Part IV — Topics in Geometric Group Theory

Definitions

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The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This graduate course will provide an introduction to the basic ideas of the subject.

Suppose Γ is a discrete group of isometries of a metric space X. We focus on the theorems we can prove about Γ by imposing geometric conditions on X. These conditions are motivated by curvature conditions in differential geometry, but apply to general metric spaces and are much easier to state. First we study the case when X is *Gromov-hyperbolic*, which corresponds to negative curvature. Then we study the case when X is *CAT(0)*, which corresponds to non-positive curvature. In order for this theory to be useful, we need a rich supply of negatively and non-positively curved spaces. We develop the theory of *non-positively curved cube complexes*, which provide many examples of CAT(0) spaces and have been the source of some dramatic developments in low-dimensional topology over the last twenty years.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasiisometries, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Part 3. We will cover the basic theory of CAT(0) spaces, working up to the Cartan–Hadamard theorem and Gromov's Link Condition. These two results together enable us to check whether the universal cover of a complex admits a CAT(1) metric.

Part 4. We will introduce cube complexes, in which Gromov's link condition becomes purely combinatorial. If there is time, we will discuss Haglund–Wise's *special* cube complexes, which combine the good geometric properties of CAT(0) spaces with some strong algebraic and topological properties.

Pre-requisites

Part IB Geometry and Part II Algebraic topology are required.

Contents

1 Cayley graphs and the word metric

1.1 The word metric

Definition (Word length)**.** Let Γ be a group and S a finite generating set. If $\gamma \in \Gamma$, the *word length* of γ is

$$\ell_S(\gamma) = \min\{n : \gamma = s_1^{\pm 1} \cdots s_n^{\pm 1} \text{ for some } s_i \in S\}.$$

Definition (Word metric)**.** Let Γ be a group and S a finite generating set. The *word metric* on Γ is given by

$$d_S(\gamma_1, \gamma_2) = \ell_s(\gamma_1^{-1}\gamma_2).$$

Definition (Cayley graph)**.** The *Cayley graph* $\mathrm{Cay}_S(\Gamma)$ is defined as follows:

- $V(\mathrm{Cay}_S(\Gamma)) = \Gamma$
- For each $\gamma \in \Gamma$ and $s \in S$, we draw an edge from γ to γs.

If we want it to be a directed and labelled graph, we can label each edge by the generator s.

Definition (Quasi-isometry)**.** Let $\lambda \geq 1$ and $c \geq 0$. A function between metric spaces $f : X \to Y$ is a (λ, c)*-quasi-isometric embedding* if for all $x_1, x_2 \in X$

$$\frac{1}{\lambda} d_X(x_1, x_2) - c \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_{,1}, x_2) + c$$

If, in addition, there is a C such that for all $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \leq C$, we say f is a *quasi-isometry*, and X is *quasi-isometric* to Y. We write $X \underset{qi}{\simeq} Y$.

Definition (Geodesic)**.** Let X be a metric space. A *geodesic* in X is an isometric embedding of a closed interval $\gamma : [a, b] \to X$.

Definition (Geodesic metric space)**.** A metric space X is called *geodesic* if every pair of points $x, y \in X$ is joined by a geodesic denoted by $[x, y]$.

Definition (Proper metric space)**.** A metric space is *proper* if closed balls in X are compact.

Definition (Proper discontinuous action)**.** An action Γ on a topological space X is *proper discontinuous* if for every compact set K, the set

$$\{g \in \Gamma : gK \cap K \neq \emptyset\}$$

is finite.

Definition (Cocompact action)**.** An action Γ on a topological space X is *cocompact* if the quotient $\Gamma \setminus X$ is compact.

1.2 Free groups

Definition (Free group)**.** Let S be a (usually finite) set. A group $F(S)$ with a map $S \to F(S)$ is called the *free group on* S if it satisfies the following universal property: for any set map $S \to G$, there is a unique group homomorphism $F(S) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & F(S) \arrow[d, dashed] \\ & G \end{tikzcd}.$$

Usually, if $|S| = r$, we just write $F(S) = F_r$.

1.3 Finitely-presented groups

Definition (Finitely-presented group)**.** A *finitely-presentable group* is a group Γ such that there are finite S and R such that $\Gamma \cong \langle S \mid R\rangle$.

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1.4 The word problem

Definition (Null-homotopic)**.** We say $w \in S^*$ is *null-homotopic* if $w = 1$ in Γ.

Definition (Algebraic area)**.** Let $w \in S^*$ be mull-homotopic. Its *algebraic area* is

$$\mathrm{Area}_{a,\mathcal{P}}(w) = \min \left\{ d : w = \prod_{i=1}^{d} g_i r_i^{\pm 1} g_i^{-1} \right\}.$$

Definition (Dehn function)**.** Then *Dehn function* is the function $\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ mapping

$$n \mapsto \max \left\{ \mathrm{Area}_{a,\mathcal{P}}(w) \mid |w|_S \leq n, w \text{ is null-homotopic} \right\}.$$

Definition ($\preccurlyeq$)**.** We write $f \preccurlyeq g$ iff for some $C > 0$, we have

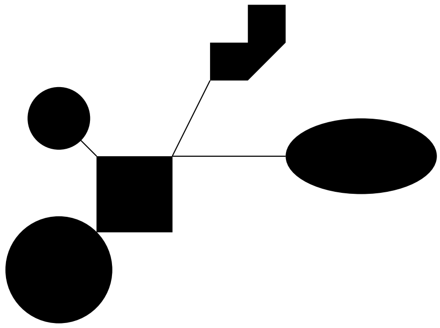
$$f(x) \leq Cg(Cx + C) + Cx + C.$$

for all x.

We write $f \approx g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

2 Van Kampen diagrams

Definition (Singular disc diagram)**.** A *(singular) disc diagram* is a compact, contractible subcomplex of $\mathbb{R}^2$, e.g.



We usually endow the disc diagram D with a base point p, and define

$$\mathrm{Area}_g(D) = \text{number of 2-cells of } D$$

and

$$\mathrm{Diam}_p(D) = \text{length of the longest embedded path in } D^{(1)}, \text{ starting at } p.$$

If we orient $\mathbb{R}^2$, then D has a well-defined boundary cycle. A disc diagram is *labelled* if each (oriented) edge is labelled by an element $s \in S^{\pm}$. The set of *face labels* is the set of boundary cycles of the 2-cells.

If all the face labels are elements of $R^{\pm}$, then we say D is a *diagram* over $\langle S \mid R \rangle$.

Definition (van Kampen diagram)**.** If $w \in \langle\langle R \rangle\rangle$, and w is the boundary cycle of a singular disc diagram D over the presentation $\langle S \mid R \rangle$, we say that D is a *van Kampen diagram* for w.

Definition (Filling disc)**.** Let (M, g) be a closed Riemannian manifold. Let $\gamma : S^1 \to M$ be a smooth null-homotopic loop. A filling disk for γ in M is a smooth map $f : D^2 \to M$ such that the diagram

$$\begin{tikzcd} S^1 \arrow[d, hook] \arrow[dr, "\gamma"] & \\ D^2 \arrow[r, "f"] & M \end{tikzcd}$$

Definition (FArea)**.**

$$\mathrm{FArea}(\gamma) = \inf\{\mathrm{Area}(f) \mid f : D^2 \to M \text{ is a filling disc for } \gamma\}.$$

Definition (Isoperimetric function)**.** The isoperimetric function of (M, g) is

$$\begin{aligned} &FiU^M : [0, \infty) \to [0, \infty) \\ &\ell \mapsto \sup\{\mathrm{FArea}(\gamma) : \gamma : S^1 \to M \text{ is a smooth null-homotopic loop, } \ell(\gamma) \leq \ell\} \end{aligned}$$

3 Bass–Serre theory

3.1 Graphs of spaces

Definition (Homotopy pushout)**.** Let X, Y, Z be spaces, and $\partial^- : Z \to X$ and $\partial^+ : Z \to Y$ be maps. We define

$$X \coprod_Z Y = (X \amalg Y \amalg Z \times [-1, 1])/\sim,$$

where we identify $\partial^\pm(z) \sim (z, \pm 1)$ for all $z \in Z$.

Definition (Graph of spaces)**.** A *graph of spaces* $\mathcal{X}$ consists of the following data

- A connected graph Ξ.
- For each vertex $v \in V(\Xi)$, a path-connected space X_v.
- For each edge $e \in E(\Xi)$, a path-connected space X_e.
- For each edge $e \in E(\Xi)$ attached to $v^\pm \in V(\Xi)$, we have π_1-injective maps $\partial_e^\pm : X_e \to X_{v^\pm}$.

The *realization* of $\mathcal{X}$ is

$$|\mathcal{X}| = X = \frac{\coprod_{v \in V(\Xi)} X_v \amalg \coprod_{e \in E(\Xi)} (X_e \times [-1, 1])}{(\forall e \in E(\Xi), \forall x \in X_e, (x, \pm 1) \sim \partial_e^\pm(x))}.$$

Definition (Graph of groups)**.** A *graph of groups* $\mathcal{G}$ consists of

- A graph Γ
- Groups G_v for all $v \in V(\Gamma)$
- Groups G_e for all $e \in E(\Gamma)$
- For each edge e with vertices $v^\pm(e)$, injective group homomorphisms

$$\partial_e^\pm : G_e \to G_{v^\pm(e)}.$$

Definition (Aspherical space)**.** A space X is *aspherical* if $\tilde{X}$ is contractible. By Whitehead's theorem and the lifting criterion, this is true iff $\pi_n(X) = 0$ for all $n \geq 2$.

3.2 The Bass–Serre tree

4 Hyperbolic groups

4.1 Definitions and examples

Definition (Geodesic triangle). A *geodesic triangle* Δ is a choice of three points x, y, z and geodesics $[x, y], [y, z], [z, x]$.

Definition (δ-slim triangle). We say Δ is δ-slim if every side of Δ is contained in the union of the δ-neighbourhoods of the other two sides.

Definition (Hyperbolic space). A metric space is *(Gromov) hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle in X is δ-slim. In this case, we say it *is δ-hyperbolic.*

4.2 Quasi-geodesics and hyperbolicity

Definition (Quasi-geodesic). A (λ, ε)-quasi-geodesic for $\lambda \geq 1, \varepsilon \geq 0$ is a (λ, ε)-quasi-isometric embedding $I \to X$, where $I \subseteq \mathbb{R}$ is a closed interval.

Definition (Hyperbolic group). A group Γ is *hyperbolic* if it acts properly discontinuously and cocompactly by isometries on a proper, geodesic hyperbolic metric space. Equivalently, if it is finitely-generated and with a hyperbolic Cayley graph.

Definition (Length of path). Let $c : [a, b] \to X$ be a continuous path. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a *dissection* $\mathcal{D}$ of $[a, b]$. Define

$$\ell(c) = \sup_{\mathcal{D}} \sum_{i=1}^{n} d(c(t_{i-1}), c(t_i)).$$

Definition (Rectifiable path). We say a path c is *rectifiable* if $\ell(c) < \infty$.

Definition (virtually-P). Let P be a property of groups. Then we say G is virtually-P if there is a finite index subgroup $G_0 \leq G$ such that G_0 is P.

Definition (k-local geodesic). Let X be a geodesic metric space, and $k > 0$. A path $c : [a, b] \to X$ is a *k-local geodesic* if

$$d(c(t), c(t')) = |t - t'|$$

for all $t, t' \in [a, b]$ with $|t - t'| \leq k$.

4.3 Dehn functions of hyperbolic groups

Definition (Dehn presentation). A finite presentation $\langle S \mid R \rangle$ for a group Γ is called *Dehn* if for every null-homotopic reduced word $w \in S^*$, there is (a cyclic conjugate of) a relator $r \in R$ such that $r = u^{-1}v$ with $\ell_S(u) < \ell_S(v)$, and $w = w_1 v w_2$ (without cancellation).

Definition (Geodesic ray). Let X be a δ-hyperbolic geodesic metric space. A *geodesic ray* is an isometric embedding $r : [0, \infty) \to X$.

5 CAT(0) spaces and groups

5.1 Some basic motivations

Definition (Group (co)homology)**.** The *(co)homology* of a group Γ is the (co)homology of $K(\Gamma, 1)$.

5.2 CAT(κ) spaces

Definition (Triangle)**.** A *triangle* with vertices $\{p, q, r\} \subseteq X$ is a choice

$$\Delta(p, q, r) = [p, q] \cup [q, r] \cup [r, p].$$

Definition (CAT(κ) space)**.** We say a space (X, d) is CAT(κ) if for any geodesic triangle $\Delta \subseteq X$ of diameter $\leq 2D_\kappa$, any $p, q \in \Delta$ and any comparison points $\bar{p}, \bar{q} \in \bar{\Delta}$,

$$d(p, q) \leq d_{M_\kappa}(\bar{p}, \bar{q}).$$

If X is locally CAT(κ), then K is said to be of *curvature* at most κ.

In particular, a locally CAT(0) space is called a *non-positively curved space*.

Definition (CAT(0) group)**.** A group is CAT(0) if it acts properly discontinuously and cocompactly by isometries on a proper CAT(0) space.

5.3 Length metrics

Definition (Length space)**.** A metric space X is called a *length space* if for all $x, y \in X$, we have

$$d(x, y) = \inf_{\gamma : x \to y} \ell(\gamma).$$

5.4 Alexandrov's lemma

5.5 Cartan–Hadamard theorem

5.6 Gromov's link condition

Definition (Euclidean cell complex)**.** A locally finite cell complex X is *Euclidean* if every cell is isometric to a convex polyhedron in Euclidean space and the attaching maps are isometries from the lower-dimensional cell to a face of the new cell.

Definition (Link)**.** Let X be a Euclidean complex, and let v be a vertex of X, and let $0 < \varepsilon \ll$ shortest 1-cell. Then the *link* of v is

$$Lk(v) = S_v(\varepsilon) = \{x \in X : d(x, v) = \varepsilon\}.$$

Definition (Residually finite group)**.** A group G is *residually finite* if for every $g \in G \setminus \{1\}$, there is a homomorphism $\varphi : G \to$ finite group such that $\varphi(g) \neq 0$.

5.7 Cube complexes

Definition (Cube complex)**.** A Euclidean complex is a *cube complex* if every cell is isometric to a cube (of any dimension).

Definition (Flag simplicial complex)**.** A simplicial complex X is *flag* if for all $n \geq 2$, each copy of $\partial\Delta^n$ in X is in fact the boundary of a copy of Δ^n in X.

Definition (right-angled Artin group)**.** Let N be a simplicial graph, i.e. a graph where the vertices determine the edges, i.e. a graph as a graph theorist would consider. Then

$$A_N = \langle V(N) \mid [u, v] = 1 \text{ for all } (u, v) \in E(N)\rangle$$

is the *right-angled Artin group*, or *graph group* of N.

Definition (Salvetti complex)**.** Given a simplicial group N, the *Salvetti complex* $\mathcal{S}_N$ is the cube complex defined as follows:

- Set $\mathcal{S}_N^{(2)}$ is the presentation complex for A_N.
- For any immersion of the 2-skeleton of a d-dimensional cube, we glue in an d-dimensional cube to $\mathcal{S}_N^{(2)}$.

Alternatively, we have a natural inclusion $\mathcal{S}_N^{(2)} \subseteq (S^1)^{|V(N)|}$, and $\mathcal{S}_N$ is the largest subcomplex whose 2-skeleton coincides with $\mathcal{S}_N^{(2)}$.

Definition (Double)**.** The *double* $D(K)$ of a simplicial complex K is defined as follows:

- The vertices are $\{v_1^+, \ldots, v_n^+, v_1^-, \ldots, v_n^-\}$, where $\{v_1, \ldots, v_n\}$ are the vertices of k.
- The simplices are those of the form $\langle v_{i_0}^{\pm}, \ldots, v_{i_k}^{\pm}\rangle$, where $\langle v_{i_0}, \ldots, v_{i_k}\rangle \in K$.

Definition (Flag complex)**.** The *flag complex* of N, written $\bar{N}$, is the only flag simplicial complex with 1-skeleton N.

5.8 Special cube complexes

Definition (Special cube complex)**.** A cube complex is *special* if its hyperplanes do not exhibit any of the following four pathologies:

- One-sidedness
- Self-intersection
- Direct self-osculation
- Inter-osculation

Definition (Virtually special group)**.** A group Γ is *virtually special* if there exists a finite index subgroup $\Gamma_0 \leq \Gamma$ such that $\Gamma_0 \cong \pi_1 X$, where X is a compact special cube complex.