Part IB — Optimisation
Theorems

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Easter 2015

Lagrangian methods
General formulation of constrained problems; the Lagrangian sufficiency theorem. Interpretation of Lagrange multipliers as shadow prices. Examples. [2]

Linear programming in the nondegenerate case
Convexity of feasible region; sufficiency of extreme points. Standardization of problems, slack variables, equivalence of extreme points and basic solutions. The primal simplex algorithm, artificial variables, the two-phase method. Practical use of the algorithm; the tableau. Examples. The dual linear problem, duality theorem in a standardized case, complementary slackness, dual variables and their interpretation as shadow prices. Relationship of the primal simplex algorithm to dual problem. Two person zero-sum games. [6]

Network problems
The Ford-Fulkerson algorithm and the max-flow min-cut theorems in the rational case. Network flows with costs, the transportation algorithm, relationship of dual variables with nodes. Examples. Conditions for optimality in more general networks; *the simplex-on-a-graph algorithm*. [3]

Practice and applications
Efficiency of algorithms. The formulation of simple practical and combinatorial problems as linear programming or network problems. [1]

Contents

1 Introduction and preliminaries

1.1 Constrained optimization

1.2 Review of unconstrained optimization

Lemma. Let f be twice differentiable. Then f is convex on a convex set S if the Hessian matrix

$$Hf_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is positive semidefinite for all $x \in S$, where this fancy term means:

Theorem. Let $X \subseteq \mathbb{R}^n$ be convex, $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on X. If $x^* \in X$ satisfy $\nabla f(x^*) = 0$ and $Hf(x)$ is positive semidefinite for all $x \in X$, then x^* minimizes f on X.

2 The method of Lagrange multipliers

Theorem (Lagrangian sufficiency)**.** Let $x^* \in X$ and $\lambda^* \in \mathbb{R}^m$ be such that

$$L(x^*, \lambda^*) = \inf_{x \in X} L(x, \lambda^*) \quad \text{and} \quad h(x^*) = b.$$

Then x^* is optimal for (P).

In words, if x^* minimizes L for a fixed λ^*, and x^* satisfies the constraints, then x^* minimizes f.

2.1 Complementary Slackness

2.2 Shadow prices

Theorem. Consider the problem

$$\text{minimize } f(x) \text{ subject to } h(x) = b.$$

Here we assume all functions are continuously differentiable. Suppose that for each $b \in \mathbb{R}^n$, $\phi(b)$ is the optimal value of f and λ^* is the corresponding Lagrange multiplier. Then

$$\frac{\partial \phi}{\partial b_i} = \lambda_i^*.$$

2.3 Lagrange duality

Theorem (Weak duality)**.** If $x \in X(b)$ (i.e. x satisfies both the functional and regional constraints) and $\lambda \in Y$, then

$$g(\lambda) \leq f(x).$$

In particular,

$$\sup_{\lambda \in Y} g(\lambda) \leq \inf_{x \in X(b)} f(x).$$

2.4 Supporting hyperplanes and convexity

Theorem. (P) satisfies strong duality iff $\phi(c) = \inf_{x \in X(c)} f(x)$ has a supporting hyperplane at b.

Theorem (Supporting hyperplane theorem)**.** Suppose that $\phi : \mathbb{R}^m \to \mathbb{R}$ is convex and $b \in \mathbb{R}^m$ lies in the interior of the set of points where ϕ is finite. Then there exists a supporting hyperplane to ϕ at b.

Theorem. Let

$$\phi(b) = \inf_{x \in X} \{f(x) : h(x) \leq b\}$$

If X, f, h are convex, then so is ϕ (assuming feasibility and boundedness).

Theorem. If a linear program is feasible and bounded, then it satisfies strong duality.

3 Solutions of linear programs

3.1 Linear programs

3.2 Basic solutions

Theorem. A vector x is a basic feasible solution of $Ax = b$ if and only if it is an extreme point of the set $X(b) = \{x' : Ax' = b, x' \geq 0\}$.

3.3 Extreme points and optimal solutions

Theorem. If (P) is feasible and bounded, then there exists an optimal solution that is a basic feasible solution.

3.4 Linear programming duality

Theorem. The dual of the dual of a linear program is the primal.

Theorem. Let x and λ be feasible for the primal and the dual of the linear program in general form. Then x and λ and optimal if and only if they satisfy complementary slackness, i.e. if

$$(c^T - \lambda^T A)x = 0 \text{ and } \lambda^T(Ax - b) = 0.$$

3.5 Simplex method

3.5.1 The simplex tableau

3.5.2 Using the Tableau

3.6 The two-phase simplex method

4 Non-cooperative games

4.1 Games and Solutions

Theorem (Nash, 1961)**.** Every bimatrix game has an equilibrium.

4.2 The minimax theorem

Theorem (von Neumann, 1928)**.** If $P \in \mathbb{R}^{m\times n}$. Then

$$\max_{x\in X} \min_{y\in Y} p(x, y) = \min_{y\in Y} \max_{x\in X} p(x, y).$$

Note that this is equivalent to

$$\max_{x\in X} \min_{y\in Y} p(x, y) = -\max_{y\in Y} \min_{x\in X} -p(x, y).$$

The left hand side is the worst payoff the row player can get if he employs the minimax strategy. The right hand side is the worst payoff the column player can get if he uses his minimax strategy.

The theorem then says that if both players employ the minimax strategy, then this is an equilibrium.

Theorem. $(x, y) \in X \times Y$ is an equilibrium of the matrix game with payoff matrix P if and only if

$$\begin{aligned} \min_{y'\in Y} p(x, y') &= \max_{x'\in X} \min_{y'\in Y} p(x', y') \\ \max_{x'\in X} p(x', y) &= \min_{y'\in Y} \max_{x'\in X} p(x', u') \end{aligned}$$

i.e. the x, y are optimizers for the $\max\min$ and $\min\max$ functions.

5 Network problems

5.1 Definitions

5.2 Minimum-cost flow problem

5.3 The transportation problem

Theorem. Every minimum cost-flow problem with finite capacities or non-negative costs has an equivalent transportation problem.

5.4 The maximum flow problem

Theorem (Max-flow min-cut theorem)**.** Let δ be an optimal solution. Then

$$\delta = \min\{C(S) : S \subseteq V, 1 \in S, n \in V \setminus S\}$$