Part IB — Complex Methods
Definitions

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Analytic functions
Definition of an analytic function. Cauchy-Riemann equations. Analytic functions as conformal mappings; examples. Application to the solutions of Laplace's equation in various domains. Discussion of $\log z$ and z^a. [5]

Contour integration and Cauchy's Theorem
[*Proofs of theorems in this section will not be examined in this course.*]
Contours, contour integrals. Cauchy's theorem and Cauchy's integral formula. Taylor and Laurent series. Zeros, poles and essential singularities. [3]

Residue calculus
Residue theorem, calculus of residues. Jordan's lemma. Evaluation of definite integrals by contour integration. [4]

Fourier and Laplace transforms
Laplace transform: definition and basic properties; inversion theorem (proof not required); convolution theorem. Examples of inversion of Fourier and Laplace transforms by contour integration. Applications to differential equations. [4]

Contents

0 Introduction

1 Analytic functions

1.1 The complex plane and the Riemann sphere

Definition (Modulus and argument)**.** The *modulus* and *argument* of a complex number $z = x + iy$ are given by

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arg z,$$

where $x = r\cos\theta, y = r\sin\theta$.

Definition (Principal value of argument)**.** The *principal value* of the argument is the value of θ in the range $(-\pi, \pi]$.

Definition (Open set)**.** An *open set* $\mathcal{D}$ is one which does not include its boundary. More technically, $\mathcal{D} \subseteq \mathbb{C}$ is open if for all $z_0 \in \mathcal{D}$, there is some $\delta > 0$ such that the disc $|z - z_0| < \delta$ is contained in $\mathcal{D}$.

Definition (Neighbourhood)**.** A *neighbourhood* of a point $z \in \mathbb{C}$ is an open set containing z.

Definition (The extended complex plane)**.** The *extended complex plane* is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. We can reach the "point at infinity" by going off in any direction in the plane, and all are equivalent. In particular, there is no concept of $-\infty$. All infinities are the same. Operations with ∞ are done in the obvious way.

1.2 Complex differentiation

Definition (Complex differentiable function)**.** A complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ is *differentiable* at z if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists (and is therefore independent of the direction of approach — but now there are infinitely many possible directions).

Definition (Analytic function)**.** We say f is *analytic* at a point z if there exists a neighbourhood of z throughout which f' exists. The terms *regular* and *holomorphic* are also used.

Definition (Entire function)**.** A complex function is *entire* if it is analytic throughout $\mathbb{C}$.

1.3 Harmonic functions

Definition (Harmonic conjugates)**.** Two functions u, v satisfying the Cauchy-Riemann equations are called *harmonic conjugates*.

Definition (Harmonic function)**.** A function satisfying Laplace's equation equation in an open set is said to be *harmonic*.

1.4 Multi-valued functions

Definition (Branch point)**.** A *branch point* of a function is a point which is impossible to encircle with a curve on which the function is both continuous and single-valued. The function is said to have a *branch point singularity* there.

1.5 Möbius map

Definition (Circline)**.** A *circline* is either a circle or a line.

1.6 Conformal maps

Definition (Conformal map)**.** A *conformal map* $f : U \to V$, where U, V are *open* subsets of $\mathbb{C}$, is one which is analytic with non-zero derivative.

1.7 Solving Laplace's equation using conformal maps

2 Contour integration and Cauchy's theorem

2.1 Contour and integrals

Definition (Curve)**.** A *curve* $\gamma(t)$ is a (continuous) map $\gamma : [0, 1] \to \mathbb{C}$.

Definition (Closed curve)**.** A *closed curve* is a curve γ such that $\gamma(0) = \gamma(1)$.

Definition (Simple curve)**.** A *simple curve* is one which does not intersect itself, except at $t = 0, 1$ in the case of a closed curve.

Definition (Contour)**.** A *contour* is a piecewise smooth curve.

Notation. The contour $-\gamma$ is the contour γ traversed in the opposite direction. Formally, we say

$$(-\gamma)(t) = \gamma(1 - t).$$

Given two contours γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$, $\gamma_1 + \gamma_2$ denotes the two contours joined end-to-end. Formally,

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t < \frac{1}{2} \\ \gamma_2(2t - 1) & t \geq \frac{1}{2} \end{cases}.$$

Definition (Contour integral)**.** The *contour integral* $\int_\gamma f(z)\,\mathrm{d}z$ is defined to be the usual real integral

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^1 f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Definition (Simply connected domain)**.** A domain $\mathcal{D}$ (an open subset of $\mathbb{C}$) is *simply connected* if it is connected and every closed curve in $\mathcal{D}$ encloses only points which are also in $\mathcal{D}$.

2.2 Cauchy's theorem

2.3 Contour deformation

2.4 Cauchy's integral formula

3 Laurent series and singularities

3.1 Taylor and Laurent series

3.2 Zeros

Definition (Zeros)**.** The *zeros* of an analytic function $f(Z)$ are the points z_0 where $f(z_0) = 0$. A zero is of *order* N if in its Taylor series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, the first non-zero coefficient is a_N.

Alternatively, it is of order N if $0 = f(z_0) = f'(z_0) = \cdots = f^{(N-1)}$, but $f^{(N)}(z_0) \neq 0$.

Definition (Simple zero)**.** A zero of order one is called a *simple zero*.

3.3 Classification of singularities

Definition (Isolated singularity)**.** Suppose that f has a singularity at $z_0 = z$. If there is a neighbourhood of z_0 within which f is analytic, except at z_0 itself, then f has an *isolated singularity* at z_0. If there is no such neighbourhood, then f has an *essential (non-isolated) singularity* at z_0.

3.4 Residues

Definition (Residue)**.** The *residue* of a function f at an isolated singularity z_0 is the coefficient a_{-1} in its Laurent expansion about z_0. There is no standard notation, but shall denote the residue by $\operatorname*{res}_{z=z_0} f(z)$.

4 The calculus of residues

4.1 The residue theorem

4.2 Applications of the residue theorem

4.3 Further applications of the residue theorem using rectangular contours

4.4 Jordan's lemma

5 Transform theory

5.1 Fourier transforms

Definition (Fourier transform)**.** The *Fourier transform* of a function $f(x)$ that decays sufficiently as $|x| \to \infty$ is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\;\mathrm{d}x,$$

and the inverse transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\;\mathrm{d}k.$$

Notation. The Fourier transform can also be denoted by $\tilde{f} = \mathcal{F}(f)$ or $\tilde{f}(k) = \mathcal{F}(f)(k)$. In a slight abuse of notation, we often write $\tilde{f}(k) = \mathcal{F}(f(x))$, but this is not correct notation, since $\mathcal{F}$ takes in a function at a parameter, not a function evaluated at a particular point.

5.2 Laplace transform

Definition (Laplace transform)**.** The *Laplace transform* of a function $f(t)$ such that $f(t) = 0$ for $t < 0$ is defined by

$$\hat{f}(p) = \int_0^{\infty} f(t) e^{-pt}\;\mathrm{d}t.$$

This exists for functions that grow no more than exponentially fast.

Notation. We sometimes write

$$\hat{f} = \mathcal{L}(f),$$

or

$$\hat{f}(p) = \mathcal{L}(f(t)).$$

5.3 Elementary properties of the Laplace transform

5.4 The inverse Laplace transform

5.5 Solution of differential equations using the Laplace transform

5.6 The convolution theorem for Laplace transforms

Definition (Convolution)**.** The *convolution* of two functions f and g is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - t') g(t')\;\mathrm{d}t'.$$