Part IA — Vectors and Matrices
Theorems with proof

Based on lectures by N. Peake
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Michaelmas 2014

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

**Complex numbers**
Review of complex numbers, including complex conjugate, inverse, modulus, argument and Argand diagram. Informal treatment of complex logarithm, \( n \)-th roots and complex powers. de Moivre’s theorem. [2]

**Vectors**
Review of elementary algebra of vectors in \( \mathbb{R}^3 \), including scalar product. Brief discussion of vectors in \( \mathbb{R}^n \) and \( \mathbb{C}^n \); scalar product and the Cauchy-Schwarz inequality. Concepts of linear span, linear independence, subspaces, basis and dimension.
Suffix notation: including summation convention, \( \delta_{ij} \) and \( \varepsilon_{ijk} \). Vector product and triple product: definition and geometrical interpretation. Solution of linear vector equations. Applications of vectors to geometry, including equations of lines, planes and spheres. [5]

**Matrices**
Elementary algebra of \( 3 \times 3 \) matrices, including determinants. Extension to \( n \times n \) complex matrices. Trace, determinant, non-singular matrices and inverses. Matrices as linear transformations; examples of geometrical actions including rotations, reflections, dilations, shears; kernel and image. [4]
Simultaneous linear equations: matrix formulation; existence and uniqueness of solutions, geometric interpretation; Gaussian elimination. [3]
Symmetric, anti-symmetric, orthogonal, hermitian and unitary matrices. Decomposition of a general matrix into isotropic, symmetric trace-free and antisymmetric parts. [1]

**Eigenvalues and Eigenvectors**
Eigenvalues and eigenvectors; geometric significance. [2]
Proof that eigenvalues of hermitian matrix are real, and that distinct eigenvalues give an orthogonal basis of eigenvectors. The effect of a general change of basis (similarity transformations). Diagonalization of general matrices: sufficient conditions; examples of matrices that cannot be diagonalized. Canonical forms for \( 2 \times 2 \) matrices. [5]
Discussion of quadratic forms, including change of basis. Classification of conics, cartesian and polar forms. [1]
Rotation matrices and Lorentz transformations as transformation groups. [1]
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0 Introduction
1 Complex numbers

1.1 Basic properties

Proposition. $\bar{z}z = a^2 + b^2 = |z|^2$.

Proposition. $z^{-1} = \bar{z}/|z|^2$.

Theorem (Triangle inequality). For all $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$ 

Alternatively, we have $|z_1 - z_2| \geq ||z_1| - |z_2||$.

1.2 Complex exponential function

Lemma. \[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} = \sum_{r=0}^{\infty} \sum_{m=0}^{r} a_{r-m,m} \]

Proof.

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} = a_{00} + a_{01} + a_{02} + \cdots \]
\[ + a_{10} + a_{11} + a_{12} + \cdots \]
\[ + a_{20} + a_{21} + a_{22} + \cdots \]
\[ = (a_{00}) + (a_{10} + a_{01}) + (a_{20} + a_{11} + a_{02}) + \cdots \]
\[ = \sum_{r=0}^{\infty} \sum_{m=0}^{r} a_{r-m,m} \]

\[ \square \]

Theorem. $\exp(z_1) \exp(z_2) = \exp(z_1 + z_2)$

Proof.

\[ \exp(z_1) \exp(z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^n z_2^m}{m! n!} \]
\[ = \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{z_1^{r-m} z_2^m}{(r-m)! m!} \]
\[ = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{m=0}^{r} \frac{r!}{(r-m)!m!} z_1^{r-m} z_2^m \]
\[ = \sum_{r=0}^{\infty} \frac{(z_1 + z_2)^r}{r!} \]

\[ \square \]

Theorem. $e^{iz} = \cos z + i \sin z$. 

5
Proof.

\[ e^{iz} = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} z^{2n} + i \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \cos z + i \sin z \]

1.3 Roots of unity

**Proposition.** If \( \omega = \exp \left( \frac{2\pi i}{n} \right) \), then \( 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0 \)

**Proof.** Two proofs are provided:

(i) Consider the equation \( z^n = 1 \). The coefficient of \( z^{n-1} \) is the sum of all roots. Since the coefficient of \( z^{n-1} \) is 0, then the sum of all roots \( = 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0 \).

(ii) Since \( \omega^n - 1 = (\omega - 1)(1 + \omega + \cdots + \omega^{n-1}) \) and \( \omega \neq 1 \), dividing by \( (\omega - 1) \), we have \( 1 + \omega + \cdots + \omega^{n-1} = (\omega^n - 1)/(\omega - 1) = 0 \).

1.4 Complex logarithm and power

1.5 De Moivre’s theorem

**Theorem** (De Moivre’s theorem).

\[ \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n. \]

**Proof.** First prove for the \( n \geq 0 \) case by induction. The \( n = 0 \) case is true since it merely reads \( 1 = 1 \). We then have

\[
(cos \theta + i \sin \theta)^{n+1} = (cos \theta + i \sin \theta)^n (cos \theta + i \sin \theta) \\
= (cos n\theta + i \sin n\theta)(cos \theta + i \sin \theta) \\
= cos((n+1)\theta) + i \sin((n+1)\theta)
\]

If \( n < 0 \), let \( m = -n \). Then \( m > 0 \) and

\[
(cos \theta + i \sin \theta)^{-m} = (cos m\theta + i \sin m\theta)^{-1} \\
= \frac{cos m\theta - i \sin m\theta}{(cos m\theta + i \sin m\theta)(cos m\theta - i \sin m\theta)} \\
= \frac{cos(-m\theta) + i \sin(-m\theta)}{cos^2 m\theta + \sin^2 m\theta} \\
= cos(-m\theta) + i \sin(-m\theta) \\
= cos m\theta + i \sin m\theta
\]
1.6 Lines and circles in $\mathbb{C}$

**Theorem** (Equation of straight line). The equation of a straight line through $z_0$ and parallel to $w$ is given by

$$z\bar{w} - \bar{z}w = z_0\bar{w} - \bar{z}_0w.$$ 

**Theorem.** The general equation of a circle with center $c \in \mathbb{C}$ and radius $\rho \in \mathbb{R}^+$ can be given by

$$z\bar{z} - cz - \bar{c}z = \rho^2 - c\bar{c}.$$
2 Vectors

2.1 Definition and basic properties

2.2 Scalar product

2.2.1 Geometric picture ($\mathbb{R}^2$ and $\mathbb{R}^3$ only)

2.2.2 General algebraic definition

2.3 Cauchy-Schwarz inequality

Theorem (Cauchy-Schwarz inequality). For all $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq |x||y|.$$ 

Proof. Consider the expression $|x - \lambda y|^2$. We must have

$$|x - \lambda y|^2 \geq 0$$

$$(x - \lambda y) \cdot (x - \lambda y) \geq 0$$

$$\lambda^2 |y|^2 - \lambda(2x \cdot y) + |x|^2 \geq 0.$$ 

Viewing this as a quadratic in $\lambda$, we see that the quadratic is non-negative and thus cannot have 2 real roots. Thus the discriminant $\Delta \leq 0$. So

$$4(x \cdot y)^2 \leq 4|y|^2|x|^2$$

$$(x \cdot y)^2 \leq |x|^2|y|^2$$

$$|x \cdot y| \leq |x||y|.$$ 

Corollary (Triangle inequality).

$$|x + y| \leq |x| + |y|.$$ 

Proof.

$$|x + y|^2 = (x + y) \cdot (x + y)$$

$$= |x|^2 + 2x \cdot y + |y|^2$$

$$\leq |x|^2 + 2|x||y| + |y|^2$$

$$= (|x| + |y|)^2.$$ 

So

$$|x + y| \leq |x| + |y|.$$ 

2.4 Vector product

Proposition.

$$a \times b = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + \cdots$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$
2.5 Scalar triple product

**Proposition.** If a parallelepiped has sides represented by vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) that form a right-handed system, then the volume of the parallelepiped is given by \( | \mathbf{a} \times \mathbf{b} \times \mathbf{c} | \).

**Proof.** The area of the base of the parallelepiped is given by \( | \mathbf{b} \times \mathbf{c} | \sin \theta = | \mathbf{b} \times \mathbf{c} | \), where \( \theta \) is the angle between \( \mathbf{a} \) and the normal to \( \mathbf{b} \) and \( \mathbf{c} \). However, since \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) form a right-handed system, we have \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \geq 0 \). Therefore the volume is \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \).

**Theorem.** \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \).

**Proof.** Let \( \mathbf{d} = \mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} \). We have

\[
\begin{align*}
\mathbf{d} \cdot \mathbf{d} &= \mathbf{d} \cdot [\mathbf{a} \times (\mathbf{b} + \mathbf{c})] - \mathbf{d} \cdot (\mathbf{a} \times \mathbf{b}) - \mathbf{d} \cdot (\mathbf{a} \times \mathbf{c}) \\
&= (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{d} \times \mathbf{a}) - \mathbf{b} \cdot (\mathbf{d} \times \mathbf{a}) - \mathbf{c} \cdot (\mathbf{d} \times \mathbf{a}) \\
&= 0
\end{align*}
\]

Thus \( \mathbf{d} = 0 \).

2.6 Spanning sets and bases

2.6.1 2D space

**Theorem.** The coefficients \( \lambda, \mu \) are unique.

**Proof.** Suppose that \( \mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} = \lambda' \mathbf{a} + \mu' \mathbf{b} \). Take the vector product with \( \mathbf{a} \) on both sides to get \( (\mu - \mu') \mathbf{a} \times \mathbf{b} = 0 \). Since \( \mathbf{a} \times \mathbf{b} \neq 0 \), then \( \mu = \mu' \). Similarly, \( \lambda = \lambda' \).

2.6.2 3D space

**Theorem.** If \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \) are non-coplanar, i.e. \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0 \), then they form a basis of \( \mathbb{R}^3 \).

**Proof.** For any \( \mathbf{r} \), write \( \mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} \). Performing the scalar product with \( \mathbf{b} \times \mathbf{c} \) on both sides, one obtains \( \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mu \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + \nu \mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda |\mathbf{a}, \mathbf{b}, \mathbf{c}| \). Thus \( \lambda = [\mathbf{r}, \mathbf{b}, \mathbf{c}]/|\mathbf{a}, \mathbf{b}, \mathbf{c}| \). The values of \( \mu \) and \( \nu \) can be found similarly. Thus each \( \mathbf{r} \) can be written as a linear combination of \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).

By the formula derived above, it follows that if \( \alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} = \mathbf{0} \), then \( \alpha = \beta = \gamma = 0 \). Thus they are linearly independent.

2.6.3 \( \mathbb{R}^n \) space

2.6.4 \( \mathbb{C}^n \) space

2.7 Vector subspaces

2.8 Suffix notation

**Proposition.** \( (\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k \)

**Proof.** By expansion of formula
Theorem. $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

Proof. Proof by exhaustion:

$$\text{RHS} = \begin{cases} +1 & \text{if } j = p \text{ and } k = q \\ -1 & \text{if } j = q \text{ and } k = p \\ 0 & \text{otherwise} \end{cases}$$

LHS: Summing over $i$, the only non-zero terms are when $j, k \neq i$ and $p, q \neq i$.

If $j = p$ and $k = q$, LHS is $(-1)^2$ or $(+1)^2 = 1$. If $j = q$ and $k = p$, LHS is $(+1)(-1)$ or $(-1)(+1) = -1$. All other possibilities result in 0.

Proposition. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

Proof. In suffix notation, we have

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i (b \times c)_i = \varepsilon_{ijk} b_j c_k a_i = \varepsilon_{jki} b_j c_k a_i = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).$$

Theorem (Vector triple product).

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$ 

Proof.

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \varepsilon_{ijk} a_j (b \times c)_k = \varepsilon_{ijk} \varepsilon_{kpq} a_j b_p c_q = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} a_j b_p c_q = a_j b_k c_j - a_j c_k b_j = (\mathbf{a} \cdot \mathbf{c})b_i - (\mathbf{a} \cdot \mathbf{b})c_i.$$ 

Proposition. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$.

Proof.

$$\text{LHS} = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})_i = \varepsilon_{ijk} a_j (b \times c)_k = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} a_j b_k c_q = a_j b_k a_j c_k - a_j b_k a_j c_j = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$$

2.9 Geometry

2.9.1 Lines

Theorem. The equation of a straight line through $\mathbf{a}$ and parallel to $\mathbf{t}$ is

$$(\mathbf{x} - \mathbf{a}) \times \mathbf{t} = \mathbf{0} \text{ or } \mathbf{x} \times \mathbf{t} = \mathbf{a} \times \mathbf{t}.$$
2.9.2 Plane

**Theorem.** The equation of a plane through \( \mathbf{b} \) with normal \( \mathbf{n} \) is given by

\[
\mathbf{x} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}.
\]

2.10 Vector equations
3 Linear maps

3.1 Examples

3.1.1 Rotation in $\mathbb{R}^3$

3.1.2 Reflection in $\mathbb{R}^3$

3.2 Linear Maps

**Theorem.** Consider a linear map $f : U \rightarrow V$, where $U, V$ are vector spaces. Then $\text{im}(f)$ is a subspace of $V$, and $\ker(f)$ is a subspace of $U$.

**Proof.** Both are non-empty since $f(0) = 0$.

If $x, y \in \text{im}(f)$, then $\exists a, b \in U$ such that $x = f(a), y = f(b)$. Then $\lambda x + \mu y = \lambda f(a) + \mu f(b) = f(\lambda a + \mu b)$. Now $\lambda a + \mu b \in U$ since $U$ is a vector space, so there is an element in $U$ that maps to $\lambda x + \mu y$. So $\lambda x + \mu y \in \text{im}(f)$ and $\text{im}(f)$ is a subspace of $V$.

Suppose $x, y \in \ker(f)$, i.e. $f(x) = f(y) = 0$. Then $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = \lambda 0 + \mu 0 = 0$. Therefore $\lambda x + \mu y \in \ker(f)$.

3.3 Rank and nullity

**Theorem (Rank-nullity theorem).** For a linear map $f : U \rightarrow V$,

$$r(f) + n(f) = \dim(U).$$

**Proof.** (Non-examinable) Write $\dim(U) = n$ and $n(f) = m$. If $m = n$, then $f$ is the zero map, and the proof is trivial, since $r(f) = 0$. Otherwise, assume $m < n$.

Suppose $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\ker(f)$, Extend this to a basis of the whole of $U$ to get $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n\}$. To prove the theorem, we need to prove that $\{f(e_{m+1}), f(e_{m+2}), \ldots f(e_n)\}$ is a basis of $\text{im}(f)$.

(i) First show that it spans $\text{im}(f)$. Take $y \in \text{im}(f)$. Thus $\exists x \in U$ such that $y = f(x)$. Then

$$y = f(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n),$$

since $e_1, \ldots, e_n$ is a basis of $U$. Thus

$$y = \alpha_1 f(e_1) + \alpha_2 f(e_2) + \cdots + \alpha_m f(e_m) + \alpha_{m+1} f(e_{m+1}) + \cdots + \alpha_n f(e_n).$$

The first $m$ terms map to 0, since $e_1, \ldots, e_m$ is the basis of the kernel of $f$. Thus

$$y = \alpha_{m+1} f(e_{m+1}) + \cdots + \alpha_n f(e_n).$$

(ii) To show that they are linearly independent, suppose

$$\alpha_{m+1} f(e_{m+1}) + \cdots + \alpha_n f(e_n) = 0.$$

Then

$$f(\alpha_{m+1} e_{m+1} + \cdots + \alpha_n e_n) = 0.$$
Thus $\alpha_{m+1}e_{m+1} + \cdots + \alpha_ne_n \in \ker(f)$. Since $\{e_1, \cdots, e_m\}$ span $\ker(f)$, there exist some $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$\alpha_{m+1}e_{m+1} + \cdots + \alpha_ne_n = \alpha_1e_1 + \cdots + \alpha_me_m.$$ 

But $e_1 \cdots e_n$ is a basis of $U$ and are linearly independent. So $\alpha_i = 0$ for all $i$. Then the only solution to the equation $\alpha_{m+1}f(e_{m+1}) + \cdots + \alpha_nf(e_n) = 0$ is $\alpha_i = 0$, and they are linearly independent by definition.

3.4 Matrices

3.4.1 Examples

3.4.2 Matrix Algebra

Proposition. 
(i) $(A^T)^T = A$.

(ii) If $x$ is a column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $x^T$ is a row vector $(x_1 \ x_2 \ \cdots \ x_n)$.

(iii) $(AB)^T = B^T A^T$ since $(AB)^T_{ij} = (AB)_{ji} = A_{jk}B_{ki} = B_{ki}A_{jk} = (B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij}$.

Proposition. $\text{tr}(BC) = \text{tr}(CB)$

Proof. $\text{tr}(BC) = B_{ik}C_{ki} = C_{ki}B_{ik} = (CB)_{kk} = \text{tr}(CB)$.

3.4.3 Decomposition of an $n \times n$ matrix

3.4.4 Matrix inverse

Proposition. $(AB)^{-1} = B^{-1}A^{-1}$

Proof. $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$.

3.5 Determinants

3.5.1 Permutations

Proposition. Any $q$-cycle can be written as a product of 2-cycles.

Proof. $(1 \ 2 \ 3 \ \cdots \ n) = (1 \ 2)(2 \ 3)(3 \ 4) \cdots (n-1 \ n)$.

Proposition.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
### 3.5.2 Properties of determinants

**Proposition.** $\det(A) = \det(A^T)$.

**Proof.** Take a single term $A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(n)}$ and let $\rho$ be another permutation in $S_n$. We have

$$A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(n)} = A_{\sigma(\rho(1))\rho(1)}A_{\sigma(\rho(2))\rho(2)}\cdots A_{\sigma(\rho(n))\rho(n)}$$

since the right hand side is just re-ordering the order of multiplication. Choose $\rho = \sigma^{-1}$ and note that $\varepsilon(\sigma) = \varepsilon(\rho)$. Then

$$\det(A) = \sum_{\rho \in S_n} \varepsilon(\rho) A_{1\rho(1)}A_{2\rho(2)}\cdots A_{n\rho(n)} = \det(A^T).$$

**Proposition.** If matrix $B$ is formed by multiplying every element in a single row of $A$ by a scalar $\lambda$, then $\det(B) = \lambda \det(A)$. Consequently, $\det(\lambda A) = \lambda^n \det(A)$.

**Proof.** Each term in the sum is multiplied by $\lambda$, so the whole sum is multiplied by $\lambda^n$.

**Proposition.** If 2 rows (or 2 columns) of $A$ are identical, the determinant is 0.

**Proof.** wlog, suppose columns 1 and 2 are the same. Then

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma) A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(n)}.$$

Now write an arbitrary $\sigma$ in the form $\sigma = \rho(1\ 2)$. Then $\varepsilon(\sigma) = \varepsilon(\rho)\varepsilon((1\ 2)) = -\varepsilon(\rho)$. So

$$\det(A) = \sum_{\rho \in S_n} -\varepsilon(\rho) A_{\rho(2)}\rho(1)A_{\rho(1)}A_{\rho(3)}\cdots A_{\rho(n)}.$$

But columns 1 and 2 are identical, so $A_{\rho(2)}1 = A_{\rho(2)}2$ and $A_{\rho(1)}2 = A_{\rho(1)}1$. So $\det(A) = -\det(A)$ and $\det(A) = 0$.

**Proposition.** If 2 rows or 2 columns of a matrix are linearly dependent, then the determinant is zero.

**Proof.** Suppose in $A$, (column $r$) $+ \lambda$(column $s$) = 0. Define

$$B_{ij} = \begin{cases} A_{ij} & j \neq r \\ A_{ij} + \lambda A_{is} & j = r \end{cases}$$

Then $\det(B) = \det(A) + \lambda \det(\text{matrix with column } r = \text{column } s) = \det(A)$. Then we can see that the $r$th column of $B$ is all zeroes. So each term in the sum contains one zero and $\det(A) = \det(B) = 0$.

**Proposition.** Given a matrix $A$, if $B$ is a matrix obtained by adding a multiple of a column (or row) of $A$ to another column (or row) of $A$, then $\det A = \det B$.

**Corollary.** Swapping two rows or columns of a matrix negates the determinant.
Proof. We do the column case only. Let $A = (a_1 \cdots a_i \cdots a_j \cdots a_n)$. Then
\[
\det(a_1 \cdots a_i \cdots a_j \cdots a_n) = \det(a_1 \cdots a_i + a_j \cdots a_n)
= \det(a_1 \cdots a_i + a_j \cdots a_i - a_j \cdots a_n)
= -\det(a_1 \cdots a_j \cdots a_i \cdots a_n)
\]
Alternatively, we can prove this from the definition directly, using the fact that the sign of a transposition is $-1$ (and that the sign is multiplicative).

Proposition. $\det(AB) = \det(A) \det(B)$.

Proof. First note that $\sum_\sigma \varepsilon(\sigma) A_{\sigma(1)}^{(1)} A_{\sigma(2)}^{(2)} \cdots (AB)_{\sigma(n)n}$
\[
= \sum_\sigma \varepsilon(\sigma) \sum_{k_1, k_2, \ldots, k_n} A_{\sigma(1)k_1} B_{k_11} \cdots A_{\sigma(n)k_n} B_{k_n n}
= \sum_{k_1, k_2, \ldots, k_n} B_{k_11} \cdots B_{k_n n} \sum_\sigma \varepsilon(\sigma) A_{\sigma(1)k_1} A_{\sigma(2)k_2} \cdots A_{\sigma(n)k_n}
\]
Now consider the many different $S$'s. If in $S$, two of $k_1$ and $k_n$ are equal, then $S$ is a determinant of a matrix with two columns the same, i.e. $S = 0$. So we only have to consider the sum over distinct $k_i$s. Thus the $k_i$s are are a permutation of $1, \cdots, n$, say $k_i = \rho(i)$. Then we can write
\[
\det AB = \sum_\rho B_{\rho(1)1} \cdots B_{\rho(n)n} \sum_\sigma \varepsilon(\sigma) A_{\sigma(1)\rho(1)} \cdots A_{\sigma(n)\rho(n)}
= \sum_\rho B_{\rho(1)1} \cdots B_{\rho(n)n} (\varepsilon(\rho) \det A)
= \det A \sum_\rho \varepsilon(\rho) B_{\rho(1)1} \cdots B_{\rho(n)n}
= \det A \det B
\]

Corollary. If $A$ is orthogonal, $\det A = \pm 1$.

Proof.
\[
AA^T = I \quad \text{det } AA^T = \det I
\]
\[
\det A \det A^T = 1 \quad (\det A)^2 = 1 \quad \det A = \pm 1
\]
Corollary. If $U$ is unitary, $|\det U| = 1$.

Proof. We have $\det U^\dagger = (\det U^T)^* = \det(U)^*$. Since $UU^\dagger = I$, we have $\det(U)\det(U)^* = 1$. \qed

Proposition. In $\mathbb{R}^3$, orthogonal matrices represent either a rotation ($\det = 1$) or a reflection ($\det = -1$).

3.5.3 Minors and Cofactors

Theorem (Laplace expansion formula). For any particular fixed $i$,

$$\det A = \sum_{j=1}^{n} A_{ji} \Delta_{ji}.\]$$

Proof.

$$\det A = \sum_{j=1}^{n} A_{ji} \sum_{j_1=1}^{n} \varepsilon_{j_1 j_2 \ldots j_n} A_{j_1 j_2} \ldots \overline{A_{ji}} \cdots A_{jn}.\]$$

Let $\sigma \in S_n$ be the permutation which moves $j_i$ to the $i$th position, and leave everything else in its natural order, i.e.

$$\sigma = \left( \begin{array}{cccccc} 1 & \cdots & i & i+1 & \cdots & n \\ 1 & \cdots & j_i & i+1 & \cdots & n \end{array} \right)$$

if $j_i > i$, and similarly for other cases. To perform this permutation, $|i - j_i|$ transpositions are made. So $\varepsilon(\sigma) = (-1)^{|i - j_i|}$.

Now consider the permutation $\rho \in S_n$

$$\rho = \left( \begin{array}{cccccc} 1 & \cdots & \tilde{j}_i & \cdots & n \\ j_1 & \cdots & \tilde{j}_i & \cdots & j_n \end{array} \right)$$

The composition $\rho \sigma$ reorders $(1, \ldots, n)$ to $(j_1, \tilde{j}_i, \ldots, j_n)$. So $\varepsilon(\rho \sigma) = \varepsilon(j_1 \ldots j_n) = \varepsilon(\rho)\varepsilon(\sigma) = (-1)^{|j_i - \tilde{j}_i|} \varepsilon(j_1 \ldots j_n)$. Hence the original equation becomes

$$\det A = \sum_{j=1}^{n} A_{ji} \sum_{j_1=1}^{n} (-1)^{|j_i - j_1|} \varepsilon_{j_1 j_2 \ldots j_n} A_{j_1 j_2} \ldots \overline{A_{ji}} \cdots A_{jn}$$

$$\begin{align*}
&= \sum_{j_i=1}^{n} A_{ji} \varepsilon_{j_1 j_2 \ldots j_n} M_{ji} \\
&= \sum_{j_i=1}^{n} A_{ji} \Delta_{ji} \\
&= \sum_{j=1}^{n} A_{ji} \Delta_{ji} \quad \square
\end{align*}$$
4 Matrices and linear equations

4.1 Simple example, $2 \times 2$

4.2 Inverse of an $n \times n$ matrix

**Lemma.** $\sum A_{ik} \Delta_{jk} = \delta_{ij} \det A$.

**Proof.** If $i \neq j$, then consider an $n \times n$ matrix $B$, which is identical to $A$ except the $j$th row is replaced by the $i$th row of $A$. So $\Delta_{jk}$ of $B = \Delta_{jk}$ of $A$, since $\Delta_{jk}$ does not depend on the elements in row $j$. Since $B$ has a duplicate row, we know that

$$0 = \det B = \sum_{k=1}^{n} B_{jk} \Delta_{jk} = \sum_{k=1}^{n} A_{ik} \Delta_{jk}.$$ 

If $i = j$, then the expression is $\det A$ by the Laplace expansion formula.

**Theorem.** If $\det A \neq 0$, then $A^{-1}$ exists and is given by

$$(A^{-1})_{ij} = \frac{\Delta_{ji}}{\det A}.$$ 

**Proof.**

$$(A^{-1})_{ik} A_{kj} = \frac{\Delta_{ki}}{\det A} A_{kj} = \delta_{ij} \frac{\det A}{\det A} = \delta_{ij}.$$ 

So $A^{-1} A = I$.

4.3 Homogeneous and inhomogeneous equations

4.3.1 Gaussian elimination

4.4 Matrix rank

**Theorem.** The column rank and row rank are equal for any $m \times n$ matrix.

**Proof.** Let $r$ be the row rank of $A$. Write the biggest set of linearly independent rows as $v^T_1, v^T_2, \cdots v^T_r$ or in component form $v^T_k = (v_{k1}, v_{k2}, \cdots, v_{kn})$ for $k = 1, 2, \cdots, r$.

Now denote the $i$th row of $A$ as $r^T_i = (A_{i1}, A_{i2}, \cdots, A_{in})$.

Note that every row of $A$ can be written as a linear combination of the $v$’s. (If $r_1$ cannot be written as a linear combination of the $v$’s, then it is independent of the $v$’s and $v$ is not the maximum collection of linearly independent rows)

Write

$$r^T_i = \sum_{k=1}^{r} C_{ik} v^T_k.$$ 

For some coefficients $C_{ik}$ with $1 \leq i \leq m$ and $1 \leq k \leq r$.

Now the elements of $A$ are

$$A_{ij} = (r^T_i)_j = \sum_{k=1}^{r} C_{ik} (v^T_k)_j.$$
or

\[
\begin{pmatrix}
A_{1j} \\
A_{2j} \\
\vdots \\
A_{mj}
\end{pmatrix}
= \sum_{k=1}^{r} v_{kj}
\begin{pmatrix}
C_{1k} \\
C_{2k} \\
\vdots \\
C_{mk}
\end{pmatrix}
\]

So every column of \( A \) can be written as a linear combination of the \( r \) column vectors \( c_k \). Then the column rank of \( A \leq r \), the row rank of \( A \).

Apply the same argument to \( A^T \) to see that the row rank is \( \leq \) the column rank.

4.5 Homogeneous problem \( Ax = 0 \)

4.5.1 Geometrical interpretation

4.5.2 Linear mapping view of \( Ax = 0 \)

4.6 General solution of \( Ax = d \)
5 Eigenvalues and eigenvectors

5.1 Preliminaries and definitions

**Theorem** (Fundamental theorem of algebra). Let \( p(z) \) be a polynomial of degree \( m \geq 1 \), i.e.
\[
p(z) = \sum_{j=0}^{m} c_j z^j,
\]
where \( c_j \in \mathbb{C} \) and \( c_m \neq 0 \).

Then \( p(z) = 0 \) has precisely \( m \) (not necessarily distinct) roots in the complex plane, accounting for multiplicity.

**Theorem.** \( \lambda \) is an eigenvalue of \( A \) iff \( \det(A - \lambda I) = 0 \).

**Proof.** (\( \Rightarrow \)) Suppose that \( \lambda \) is an eigenvalue and \( x \) is the associated eigenvector. We can rearrange the equation in the definition above to
\[
(A - \lambda I)x = 0
\]
and thus
\[
x \in \ker(A - \lambda I)
\]
But \( x \neq 0 \). So \( \ker(A - \lambda I) \) is non-trivial and \( \det(A - \lambda I) = 0 \). The (\( \Leftarrow \)) direction is similar.

5.2 Linearly independent eigenvectors

**Theorem.** Suppose \( n \times n \) matrix \( A \) has distinct eigenvalues \( \lambda_1, \lambda_2, \cdots, \lambda_n \). Then the corresponding eigenvectors \( x_1, x_2, \cdots, x_n \) are linearly independent.

**Proof.** Proof by contradiction: Suppose \( x_1, x_2, \cdots, x_n \) are linearly dependent. Then we can find non-zero constants \( d_i \) for \( i = 1, 2, \cdots, r \), such that
\[
d_1x_1 + d_2x_2 + \cdots + d_rx_r = 0.
\]
Suppose that this is the shortest non-trivial linear combination that gives \( 0 \) (we may need to re-order \( x_i \)).

Now apply \( (A - \lambda_1 I) \) to the whole equation to obtain
\[
d_1(\lambda_1 - \lambda_1)x_1 + d_2(\lambda_2 - \lambda_1)x_2 + \cdots + d_r(\lambda_r - \lambda_1)x_r = 0.
\]
We know that the first term is \( 0 \), while the others are not (since we assumed \( \lambda_i \neq \lambda_j \) for \( i \neq j \)). So
\[
d_2(\lambda_2 - \lambda_1)x_2 + \cdots + d_r(\lambda_r - \lambda_1)x_r = 0,
\]
and we have found a shorter linear combination that gives \( 0 \). Contradiction.

5.3 Transformation matrices

5.3.1 Transformation law for vectors

**Theorem.** Denote vector as \( u \) with respect to \( \{e_i\} \) and \( \tilde{u} \) with respect to \( \{\tilde{e}_i\} \).

Then
\[
u = P\tilde{u} \quad \text{and} \quad \tilde{u} = P^{-1}u
\]
5.3.2 Transformation law for matrix
Theorem. 
\[ \tilde{A} = P^{-1}AP. \]

5.4 Similar matrices

**Proposition.** Similar matrices have the following properties:

(i) Similar matrices have the same determinant.

(ii) Similar matrices have the same trace.

(iii) Similar matrices have the same characteristic polynomial.

**Proof.** They are proven as follows:

(i) \[ \det B = \det(P^{-1}AP) = (\det A)(\det P)^{-1}(\det P) = \det A \]

(ii) \[ \text{tr } B = \text{tr } P^{-1}AP = P^{-1}A_{ji}P_{ji} = A_{jk}P_{kj}P_{i}^{-1} = A_{jk}(PP^{-1})_{kj} = A_{jk}\delta_{kj} = A_{jj} = \text{tr } A \]

(iii) \[ p_B(\lambda) = \det(B - \lambda I) = \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP) = \det(P^{-1}(A - \lambda I)P) = \det(A - \lambda I) = p_A(\lambda) \]

5.5 Diagonalizable matrices

**Theorem.** Let \( \lambda_1, \lambda_2, \cdots, \lambda_r \), with \( r \leq n \) be the distinct eigenvalues of \( A \). Let \( B_1, B_2, \cdots, B_r \) be the bases of the eigenspaces \( E_{\lambda_1}, E_{\lambda_2}, \cdots, E_{\lambda_r} \) correspondingly.

Then the set \( B = \bigcup_{i=1}^{r} B_i \) is linearly independent.

**Proof.** Write \( B_1 = \{ x^{(1)}_1, x^{(1)}_2, \cdots, x^{(1)}_{m(\lambda_1)} \} \). Then \( m(\lambda_1) = \dim(E_{\lambda_1}) \), and similarly for all \( B_i \).
Consider the following general linear combination of all elements in $B$. Consider the equation

$$\sum_{i=1}^{r} \sum_{j=1}^{m(\lambda_i)} \alpha_{ij} x_j^{(i)} = 0.$$  

The first sum is summing over all eigenspaces, and the second sum sums over the basis vectors in $B_i$. Now apply the matrix $\prod_{k=1,2,\ldots,K,\ldots,r}(A - \lambda_k I)$ to the above sum, for some arbitrary $K$. We obtain

$$\sum_{j=1}^{m(\lambda_K)} \alpha_{Kj} \left[ \prod_{k=1,2,\ldots,K,\ldots,r}(\lambda - \lambda_k) \right] x_j^{(K)} = 0.$$  

Since the $x_j^{(K)}$ are linearly independent ($B_K$ is a basis), $\alpha_{Kj} = 0$ for all $j$. Since $K$ was arbitrary, all $\alpha_{ij}$ must be zero. So $B$ is linearly independent.

**Proposition.** $A$ is diagonalizable iff all its eigenvalues have zero defect.

### 5.6 Canonical (Jordan normal) form

**Theorem.** Any $2 \times 2$ complex matrix $A$ is similar to exactly one of

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

**Proof.** For each case:

(i) If $A$ has two distinct eigenvalues, then eigenvectors are linearly independent. Then we can use $P$ formed from eigenvectors as its columns

(ii) If $\lambda_1 = \lambda_2 = \lambda$ and $\dim E_\lambda = 2$, then write $E_\lambda = \text{span}\{u, v\}$, with $u, v$ linearly independent. Now use $\{u, v\}$ as a new basis of $\mathbb{C}^2$ and

$$\tilde{A} = P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$$

Note that since $P^{-1}AP = \lambda I$, we have $A = P(\lambda I)P^{-1} = \lambda I$. So $A$ is isotropic, i.e. the same with respect to any basis.

(iii) If $\lambda_1 = \lambda_2 = \lambda$ and $\dim(E_\lambda) = 1$, then $E_\lambda = \text{span}\{v\}$. Now choose basis of $\mathbb{C}^2$ as $\{v, w\}$, where $w \in \mathbb{C}^2 \setminus E_\lambda$.

We know that $Aw \in \mathbb{C}^2$. So $Aw = \alpha v + \beta w$. Hence, if we change basis to $\{v, w\}$, then $\tilde{A} = P^{-1}AP = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}$.

However, $A$ and $\tilde{A}$ both have eigenvalue $\lambda$ with algebraic multiplicity 2. So we must have $\beta = \lambda$. To make $\alpha = 1$, let $u = (\tilde{A} - \lambda I)w$. We know $u \neq 0$ since $w$ is not in the eigenspace. Then

$$(\tilde{A} - \lambda I)u = (\tilde{A} - \lambda I)^2 w = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} w = 0.$$
So \( u \) is an eigenvector of \( \hat{A} \) with eigenvalue \( \lambda \).

We have \( u = \hat{A}w - \lambda w \). So \( \hat{A}w = u + \lambda w \).

Change basis to \( \{ u, w \} \). Then \( A \) with respect to this basis is
\[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}
\]

This is a two-stage process: \( P \) sends basis to \( \{ v, w \} \) and then matrix \( Q \) sends to basis \( \{ u, w \} \). So the similarity transformation is \( Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ) \). 

**Proposition.** (Without proof) The canonical form, or Jordan normal form, exists for any \( n \times n \) matrix \( A \). Specifically, there exists a similarity transform such that \( A \) is similar to a matrix to \( \hat{A} \) that satisfies the following properties:

(i) \( \hat{A}_{\alpha\alpha} = \lambda_\alpha \), i.e. the diagonal composes of the eigenvalues.

(ii) \( \hat{A}_{\alpha,\alpha+1} = 0 \) or 1.

(iii) \( \hat{A}_{ij} = 0 \) otherwise.

### 5.7 Cayley-Hamilton Theorem

**Theorem** (Cayley-Hamilton theorem). Every \( n \times n \) complex matrix satisfies its own characteristic equation.

**Proof.** We will only prove for diagonalizable matrices here. So suppose for our matrix \( A \), there is some \( P \) such that \( D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = P^{-1}AP \). Note that
\[
D^i = (P^{-1}AP)(P^{-1}AP) \cdots (P^{-1}AP) = P^{-1}A^i P.
\]

Hence
\[
p_D(D) = p_D(P^{-1}AP) = P^{-1}[p_D(A)]P.
\]

Since similar matrices have the same characteristic polynomial. So
\[
p_A(D) = P^{-1}[p_A(A)]P.
\]

However, we also know that \( D^i = \text{diag}(\lambda_1^i, \lambda_2^i, \cdots, \lambda_n^i) \). So
\[
p_A(D) = \text{diag}(p_A(\lambda_1), p_A(\lambda_2), \cdots, p_A(\lambda_n)) = \text{diag}(0, 0, \cdots, 0)
\]

since the eigenvalues are roots of \( p_A(\lambda) = 0 \). So \( 0 = p_A(D) = P^{-1}p_A(A)P \) and thus \( p_A(A) = 0 \). 

### 5.8 Eigenvalues and eigenvectors of a Hermitian matrix

#### 5.8.1 Eigenvalues and eigenvectors

**Theorem.** The eigenvalues of a Hermitian matrix \( H \) are real.

**Proof.** Suppose that \( H \) has eigenvalue \( \lambda \) with eigenvector \( v \neq 0 \). Then
\[
Hv = \lambda v.
\]

We pre-multiply by \( v^\dagger \), a \( 1 \times n \) row vector, to obtain
\[
v^\dagger Hv = \lambda v^\dagger v \quad (\ast)
\]
We take the Hermitian conjugate of both sides. The left hand side is
\[(v^\dagger H v)^\dagger = v^\dagger H^\dagger v = v^\dagger H v\]
since \(H\) is Hermitian. The right hand side is
\[(\lambda v^\dagger v)^\dagger = \lambda^* v^\dagger v\]
So we have
\[v^\dagger H v = \lambda^* v^\dagger v.\]
From (\*), we know that \(\lambda v^\dagger v = \lambda^* v^\dagger v.\) Since \(v \neq 0\), we know that \(v^\dagger v = v \cdot v \neq 0.\) So \(\lambda = \lambda^*\) and \(\lambda\) is real.

**Theorem.** The eigenvectors of a Hermitian matrix \(H\) corresponding to distinct eigenvalues are orthogonal.

**Proof.** Let
\[
Hv_i = \lambda_i v_i, \quad \text{(i)}
\]
\[
Hv_j = \lambda_j v_j. \quad \text{(ii)}
\]
Pre-multiply (i) by \(v_j^\dagger\) to obtain
\[
v_j^\dagger H v_i = \lambda_i v_j^\dagger v_i. \quad \text{(iii)}
\]
Pre-multiply (ii) by \(v_i^\dagger\) and take the Hermitian conjugate to obtain
\[
v_i^\dagger H v_i = \lambda_j v_i^\dagger v_i. \quad \text{(iv)}
\]
Equating (iii) and (iv) yields
\[
\lambda_i v_j^\dagger v_i = \lambda_j v_j^\dagger v_i.
\]
Since \(\lambda_i \neq \lambda_j\), we must have \(v_j^\dagger v_i = 0.\) So their inner product is zero and are orthogonal.

**5.8.2 Gram-Schmidt orthogonalization (non-examinable)**

**5.8.3 Unitary transformation**

**5.8.4 Diagonalization of \(n \times n\) Hermitian matrices**

**Theorem.** An \(n \times n\) Hermitian matrix has precisely \(n\) orthogonal eigenvectors.

**Proof.** (Non-examinable) Let \(\lambda_1, \lambda_2, \ldots, \lambda_r\) be the distinct eigenvalues of \(H\) (\(r \leq n\)), with a set of corresponding orthonormal eigenvectors \(B = \{v_1, v_2, \ldots, v_r\}\). Extend to a basis of the whole of \(\mathbb{C}^n\)
\[
B' = \{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_{n-r}\}
\]
Now use Gram-Schmidt to create an orthonormal basis
\[
\tilde{B} = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{n-r}\}.
\]
Now write
\[ P = \begin{pmatrix}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow \\
v_1 & v_2 & \cdots & v_r & u_1 \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow
\end{pmatrix} \]

We have shown above that this is a unitary matrix, i.e. \( P^{-1} = P^\dagger \). So if we change basis, we have
\[ P^{-1}HP = P^\dagger HP \]
\[
= \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_r & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1,n-r} \\
0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2,n-r} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & c_{n-r,1} & c_{n-r,2} & \cdots & c_{n-r,n-r}
\end{pmatrix}
\]
Here \( C \) is an \((n-r) \times (n-r)\) Hermitian matrix. The eigenvalues of \( C \) are also eigenvalues of \( H \) because \( \det(H - \lambda I) = \det(P^\dagger HP - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_r - \lambda) \det(C - \lambda I) \). So the eigenvalues of \( C \) are the eigenvalues of \( H \).

We can keep repeating the process on \( C \) until we finish all rows. For example, if the eigenvalues of \( C \) are all distinct, there are \( n-r \) orthonormal eigenvectors \( w_j \) (for \( j = r+1, \ldots, n \)) of \( C \). Let
\[
Q = \begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
w_{r+1} \\
w_{r+2} \\
\cdots \\
w_n
\end{pmatrix}
\]
with other entries 0. (where we have a \( r \times r \) identity matrix block on the top left corner and a \((n-r) \times (n-r)\) with columns formed by \( w_j \).)

Since the columns of \( Q \) are orthonormal, \( Q \) is unitary. So \( Q^\dagger P^\dagger HPQ = \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_r, \lambda_{r+1}, \cdots, \lambda_n) \), where the first \( r \) \( \lambda \)s are distinct and the remaining ones are copies of previous ones.

The \( n \) linearly-independent eigenvectors are the columns of \( PQ \).

\[ \square \]

5.8.5 Normal matrices

Proposition.

(i) If \( \lambda \) is an eigenvalue of \( N \), then \( \lambda^* \) is an eigenvalue of \( N^\dagger \).

(ii) The eigenvectors of distinct eigenvalues are orthogonal.

(iii) A normal matrix can always be diagonalized with an orthonormal basis of eigenvectors.
6 Quadratic forms and conics

Theorem. Hermitian forms are real.

Proof. \((x^\dagger Hx)^* = (x^\dagger Hx)^\dagger = x^\dagger H^\dagger x = x^\dagger Hx\). So \((x^\dagger Hx)^* = x^\dagger Hx\) and it is real.

6.1 Quadrics and conics

6.1.1 Quadrics

6.1.2 Conic sections \((n = 2)\)

6.2 Focus-directrix property
7 Transformation groups

7.1 Groups of orthogonal matrices

**Proposition.** The set of all $n \times n$ orthogonal matrices $P$ forms a group under matrix multiplication.

**Proof.**

0. If $P, Q$ are orthogonal, then consider $R = PQ$. $RR^T = (PQ)(PQ)^T = P(QQ^T)P^T = PP^T = I$. So $R$ is orthogonal.

1. $I$ satisfies $II^T = I$. So $I$ is orthogonal and is an identity of the group.

2. Inverse: if $P$ is orthogonal, then $P^{-1} = P^T$ by definition, which is also orthogonal.

3. Matrix multiplication is associative since function composition is associative.

7.2 Length preserving matrices

**Theorem.** Let $P \in O(n)$. Then the following are equivalent:

(i) $P$ is orthogonal

(ii) $|Px| = |x|$

(iii) $(Px)^T(Py) = x^Ty$, i.e. $(Px) \cdot (Py) = x \cdot y$.

(iv) If $(v_1, v_2, \cdots, v_n)$ are orthonormal, so are $(Pv_1, Pv_2, \cdots, Pv_n)$

(v) The columns of $P$ are orthonormal.

**Proof.** We do them one by one:

(i) $\Rightarrow$ (ii): $|Px|^2 = (Px)^T(Px) = x^TP^TPx = x^T x = |x|^2$

(ii) $\Rightarrow$ (iii): $|P(x + y)|^2 = |x + y|^2$. The right hand side is

$$(x^T + y^T)(x + y) = x^T x + y^T y + x^T y + y^T x = |x|^2 + |y|^2 + 2x^T y.$$  

Similarly, the left hand side is

$$|Px + Py|^2 = |Px|^2 + |Py|^2 + 2(Px)^T Py = |x|^2 + |y|^2 + 2(Px)^T Py.$$  

So $(Px)^T Py = x^T y$.

(iii) $\Rightarrow$ (iv): $(Pv_i)^T Pv_j = v_i^T v_j = \delta_{ij}$. So $Pv_i$’s are also orthonormal.

(iv) $\Rightarrow$ (v): Take the $v_i$’s to be the standard basis. So the columns of $P$, being $Pe_i$, are orthonormal.

(v) $\Rightarrow$ (i): The columns of $P$ are orthonormal. Then $(PP^T)_{ij} = P_{ik}P_{jk} = (P_i) \cdot (P_j) = \delta_{ij}$, viewing $P_i$ as the $i$th column of $P$. So $PP^T = I$.  

7.3 Lorentz transformations

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