Part IV — Topics in Geometric Group Theory
Theorems

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The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This graduate course will provide an introduction to the basic ideas of the subject.

Suppose Γ is a discrete group of isometries of a metric space X. We focus on the theorems we can prove about Γ by imposing geometric conditions on X. These conditions are motivated by curvature conditions in differential geometry, but apply to general metric spaces and are much easier to state. First we study the case when X is *Gromov-hyperbolic*, which corresponds to negative curvature. Then we study the case when X is *CAT(0)*, which corresponds to non-positive curvature. In order for this theory to be useful, we need a rich supply of negatively and non-positively curved spaces. We develop the theory of *non-positively curved cube complexes*, which provide many examples of CAT(0) spaces and have been the source of some dramatic developments in low-dimensional topology over the last twenty years.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasiisometries, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Part 3. We will cover the basic theory of CAT(0) spaces, working up to the Cartan–Hadamard theorem and Gromov's Link Condition. These two results together enable us to check whether the universal cover of a complex admits a CAT(1) metric.

Part 4. We will introduce cube complexes, in which Gromov's link condition becomes purely combinatorial. If there is time, we will discuss Haglund–Wise's *special* cube complexes, which combine the good geometric properties of CAT(0) spaces with some strong algebraic and topological properties.

Pre-requisites

Part IB Geometry and Part II Algebraic topology are required.

Contents

1 Cayley graphs and the word metric

1.1 The word metric

Theorem. For any two finite generating sets S, S' of a group Γ, the identity map $(\Gamma, d_S) \to (\Gamma, d_{S'})$ is a quasi-isometry.

Lemma (Schwarz–Milnor lemma)**.** Let X be a proper geodesic metric space, and let Γ act properly discontinuously, cocompactly on X by isometries. Then

(i) Γ is finitely-generated.

(ii) For any $x_0 \in X$, the orbit map

$$\begin{aligned} \Gamma &\to X \\ \gamma &\mapsto \gamma x_0 \end{aligned}$$

is a quasi-isometry $(\Gamma, d_s) \underset{qi}{\simeq} (X, d)$.

1.2 Free groups

Corollary. $\pi_1(X_r)$ has the universal property of $F(S)$, with $S = \{a_1, \ldots, a_r\}$. So $\pi_1(X_r) \cong F_r$.

1.3 Finitely-presented groups

1.4 The word problem

Lemma. Let $\Gamma = \langle S \mid R \rangle$. Then the elements of $\ker(F(S) \to \Gamma)$ are precisely those of the form

$$\prod_{i=1}^{d} g_i r_i^{\pm 1} g_i^{-1},$$

where $g_i \in F(S)$ and $r_i \in R$.

Proposition. The word problem for $\mathcal{P}$ is solvable iff $\delta_{\mathcal{P}}$ is a computable function.

Proposition. If P and Q are two finite presentations for Γ, then $\delta_{\mathcal{P}} \approx \delta_{\mathcal{Q}}$.

Lemma. If $R' \subseteq \langle\langle R \rangle\rangle$ is a finite set, and

$$\mathcal{P} = \langle S \mid R \rangle, \quad \mathcal{Q} = \langle S \mid R \cup R' \rangle,$$

then $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{Q}}$.

Lemma. Let $\mathcal{P} = \langle S \mid R \rangle$, and let

$$\mathcal{Q} = \langle S \amalg T \mid R \amalg R' \rangle,$$

where

$$R' = \{t w_t^{-1} : t \in T, w_t \in F(S)\}.$$

Then $\delta_{\mathcal{P}} \approx \delta_{\mathcal{Q}}$.

2 Van Kampen diagrams

Lemma (van Kampen's lemma)**.** Let $\mathcal{P} = \langle S \mid R\rangle$ be a presentation and $w \in S^*$. Then the following are equivalent:

(i) $w = 1$ in Γ presented by $\mathcal{P}$ (i.e. w is null-homotopic)

(ii) There is a van Kampen diagram for w given $\mathcal{P}$.

If so, then

$$\mathrm{Area}_a(w) = \min\{\mathrm{Area}_g(D) : D \text{ is a van Kampen diagram for } w \text{ over } \mathcal{P}\}.$$

Corollary. If w is null-homotopic, then we can write w in the form

$$w = \prod_{i=1}^{d} g_i r_i^{\pm} g_i^{-1},$$

where

$$|g_i|_S \leq \mathrm{Diam}\, D$$

with D a minimal van Kampen diagram for w. We can further bound this by

$$(\max |r_i|_S)\, \mathrm{Area}(D) + |w|_S \leq \text{constant} \cdot \delta_{\mathcal{P}}(|w|_S) + |w|_S.$$

Proposition. The word problem for a presentation $\mathcal{P}$ is solvable iff $\delta_{\mathcal{P}}$ is computable.

Theorem (Novikov–Boone theorem)**.** There exists a finitely-presented group with an unsolvable word problem.

Corollary. $\delta_{\mathcal{P}}$ is sometimes non-computable.

Theorem. Let $n \geq 4$ and $\Gamma = \langle S \mid R\rangle$ be a finitely-presented group. Then we can construct a closed, smooth, orientable manifold M^n such that $\pi_1 M \cong \Gamma$.

Theorem (Douglas, Radú, Murray)**.** If γ is embedded, then there is a least-area filling disc.

Theorem (Filling theorem)**.** let M be a closed Riemannian manifold. Then $FiU^M \simeq \delta_{\pi_1 M}$.

3 Bass–Serre theory

3.1 Graphs of spaces

Proposition. For all groups G there exists an aspherical space $BG = K(G, 1)$ such that $\pi_1(K(G, 1)) \cong G$. Moreover, for any two choices of $K(G, 1)$ and $K(H, 1)$, and for every homomorphism $f : G \to H$, there is a unique map (up to homotopy) $\bar{f} : K(G, 1) \to K(H, 1)$ that induces this homomorphism on π_1. In particular, $K(G, 1)$ is well-defined up to homotopy equivalence.

Moreover, we can choose $K(G, 1)$ functorially, namely there are choices of $K(G, 1)$ for each G and choices of $\bar{f}$ such that $\overline{f_1 \circ f_2} = \bar{f}_1 \circ \bar{f}_2$ and $\overline{\mathrm{id}_G} = \mathrm{id}_{K(G,1)}$ for all f, G, H.

3.2 The Bass–Serre tree

Lemma. If $\mathcal{X}$ is a graph of spaces and $\hat{X} \to X$ is a covering map, then $\hat{X}$ naturally has the structure of a graph of spaces $\hat{\mathcal{X}}$, and p respects that structure.

Lemma (Britton's lemma)**.** For any vertex Ξ, the natural map $G_v \to G$ is injective.

Theorem (Normal form theorem)**.** Every element can be represented by a reduced loop, and the only reduced loop representing the identity is the trivial loop.

4 Hyperbolic groups

4.1 Definitions and examples

4.2 Quasi-geodesics and hyperbolicity

Theorem (Morse lemma)**.** For all $\delta \geq 0, \lambda \geq 1$ there is $R(\delta, \lambda, \varepsilon)$ such that the following holds:

If X is a δ-hyperbolic metric space, and $c : [a, b] \to X$ is a (λ, ε)-quasigeodesic from p to q, and $[p, q]$ is a choice of geodesic from p to q, then

$$d_{\text{Haus}}([p, q], \text{im}(c)) \leq R(\delta, \lambda, \varepsilon),$$

where

$$d_{\text{Haus}}(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_\varepsilon(B) \text{ and } B \subseteq N_\varepsilon(A)\}$$

is the *Hausdorff distance*.

Corollary. There is an $M(\delta, \lambda, \varepsilon)$ such that a geodesic metric space X is δ-hyperbolic iff any (λ, ε)-quasigeodesic triangle is M-slim.

Corollary. Suppose X, X' are geodesic metric spaces, and $f : X \to X''$ is a quasi-isometric embedding. If X' is hyperbolic, then so is X.

In particular, hyperbolicity is a quasi-isometrically invariant property, when restricted to geodesic metric spaces.

Lemma. Let X be a geodesic space. For any (λ, ε)-quasigeodesic $c : [a, b] \to X$, there exists a continuous, rectifiable (λ, ε')-quasigeodesic $c' : [a, b] \to X$ with $\varepsilon' = 2(\lambda + \varepsilon)$ such that

(i) $c'(a) = c(a)$, $c'(b) = c(b)$.

(ii) For all $a \leq t < t' \leq b$, we have

$$\ell(c'|_{[t,t']}) \leq k_1 d(c'(t), c'(t')) + k_2$$

where $k_1 = \lambda(\lambda + \varepsilon)$ and $k_2 = (\lambda\varepsilon' + 3)(\lambda + 3)$.

(iii) $d_{\text{Haus}}(\operatorname{im} c, \operatorname{im} c') \leq \lambda + \varepsilon$.

Lemma. let X be δ-hyperbolic, and $c : [a, b] \to X$ a continuous, rectifiable path in X joining p to q. For $[p, q]$ a geodesic, for any $x \in [p, q]$, we have

$$d(x, \operatorname{im} c) \leq \delta |\log_2 \ell(c)| + 1.$$

Theorem (Gromov)**.** A random group is infinite and hyperbolic.

Theorem. Let X be δ-hyperbolic and $c : [a, b] \to X$ be a k-local geodesic where $k > 8\delta$. Then c is a (λ, ε)-quasigeodesic for some $\lambda = \lambda(\delta, k)$ and $\varepsilon = \varepsilon(\delta, k)$.

Lemma. Let X be δ-hyperbolic and $k > 8\delta$. If $c : [a, b] \to X$ is a k-local geodesic, then $\operatorname{im} c$ is contained in the 2δ-neighbourhood of $[c(a), c(b)]$.

4.3 Dehn functions of hyperbolic groups

Corollary. Let X be δ-hyperbolic. Then there exists a constant $C = C(\delta)$ such that any non-constant loop in X is *not* C-locally geodesic.

Lemma. If Γ has a Dehn presentation, then δ_Γ is linear.

Theorem. Every hyperbolic group Γ is finitely-presented and admits a Dehn presentation.

In particular, the Dehn function is linear, and the word problem is solvable.

Theorem (Gromov, Bowditch, etc)**.** If Γ is a finitely-presented group and $\delta_\Gamma \precnsim n^2$, then Γ is hyperbolic.

Theorem. If Γ is finitely-generated, then the following are equivalent:

(i) Γ is hyperbolic.

(ii) Γ has a Dehn presentation.

(iii) Γ satisfies a linear isoperimetric inequality.

(iv) Γ has a subquadratic isoperimetric inequality.

Lemma (Ping-pong lemma)**.** Let Γ be hyperbolic and torsion-free (for convenience of statement). If $\gamma_1, \gamma_2 \in \Gamma$ do not commute, then for large enough n, the subgroup $\langle \gamma_1^n, \gamma_2^n \rangle \cong F_2$ and is quasi-convex.

Proposition. Let Γ be hyperbolic, and $\gamma \in \Gamma$. Then $C(\gamma)$ is quasiconvex. In particular, it is hyperbolic.

Corollary. Γ does not contain a copy of $\mathbb{Z}^2$.

Theorem (Casson–Jungreis, Gabai)**.** If Γ is hyperbolic and $\partial_\infty \Gamma \cong S^1$, then Γ is virtually $\pi_1 \Sigma$ for some closed hyperbolic Σ.

5 CAT(0) spaces and groups

5.1 Some basic motivations

Theorem (Novikov–Boone theorem)**.** There exists a finitely-presented group with an unsolvable word problem.

Theorem (Gordon)**.** There exists a sequence of finitely generated groups Γ_n such that $H_2(\Gamma_n)$ is not computable.

Theorem (Cartan–Hadamard theorem)**.** Let M be a non-positively curved compact manifold. Then $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$. In particular, it is contractible. Thus, $M = K(\pi_1 M, 1)$.

Theorem (Poincaré duality)**.** Let M be an orientable compact n-manifold. Then
$$H_k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R}).$$

5.2 CAT(κ) spaces

Lemma (Convexity of the metric)**.** Let X be a CAT(0) space, and $\gamma, \delta : [0, 1] \to X$ be geodesics (reparameterized). Then for all $t \in [0, 1]$, we have
$$d(\gamma(t), \delta(t)) \leq (1 - t)d(\gamma(0), \delta(0)) + td(\gamma(1), \delta(1)).$$

Lemma. If X is CAT(0), then X is *uniquely geodesic*, i.e. each pair of points is joined by a unique geodesic.

Lemma. Let X be a proper, uniquely geodesic metric space. Then geodesics in X vary continuously with their end points in the compact-open topology (which is the same as the uniform convergence topology).

Proposition. Any proper CAT(0) space X is contractible.

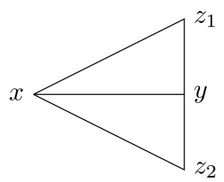
Proposition. Any CAT(0) group Γ satisfies a quadratic isoperimetric inequality, that is $\delta_\Gamma \simeq n$ or $\sim n^2$.

5.3 Length metrics

Theorem (Hopf–Rinow theorem)**.** If a length space X is complete and locally compact, then X is proper and geodesic.

5.4 Alexandrov's lemma

Lemma (Alexandrov's lemma)**.** Suppose the triangles $\Delta_1 = \Delta(x, y, z_1)$ and $\Delta_2 = \Delta(x, y, z_2)$ in a metric space satisfy the CAT(0) condition, and $y \in [z_1, z_2]$.



Then $\Delta = \Delta(x, z_1, z_2)$ also satisfies the CAT(0) condition.

Proposition. If X_1, X_2 are both locally compact, complete CAT(0) spaces and Y is isometric to closed, subspaces of both X_1 and X_2. Then $X_1 \cup_Y X_2$, equipped with the induced length metric, is CAT(0).

5.5 Cartan–Hadamard theorem

Theorem (Cartan–Hadamard theorem)**.** If X is a complete, connected length space of non-positive curvature, then the universal cover $\tilde{X}$, equipped with the induced length metric, is CAT(0).

Corollary. A (torsion free) group Γ is CAT(0) iff it is the π_1 of a complete, connected space X of non-positive curvature.

Lemma. If X is proper, non-positively curved and uniquely geodesic, then X is CAT(0).

Theorem. Let X be a proper length space of non-positive curvature, and $p, q \in X$. Then each homotopy class of paths from p to q contains a *unique* (local) geodesic representative.

5.6 Gromov's link condition

Theorem (Gromov's link criterion)**.** A Euclidean complex X is non-positively-curved iff for every vertex v of X, $Lk(v)$ is CAT(1).

Corollary. If X is a 2-dimensional Euclidean complex, then for all vertices v, $Lk(v)$ is a metric graph, and X is CAT(0) iff $Lk(v)$ has no loop of length $< 2\pi$ for all v.

Theorem (Mal'cev)**.** Every finitely generated linear subgroup (i.e. a subgroup of $\mathrm{GL}_n(\mathbb{C})$) is resudially finite.

Theorem (Mal'cev)**.** Every finitely generated residually finite group is Hopfian.

Lemma (Scott's criterion)**.** Let X be a cell complex, and $G = \pi_1 X$. Then G is residually finite if and only if the following holds:

Let $p : \tilde{X} \to X$ be the universal cover. For all compact subcomplexes $K \subseteq \tilde{X}$, there is a finite-sheeted cover $X' \to X$ such that the natural covering map $p' : \tilde{X} \to X'$ is injective on K.

5.7 Cube complexes

Theorem (Gromov)**.** A cube complex is non-positively curved iff every link is flag.

Lemma. For any (simplicial) graph N, the link of the unique vertex of $\mathcal{S}_N$ is $D(\bar{N})$. In particular, $\mathcal{S}_N$ is non-positively curved.

Theorem. Right-angled Artin groups embed into $\mathrm{GL}_n\mathbb{Z}$ (where n depends on N).

5.8 Special cube complexes

Theorem (Haglund–Wise). If X is a compact special cube complex, then there exists a graph N and a local isometry of cube complexes

$$\varphi_X : X \looparrowright \mathcal{S}_N.$$

Corollary. $\pi_1 X \hookrightarrow A_N$.

Corollary. If X is a special cube complex, then $\pi_1 X$ is linear, residually finite, Hopfian, etc.