Analytic functions
Complex differentiation and the Cauchy–Riemann equations. Examples. Conformal mappings. Informal discussion of branch points, examples of \( \log z \) and \( z^c \). [3]

Contour integration and Cauchy’s theorem

Expansions and singularities

The residue theorem
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0 Introduction
1 Complex differentiation

1.1 Differentiation

Proposition. Let $f$ be defined on an open set $U \subseteq \mathbb{C}$. Let $w = c + id \in U$ and write $f = u + iv$. Then $f$ is complex differentiable at $w$ if and only if $u$ and $v$, viewed as a real function of two real variables, are differentiable at $(c,d)$, and

$$u_x = v_y,$$
$$u_y = -v_x.$$ 

These equations are the Cauchy–Riemann equations. In this case, we have

$$f'(w) = u_x(c,d) + iv_x(c,d) = v_y(c,d) - iu_y(c,d).$$

1.2 Conformal mappings

Theorem (Riemann mapping theorem). Let $U \subseteq \mathbb{C}$ be the bounded domain enclosed by a simple closed curve, or more generally any simply connected domain not equal to all of $\mathbb{C}$. Then $U$ is conformally equivalent to $D = \{ z : |z| < 1 \} \subseteq \mathbb{C}$.

1.3 Power series

Proposition. The uniform limit of continuous functions is continuous.

Proposition (Weierstrass M-test). For a sequence of functions $f_n$, if we can find $(M_n) \subseteq \mathbb{R}_{>0}$ such that $|f_n(x)| < M_n$ for all $x$ in the domain, then $\sum M_n$ converges implies $\sum f_n(x)$ converges uniformly on the domain.

Proposition. Given any constants $(c_n)_{n \geq 0} \subseteq \mathbb{C}$, there is a unique $R \in [0, \infty]$ such that the series $z \mapsto \sum_{n=0}^{\infty} c_n (z-a)^n$ converges absolutely if $|z-a| < R$ and diverges if $|z-a| > R$. Moreover, if $0 < r < R$, then the series converges uniformly on $\{ z : |z-a| < r \}$. This $R$ is known as the radius of convergence.

Theorem. Let

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

be a power series with radius of convergence $R > 0$. Then

(i) $f$ is holomorphic on $B(a;R) = \{ z : |z-a| < R \}$.
(ii) $f'(z) = \sum n c_n (z-1)^{n-1}$, which also has radius of convergence $R$.
(iii) Therefore $f$ is infinitely complex differentiable on $B(a;R)$. Furthermore,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$ 

Corollary. Given a power series

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n$$

with radius of convergence $R > 0$, and given $0 < \varepsilon < R$, if $f$ vanishes on $B(a,\varepsilon)$, then $f$ vanishes identically.
1.4 Logarithm and branch cuts

**Proposition.** On \( \{ z \in \mathbb{C} : z \not\in \mathbb{R}_{\leq 0} \} \), the principal branch \( \log : U \to \mathbb{C} \) is a holomorphic function. Moreover,

\[
\frac{d}{dz} \log z = \frac{1}{z}.
\]

If \( |z| < 1 \), then

\[
\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.
\]
2 Contour integration

2.1 Basic properties of complex integration

Lemma. Suppose \( f : [a, b] \to \mathbb{C} \) is continuous (and hence integrable). Then
\[
\left| \int_a^b f(t) \, dt \right| \leq (b - a) \sup_t |f(t)|
\]
with equality if and only if \( f \) is constant.

Theorem (Fundamental theorem of calculus). Let \( f : U \to \mathbb{C} \) be continuous with antiderivative \( F \). If \( \gamma : [a, b] \to U \) is piecewise \( C^1 \)-smooth, then
\[
\int_\gamma f(z) \, dz = F(\gamma(b)) - F(\gamma(a)).
\]

2.2 Cauchy’s theorem

Proposition. Let \( U \subseteq \mathbb{C} \) be a domain (i.e. path-connected non-empty open set), and \( f : U \to \mathbb{C} \) be continuous. Moreover, suppose
\[
\int_\gamma f(z) \, dz = 0
\]
for any closed piecewise \( C^1 \)-smooth path \( \gamma \) in \( U \). Then \( f \) has an antiderivative.

Proposition. If \( U \) is a star domain, and \( f : U \to \mathbb{C} \) is continuous, and if
\[
\int_{\partial T} f(z) \, dz = 0
\]
for all triangles \( T \subseteq U \), then \( f \) has an antiderivative on \( U \).

Theorem (Cauchy’s theorem for a triangle). Let \( U \) be a domain, and let \( f : U \to \mathbb{C} \) be holomorphic. If \( T \subseteq U \) is a triangle, then \( \int_{\partial T} f(z) \, dz = 0 \).

Corollary (Convex Cauchy). If \( U \) is a convex or star-shaped domain, and \( f : U \to \mathbb{C} \) is holomorphic, then for any closed piecewise \( C^1 \) paths \( \gamma \in U \), we must have
\[
\int_\gamma f(z) \, dz = 0.
\]

2.3 The Cauchy integral formula

Theorem (Cauchy integral formula). Let \( U \) be a domain, and \( f : U \to \mathbb{C} \) be holomorphic. Suppose there is some \( B(z_0; r) \subseteq U \) for some \( z_0 \) and \( r > 0 \). Then for all \( z \in B(z_0; r) \), we have
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0; r)} \frac{f(w)}{w - z} \, dw.
\]

Corollary (Local maximum principle). Let \( f : B(z, r) \to \mathbb{C} \) be holomorphic. Suppose \( |f(w)| \leq |f(z)| \) for all \( w \in B(z, r) \). Then \( f \) is constant. In other words, a non-constant function cannot achieve an interior local maximum.
Theorem (Liouville’s theorem). Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function (i.e. holomorphic everywhere). If \( f \) is bounded, then \( f \) is constant.

Corollary (Fundamental theorem of algebra). A non-constant complex polynomial has a root in \( \mathbb{C} \).

### 2.4 Taylor’s theorem

**Theorem** (Taylor’s theorem). Let \( f : B(a, r) \to \mathbb{C} \) be holomorphic. Then \( f \) has a convergent power series representation

\[
f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n
\]
on \( B(a, r) \). Moreover,

\[
c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\partial B(a, \rho)} \frac{f(z)}{(z - a)^{n+1}} \, dz
\]
for any \( 0 < \rho < r \).

**Corollary.** If \( f : B(a, r) \to \mathbb{C} \) is holomorphic on a disc, then \( f \) is infinitely differentiable on the disc.

**Corollary.** If \( f : U \to \mathbb{C} \) is a complex-valued function, then \( f = u + iv \) is holomorphic at \( p \in U \) if and only if \( u, v \) satisfy the Cauchy–Riemann equations, and that \( u_x, u_y, v_x, v_y \) are continuous in a neighbourhood of \( p \).

**Corollary** (Morera’s theorem). Let \( U \subseteq \mathbb{C} \) be a domain. Let \( f : U \to \mathbb{C} \) be continuous such that

\[
\int_{\gamma} f(z) \, dz = 0
\]
for all piecewise-\( C^1 \) closed curves \( \gamma \in U \). Then \( f \) is holomorphic on \( U \).

**Corollary.** Let \( U \subseteq \mathbb{C} \) be a domain, \( f_n : U \to \mathbb{C} \) be a holomorphic function. If \( f_n \to f \) uniformly, then \( f \) is in fact holomorphic, and

\[
f'(z) = \lim_{n} f'_n(z).
\]

### 2.5 Zeroes

**Lemma** (Principle of isolated zeroes). Let \( f : B(a, r) \to \mathbb{C} \) be holomorphic and not identically zero. Then there exists some \( 0 < \rho < r \) such that \( f(z) \neq 0 \) in the punctured neighbourhood \( B(a, \rho) \setminus \{a\} \).

**Corollary** (Identity theorem). Let \( U \subseteq \mathbb{C} \) be a domain, and \( f, g : U \to \mathbb{C} \) be holomorphic. Let \( S = \{z \in U : f(z) = g(z)\} \). Suppose \( S \) contains a non-isolated point, i.e. there exists some \( w \in S \) such that for all \( \varepsilon > 0 \), \( S \cap B(w, \varepsilon) \neq \{w\} \). Then \( f = g \) on \( U \).
2.6 Singularities

**Proposition** (Removal of singularities). Let $U$ be a domain and $z_0 \in U$. If $f: U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, and $f$ is bounded near $z_0$, then there exists an $a$ such that $f(z) \to a$ as $z \to z_0$.

Furthermore, if we define

$$g(z) = \begin{cases} f(z) & z \in U \setminus \{z_0\}, \\ a & z = z_0 \end{cases},$$

then $g$ is holomorphic on $U$.

**Proposition.** Let $U$ be a domain, $z_0 \in U$ and $f: U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then there is a unique $k \in \mathbb{Z} \geq 1$ and a unique holomorphic function $g: U \to \mathbb{C}$ such that $g(z_0) \neq 0$, and

$$f(z) = \frac{g(z)}{(z - z_0)^k}.$$

**Theorem** (Casorati-Weierstrass theorem). Let $U$ be a domain, $z_0 \in U$, and suppose $f: U \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at $z_0$. Then for all $w \in \mathbb{C}$, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w$.

In other words, on any punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of $f$ is dense in $\mathbb{C}$.

**Theorem** (Picard’s theorem). If $f$ has an isolated essential singularity at $z_0$, then there is some $b \in \mathbb{C}$ such that on each punctured neighbourhood $B(z_0; \varepsilon) \setminus \{z_0\}$, the image of $f$ contains $\mathbb{C} \setminus \{b\}$.

2.7 Laurent series

**Theorem** (Laurent series). Let $0 \leq r < R < \infty$, and let

$$A = \{z \in \mathbb{C} : r < |z - a| < R\}$$

denote an annulus on $\mathbb{C}$.

Suppose $f: A \to \mathbb{C}$ is holomorphic. Then $f$ has a (unique) convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \oint_{\partial A(a,\rho)} \frac{f(z)}{(z - a)^{n+1}} \, dz$$

for $r < \rho < R$. Moreover, the series converges uniformly on compact subsets of the annulus.

**Lemma.** Let $f: A \to \mathbb{C}$ be holomorphic, $A = \{r < |z - a| < R\}$, with

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n$$

Then the coefficients $c_n$ are uniquely determined by $f$. 

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3 Residue calculus

3.1 Winding numbers

**Lemma.** Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous closed curve, and pick a point $w \in \mathbb{C} \setminus \text{image}(\gamma)$. Then there are continuous functions $r : [a, b] \to \mathbb{R} > 0$ and $\theta : [a, b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$.

**Lemma.** Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise $C^1$-smooth closed path, and $w \not\in \text{image}(\gamma)$. Then

$$I(\gamma, w) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-w} \, dz.$$

3.2 Homotopy of closed curves

**Proposition.** Let $\phi, \psi : [a, b] \to U$ be homotopic (piecewise $C^1$) closed paths in a domain $U$. Then there exists some $\phi = \phi_0, \phi_1, \ldots, \phi_N = \psi$ such that each $\phi_i$ is piecewise $C^1$ closed and $\phi_{i+1}$ is obtained from $\phi_i$ by elementary deformation.

**Corollary.** Let $U$ be a domain, $f : U \to \mathbb{C}$ be holomorphic, and $\gamma_1, \gamma_2$ be homotopic piecewise $C^1$-smooth closed curves in $U$. Then

$$\int_{\gamma_1} f(z) \, dz = \int_{\gamma_2} f(z) \, dz.$$

**Corollary** (Cauchy’s theorem for simply connected domains). Let $U$ be a simply connected domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $\gamma$ is any piecewise $C^1$-smooth closed curve in $U$, then

$$\int_{\gamma} f(z) \, dz = 0.$$

3.3 Cauchy’s residue theorem

**Theorem** (Cauchy’s residue theorem). Let $U$ be a simply connected domain, and $\{z_1, \ldots, z_k\} \subseteq U$. Let $f : U \setminus \{z_1, \ldots, z_k\} \to \mathbb{C}$ be holomorphic. Let $\gamma : [a, b] \to U$ be a piecewise $C^1$-smooth closed curve such that $z_i \not\in \text{image}(\gamma)$ for all $i$. Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \sum_{j=1}^{k} I(\gamma, z_i) \text{Res}(f; z_i).$$

3.4 Overview

3.5 Applications of the residue theorem

**Lemma.** Let $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic with a pole at $a$, i.e $f$ is meromorphic on $U$.

(i) If the pole is simple, then

$$\text{Res}(f, a) = \lim_{z \to a} (z - a)f(z).$$
(ii) If near $a$, we can write

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(a) \neq 0$ and $h$ has a simple zero at $a$, and $g, h$ are holomorphic on $B(a, \varepsilon) \setminus \{a\}$, then

$$\text{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

(iii) If

$$f(z) = \frac{g(z)}{(z-a)^k}$$

near $a$, with $g(a) \neq 0$ and $g$ is holomorphic, then

$$\text{Res}(f, a) = \frac{g^{(k-1)}(a)}{(k-1)!}.$$

**Lemma.** Let $f : B(a, r) \setminus \{a\} \to \mathbb{C}$ be holomorphic, and suppose $f$ has a simple pole at $a$. We let $\gamma_{\varepsilon} : [\alpha, \beta] \to \mathbb{C}$ be given by

$$t \mapsto a + \varepsilon e^{it}.$$

Then

$$\lim_{\varepsilon \to 0} \int_{\gamma_{\varepsilon}} f(z) \, dz = (\beta - \alpha) \cdot i \cdot \text{Res}(f, a).$$

**Lemma** (Jordan’s lemma). Let $f$ be holomorphic on a neighbourhood of infinity in $\mathbb{C}$, i.e. on $\{|z| > r\}$ for some $r > 0$. Assume that $zf(z)$ is bounded in this region. Then for $\alpha > 0$, we have

$$\int_{\gamma_R} f(z) e^{i\alpha z} \, dz \to 0$$

as $R \to \infty$, where $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$ is the semicircle (which is not closed).
3.6 Rouché’s theorem

**Theorem** (Argument principle). Let $U$ be a simply connected domain, and let $f$ be meromorphic on $U$. Suppose in fact $f$ has finitely many zeroes $z_1, \cdots, z_k$ and finitely many poles $w_1, \cdots, w_\ell$. Let $\gamma$ be a piecewise-$C^1$ closed curve such that $z_i, w_j \not\in \text{image}(\gamma)$ for all $i, j$. Then

$$I(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^k \text{ord}(f; z_i) I(\gamma, z_i) - \sum_{j=1}^\ell \text{ord}(f, w_j) I(\gamma, w_j).$$

**Corollary** (Rouché’s theorem). Let $U$ be a domain and $\gamma$ a closed curve which bounds a domain in $U$ (the key case is when $U$ is simply connected and $\gamma$ is a simple closed curve). Let $f, g$ be holomorphic on $U$, and suppose $|f| > |g|$ for all $z \in \text{image}(\gamma)$. Then $f$ and $f + g$ have the same number of zeroes in the domain bound by $\gamma$, when counted with multiplicity.

**Lemma.** The local degree is given by

$$\text{deg}(f, a) = I(f \circ \gamma, f(a)),$$

where

$$\gamma(t) = a + re^{it},$$

with $0 \leq t \leq 2\pi$, for $r > 0$ sufficiently small.

**Proposition** (Local degree theorem). Let $f : B(a, r) \to \mathbb{C}$ be holomorphic and non-constant. Then for $r > 0$ sufficiently small, there is $\varepsilon > 0$ such that for any $w \in B(f(a), \varepsilon) \setminus \{f(a)\}$, the equation $f(z) = w$ has exactly $\text{deg}(f, a)$ distinct solutions in $B(a, r)$.

**Corollary** (Open mapping theorem). Let $U$ be a domain and $f : U \to \mathbb{C}$ is holomorphic and non-constant, then $f$ is an open map, i.e. for all open $V \subseteq U$, we get that $f(V)$ is open.

**Corollary.** Let $U \subseteq \mathbb{C}$ be a simply connected domain, and $U \neq \mathbb{C}$. Then there is a non-constant holomorphic function $U \to B(0, 1)$. 