Part IA — Analysis I

Theorems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Limits and convergence
Sequences and series in $\mathbb{R}$ and $\mathbb{C}$. Sums, products and quotients. Absolute convergence; absolute convergence implies convergence. The Bolzano-Weierstrass theorem and applications (the General Principle of Convergence). Comparison and ratio tests, alternating series test. [6]

Continuity
Continuity of real- and complex-valued functions defined on subsets of $\mathbb{R}$ and $\mathbb{C}$. The intermediate value theorem. A continuous function on a closed bounded interval is bounded and attains its bounds. [3]

Differentiability
Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Derivative of sums and products. The chain rule. Derivative of the inverse function. Rolle's theorem; the mean value theorem. One-dimensional version of the inverse function theorem. Taylor's theorem from $\mathbb{R}$ to $\mathbb{R}$; Lagrange's form of the remainder. Complex differentiation. [5]

Power series
Complex power series and radius of convergence. Exponential, trigonometric and hyperbolic functions, and relations between them. *Direct proof of the differentiability of a power series within its circle of convergence*. [4]

Integration
Definition and basic properties of the Riemann integral. A non-integrable function. Integrability of monotonic functions. Integrability of piecewise-continuous functions. The fundamental theorem of calculus. Differentiation of indefinite integrals. Integration by parts. The integral form of the remainder in Taylor's theorem. Improper integrals. [6]

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0 Introduction

1 The real number system

Lemma. Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Lemma (Archimedean property v1)**.** Let $\mathbb{F}$ be an ordered field with the least upper bound property. Then the set $\{1, 2, 3, \cdots\}$ is not bounded above.

2 Convergence of sequences

2.1 Definitions

Lemma (Archimedean property v2)**.** $1/n \to 0$.

Lemma. Every eventually bounded sequence is bounded.

2.2 Sums, products and quotients

Lemma (Sums of sequences)**.** If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

Lemma (Scalar multiplication of sequences)**.** Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Lemma. Let (a_n) be bounded and $b_n \to 0$. Then $a_n b_n \to 0$.

Lemma. Every convergent sequence is bounded.

Lemma (Product of sequences)**.** Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Lemma (Quotient of sequences)**.** Let (a_n) be a sequence such that $(\forall n)\, a_n \neq 0$. Suppose that $a_n \to a$ and $a \neq 0$. Then $1/a_n \to 1/a$.

Corollary. If $a_n \to a, b_n \to b$, $b_n, b \neq 0$, then $a_n/b_n = a/b$.

Lemma (Sandwich rule)**.** Let (a_n) and (b_n) be sequences that both converge to a limit x. Suppose that $a_n \leq c_n \leq b_n$ for every n. Then $c_n \to x$.

2.3 Monotone-sequences property

Lemma. Least upper bound property $\Rightarrow$ monotone-sequences property.

Lemma. Let (a_n) be a sequence and suppose that $a_n \to a$. If $(\forall n)\, a_n \leq x$, then $a \leq x$.

Lemma. Monotone-sequences property $\Rightarrow$ Archimedean property.

Lemma. Monotone-sequences property $\Rightarrow$ least upper bound property.

Lemma. A sequence can have at most 1 limit.

Lemma (Nested intervals property)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property. Let $I_1 \supseteq I_2 \supseteq \cdots$ be closed bounded non-empty intervals. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proposition. $\mathbb{R}$ is uncountable.

Theorem (Bolzano-Weierstrass theorem)**.** Let $\mathbb{F}$ be an ordered field with the monotone sequences property (i.e. $\mathbb{F} = \mathbb{R}$).

Then every bounded sequence has a convergent subsequence.

2.4 Cauchy sequences

Lemma. Every convergent sequence is Cauchy.

Lemma. Let (a_n) be a Cauchy sequence with a subsequence (a_{n_k}) that converges to a. Then $a_n \to a$.

Theorem (The general principle of convergence)**.** Let $\mathbb{F}$ be an ordered field with the monotone-sequence property. Then every Cauchy sequence of $\mathbb{F}$ converges.

Lemma. Let $\mathbb{F}$ be an ordered field with the Archimedean property such that every Cauchy sequence converges. The $\mathbb{F}$ satisfies the monotone-sequences property.

2.5 Limit supremum and infimum

Lemma. Let (a_n) be a sequence. The following two statements are equivalent:

- $a_n \to a$
- $\limsup a_n = \liminf a_n = a$.

3 Convergence of infinite sums

3.1 Infinite sums

Lemma. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$.

Lemma. Suppose that $a_n \geq 0$ for every n and the partial sums S_n are bounded above. Then $\sum_{n=1}^{\infty} a_n$ converges.

Lemma (Comparison test)**.** Let (a_n) and (b_n) be non-negative sequences, and suppose that $\exists C, N$ such that $\forall n \geq N$, $a_n \leq Cb_n$. Then if $\sum b_n$ converges, then so does $\sum a_n$.

3.2 Absolute convergence

Lemma. Let $\sum a_n$ converge absolutely. Then $\sum a_n$ converges.

Theorem. If $\sum a_n$ converges absolutely, then it converges unconditionally.

Theorem. If $\sum a_n$ converges unconditionally, then it converges absolutely.

Lemma. Let $\sum a_n$ be a series that converges absolutely. Then for any bijection $\pi : \mathbb{N} \to \mathbb{N}$,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\pi(n)}.$$

3.3 Convergence tests

Lemma (Alternating sequence test)**.** Let (a_n) be a decreasing sequence of non-negative reals, and suppose that $a_n \to 0$. Then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, i.e. $a_1 - a_2 + a_3 - a_4 + \cdots$ converges.

Lemma (Ratio test)**.** We have three versions:

(i) If $\exists c < 1$ such that
$$\frac{|a_{n+1}|}{|a_n|} \leq c,$$
for all n, then $\sum a_n$ converges.

(ii) If $\exists c < 1$ and $\exists N$ such that
$$(\forall n \geq N)\, \frac{|a_{n+1}|}{|a_n|} \leq c,$$
then $\sum a_n$ converges. Note that just because the ratio is always less than 1, it doesn't necessarily converge. It has to be always less than a fixed number c. Otherwise the test will say that $\sum 1/n$ converges.

(iii) If $\exists \rho \in (-1, 1)$ such that
$$\frac{a_{n+1}}{a_n} \to \rho,$$
then $\sum a_n$ converges. Note that we have the *open* interval $(-1, 1)$. If $\frac{|a_{n+1}|}{|a_n|} \to 1$, then the test is inconclusive!

Theorem (Condensation test)**.** Let (a_n) be a decreasing non-negative sequence. Then $\sum_{n=1}^{\infty} a_n < \infty$ if and only if

$$\sum_{k=1}^{\infty} 2^k a_{2^k} < \infty.$$

Theorem (Integral test)**.** Let $f : [1, \infty] \to \mathbb{R}$ be a decreasing non-negative function. Then $\sum_{n=1}^{\infty} f(n)$ converges iff $\int_1^{\infty} f(x)\, \mathrm{d}x < \infty$.

3.4 Complex versions

Lemma (Abel's test)**.** Let $a_1 \geq a_2 \geq \cdots \geq 0$, and suppose that $a_n \to 0$. Let $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$. Then $\sum a_n z^n$ converges.

4 Continuous functions

4.1 Continuous functions

Lemma. The following two statements are equivalent for a function $f : A \to \mathbb{R}$.

- f is continuous
- If (a_n) is a sequence in A with $a_n \to a$, then $f(a_n) \to f(a)$.

Lemma. Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$ be continuous functions. Then

(i) $f + g$ is continuous

(ii) fg is continuous

(iii) if g never vanishes, then f/g is continuous.

Lemma. Let $A, B \subseteq \mathbb{R}$ and $f : A \to B$, $g : B \to \mathbb{R}$. Then if f and g are continuous, $g \circ f : A \to \mathbb{R}$ is continuous.

Theorem (Maximum value theorem)**.** Let $[a, b]$ be a closed interval in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains its bounds, i.e. $f(x) = \sup f$ for some x, and $f(y) = \inf f$ for some y.

Theorem (Intermediate value theorem)**.** Let $a < b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < 0 < f(b)$. Then there exists an $x \in (a, b)$ such that $f(x) = 0$.

Corollary. Let $f : [a, b] \to [c, d]$ be a continuous strictly increasing function with $f(a) = c$, $f(b) = d$. Then f is invertible and its inverse is continuous.

4.2 Continuous induction*

Proposition (Continuous induction v1)**.** Let $a < b$ and let $A \subseteq [a, b]$ have the following properties:

(i) $a \in A$

(ii) If $x \in A$ and $x \neq b$, then $\exists y \in A$ with $y > x$.

(iii) If $\forall \varepsilon > 0$, $\exists y \in A : y \in (x - \varepsilon, x]$, then $x \in A$.

Then $b \in A$.

Proposition (Continuous induction v2)**.** Let $A \subseteq [a, b]$ and suppose that

(i) $a \in A$

(ii) If $[a, x] \subseteq A$ and $x \neq b$, then there exists $y > x$ such that $[a, y] \subseteq A$.

(iii) If $[a, x) \subseteq A$, then $[a, x] \subseteq A$.

Then $A = [a, b]$

Theorem (Intermediate value theorem)**.** Let $a < b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < 0 < f(b)$. Then there exists an $x \in (a, b)$ such that $f(x) = 0$.

Theorem. Let $[a, b]$ be a closed interval in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded.

Theorem (Heine-Borel*)**.** Every cover of a closed, bounded interval $[a, b]$ by open intervals has a finite subcover. We say closed intervals are *compact* (cf. Metric and Topological Spaces).

Theorem. Let $[a, b]$ be a closed interval in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains it bounds, i.e. $f(x) = \sup f$ for some x, and $f(y) = \inf f$ for some y.

5 Differentiability

5.1 Limits

Proposition. If $f(x) \to \ell$ and $g(x) \to m$ as $x \to a$, then $f(x) + g(x) \to \ell + m$, $f(x)g(x) \to \ell m$, and $\frac{f(x)}{g(x)} \to \frac{\ell}{m}$ if g and m don't vanish.

5.2 Differentiation

Proposition.

$$f(x + h) = f(x) + hf'(x) + o(h).$$

Proposition. If $f(x + h) = f(x) + hf'(x) + o(h)$, then f is differentiable at x with derivative $f'(x)$.

Lemma (Sum and product rule)**.** Let f, g be differentiable at x. Then $f + g$ and fg are differentiable at x, with

$$\begin{aligned}(f + g)'(x) &= f'(x) + g'(x)\\ (fg)'(x) &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

Lemma (Chain rule)**.** If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x with derivative $g'(f(x))f'(x)$.

Lemma (Quotient rule)**.** If f and g are differentiable at x, and $g(x) \neq 0$, then f/g is differentiable at x with derivative

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

Lemma. If f is differentiable at x, then it is continuous at x.

Theorem. Let $f : [a, b] \to [c, d]$ be differentiable on (a, b), continuous on $[a, b]$, and strictly increasing. Suppose that $f'(x)$ never vanishes. Suppose further that $f(a) = c$ and $f(b) = d$. Then f has an inverse g and for each $y \in (c, d)$, g is differentiable at y with derivative $1/f'(g(y))$.

In human language, this states that if f is invertible, then the derivative of f^{-1} is $1/f'$.

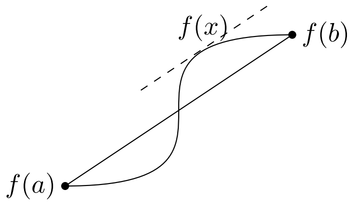
5.3 Differentiation theorems

Theorem (Rolle's theorem)**.** Let f be continuous on a closed interval $[a, b]$ (with $a < b$) and differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

Corollary (Mean value theorem)**.** Let f be continuous on $[a, b]$ ($a < b$), and differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Note that $\frac{f(b)-f(a)}{b-a}$ is the slope of the line joining $f(a)$ and $f(b)$.



Theorem (Local version of inverse function theorem)**.** Let f be a function with continuous derivative on (a, b).

Let $x \in (a, b)$ and suppose that $f'(x) \neq 0$. Then there is an open interval (u, v) containing x on which f is invertible (as a function from (u, v) to $f((u, v))$). Moreover, if g is the inverse, then $g'(f(z)) = \frac{1}{f'(z)}$ for every $z \in (u, v)$.

This says that if f has a non-zero derivative, then it has an inverse locally and the derivative of the inverse is $1/f'$.

Theorem (Higher-order Rolle's theorem)**.** Let f be continuous on $[a, b]$ $(a < b)$ and n-times differentiable on an open interval containing $[a, b]$. Suppose that

$$f(a) = f'(a) = f^{(2)}(a) = \cdots = f^{(n-1)}(a) = f(b) = 0.$$

Then $\exists x \in (a, b)$ such that $f^{(n)}(x) = 0$.

Corollary. Suppose that f and g are both differentiable on an open interval containing $[a, b]$ and that $f^{(k)}(a) = g^{(k)}(a)$ for $k = 0, 1, \cdots, n - 1$, and also $f(b) = g(b)$. Then there exists $x \in (a, b)$ such that $f^{(n)}(x) = g^{(n)}(x)$.

Theorem (Taylor's theorem with the Lagrange form of remainder)**.**

$$f(a + h) = \underbrace{f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)}_{(n-1)\text{-degree approximation to } f \text{ near } a} + \underbrace{\frac{h^n}{n!}f^{(n)}(x)}_{\text{error term}}.$$

for some $x \in (a, a + h)$.

5.4 Complex differentiation

6 Complex power series

Lemma. Suppose that $\sum a_n z^n$ converges and $|w| < |z|$, then $\sum a_n w^n$ converges (absolutely).

Lemma. The radius of convergence of a power series $\sum a_n z^n$ is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

Often $\sqrt[n]{|a_n|}$ converges, so we only have to find the limit.

6.1 Exponential and trigonometric functions

Proposition. The derivative of e^z is e^z.

Theorem. Let $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ be two absolutely convergent series, and let (c_n) be the convolution of the sequences (a_n) and (b_n). Then $\sum_{n=0}^\infty c_n$ converges (absolutely), and

$$\sum_{n=0}^\infty c_n = \left(\sum_{n=0}^\infty a_n\right)\left(\sum_{n=0}^\infty b_n\right).$$

Corollary.

$$e^z e^w = e^{z+w}.$$

Proposition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\sin z &= \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z\\
\frac{\mathrm{d}}{\mathrm{d}z}\cos z &= \frac{ie^{iz} - ie^{-iz}}{2} = -\sin z\\
\sin^2 z + \cos^2 z &= \frac{e^{2iz} + 2 + e^{-2iz}}{4} + \frac{e^{2iz} - 2 + e^{-2iz}}{-4} = 1.
\end{aligned}$$

Proposition.

$$\begin{aligned}
\cos(z + w) &= \cos z\cos w - \sin z\sin w\\
\sin(z + w) &= \sin z\cos w + \cos z\sin w
\end{aligned}$$

Proposition.

$$\begin{aligned}
\cos\left(z + \frac{\pi}{2}\right) &= -\sin z\\
\sin\left(z + \frac{\pi}{2}\right) &= \cos z\\
\cos(z + \pi) &= -\cos z\\
\sin(z + \pi) &= -\sin z\\
\cos(z + 2\pi) &= \cos z\\
\sin(z + 2\pi) &= \sin z
\end{aligned}$$

6.2 Differentiating power series

Lemma. Let a and b be complex numbers. Then

$$b^n - a^n - n(b-a)a^{n-1} = (b-a)^2(b^{n-2} + 2ab^{n-3} + 3a^2b^{n-4} + \cdots + (n-1)a^{n-2}).$$

Lemma. Let $a_n z^n$ have radius of convergence R, and let $|z| < R$. Then $\sum n a_n z^{n-1}$ converges (absolutely).

Corollary. Under the same conditions,

$$\sum_{n=2}^{\infty} \binom{n}{2} a_n z^{n-2}$$

converges absolutely.

Theorem. Let $\sum a_n z^n$ be a power series with radius of convergence R. For $|z| < R$, let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Then f is differentiable with derivative g.

6.3 Hyperbolic trigonometric functions

Proposition.

$$\frac{\mathrm{d}}{\mathrm{d}z} \cosh z = \sinh z$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \sinh z = \cosh z$$

Proposition.

$$\cosh iz = \cos z$$
$$\sinh iz = i \sin z$$

Proposition.

$$\cosh^2 z - \sinh^2 z = 1,$$

7 The Riemann Integral

7.1 Riemann Integral

Lemma. If $\mathcal{D}_2$ refines $\mathcal{D}_1$, then

$$U_{\mathcal{D}_2} f \leq U_{\mathcal{D}_1} f \text{ and } L_{\mathcal{D}_2 f} \geq L_{\mathcal{D}_1} f.$$

Corollary. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two dissections of $[a, b]$. Then

$$U_{\mathcal{D}_1} f \geq L_{\mathcal{D}_2} f.$$

Proposition (Riemann's integrability criterion)**.** This is sometimes known as Cauchy's integrability criterion.

Let $f : [a, b] \to \mathbb{R}$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a dissection $\mathcal{D}$ such that

$$U_{\mathcal{D}} - L_{\mathcal{D}} < \varepsilon.$$

Proposition. Let $f : [a, b] \to \mathbb{R}$ be integrable, and $\lambda \geq 0$. Then λf is integrable, and

$$\int_a^b \lambda f(x)\ \mathrm{d}x = \lambda \int_a^b f(x)\ \mathrm{d}x.$$

Proposition. Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $-f$ is integrable, and

$$\int_a^b -f(x)\ \mathrm{d}x = -\int_a^b f(x)\ \mathrm{d}x.$$

Proposition. Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then $f + g$ is integrable, and

$$\int_a^b (f(x) + g(x))\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x.$$

Proposition. Let $f, g : [a, b] \to \mathbb{R}$ be integrable, and suppose that $f(x) \leq g(x)$ for every x. Then

$$\int_a^b f(x)\ \mathrm{d}x \leq \int_a^b g(x)\ \mathrm{d}x.$$

Proposition. Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable.

Proposition (Additivity property)**.** Let $f : [a, c] \to \mathbb{R}$ be integrable, and let $b \in (a, c)$. Then the restrictions of f to $[a, b]$ and $[b, c]$ are Riemann integrable, and

$$\int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x$$

Similarly, if f is integrable on $[a, b]$ and $[b, c]$, then it is integrable on $[a, c]$ and the above equation also holds.

Proposition. Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then fg is integrable.

Theorem. Every continuous function f on a closed bounded interval $[a, b]$ is Riemann integrable.

Theorem (non-examinable)**.** Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

Theorem. Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.

Lemma. Let $a < b$ and let f be a bounded function from $[a, b] \to \mathbb{R}$ that is continuous on (a, b). Then f is integrable.

Corollary. Every piecewise continuous and bounded function on $[a, b]$ is integrable.

Lemma. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, and for each n, let $\mathcal{D}_n$ be the dissection $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i = a + \frac{i(b-a)}{n}$ for each i. Then

$$U_{\mathcal{D}_n} f \to \int_a^b f(x)\,\mathrm{d}x$$

and

$$L_{\mathcal{D}_n} f \to \int_a^b f(x)\,\mathrm{d}x.$$

Theorem (Fundamental theorem of calculus, part 1)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous, and for $x \in [a, b]$, define

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then F is differentiable and $F'(x) = f(x)$ for every x.

Corollary. If f is continuously differentiable on $[a, b]$, then

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a).$$

Theorem (Fundamental theorem of calculus, part 2)**.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, and suppose that f' is integrable. Then

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a).$$

Theorem (Integration by parts)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable such that everything below exists. Then

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,\mathrm{d}x.$$

Theorem (Taylor's theorem with the integral form of the remainder)**.** Let f be $n+1$ times differentiable on $[a, b]$ with with $f^{(n+1)}$ continuous. Then

$$\begin{aligned} f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f^{(2)}(a) + \cdots \\ + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t. \end{aligned}$$

Theorem (Integration by substitution)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $g : [u, v] \to \mathbb{R}$ be continuously differentiable, and suppose that $g(u) = a, g(v) = b$, and f is defined everywhere on $g([u, v])$ (and still continuous). Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_u^v f(g(t))g'(t)\,\mathrm{d}t.$$

7.2 Improper integrals

Theorem (Integral test)**.** Let $f : [1, \infty] \to \mathbb{R}$ be a decreasing non-negative function. Then $\sum_{n=1}^{\infty} f(n)$ converges iff $\int_1^{\infty} f(x)\,\mathrm{d}x < \infty$.