^{\sigma_n} \times T^{\rho_1, \cdots, \rho_m}{}_{\sigma_1, \cdots, \sigma_n}.$$

Definition (4-derivative)**.** The *4-derivative* is

$$\partial_\mu = \frac{\partial}{\partial X^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right).$$

5.2 Conserved currents

5.3 Gauge potentials and electromagnetic fields

5.4 Maxwell Equations

5.5 The Lorentz force law