Part IV — Topics in Number Theory
Theorems with proof

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The "Langlands programme" is a far-ranging series of conjectures describing the connections between automorphic forms on the one hand, and algebraic number theory and arithmetic algebraic geometry on the other. In these lectures we will give an introduction to some aspects of this programme.

Pre-requisites

The course will follow on naturally from the Michaelmas term courses *Algebraic Number Theory* and *Modular Forms and L-Functions*, and knowledge of them will be assumed. Some knowledge of algebraic geometry will be required in places.

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1.1 Preliminaries

Theorem (Galois theory)**.** There are bijections

$$\left\{\begin{matrix}\text{closed subgroups of}\\ \Gamma_K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subfields}\\ K \subseteq L \subseteq \bar{K}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{open subgroups of}\\ \Gamma_K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite subfields}\\ K \subseteq L \subseteq \bar{K}\end{matrix}\right\}$$

Theorem. $\ker \hat{t} = P_F$.

1.2 Local class field theory

Theorem (Local class field theory)**.**

(i) Let F be a local field. Then there is a continuous homomorphism, the *local Artin map*

$$\mathrm{Art}_F : F^\times \to \Gamma_F^{\mathrm{ab}}$$

with dense image characterized by the properties

(a) The following diagram commutes:

$$\begin{array}{ccccc} F^\times & \xrightarrow{\mathrm{Art}_F} & \Gamma_F^{\mathrm{ab}} & \twoheadrightarrow & \Gamma_F/I_F \\ \downarrow{\scriptstyle v_F} & & & & \downarrow{\scriptstyle \sim} \\ \mathbb{Z} & & \hookrightarrow & & \hat{\mathbb{Z}} \end{array}$$

(b) If F'/F is finite, then the following diagram commutes:

$$\begin{array}{ccc} (F')^\times & \xrightarrow{\mathrm{Art}_{F'}} & \Gamma_{F'}^{\mathrm{ab}} = \mathrm{Gal}(F'^{\mathrm{ab}}/F') \\ \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \text{restriction}} \\ F^\times & \xrightarrow{\mathrm{Art}_F} & \Gamma_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F) \end{array}$$

(ii) Moreover, the *existence theorem* says Art_F^{-1} induces a bijection

$$\left\{\begin{matrix}\text{open finite index}\\ \text{subgroups of } F^\times\end{matrix}\right\} \longleftrightarrow \left\{ \text{open subgroups of } \Gamma_F^{\mathrm{ab}} \right\}$$

Of course, open subgroups of Γ_F^{ab} further corresponds to finite abelian extensions of F.

(iii) Further, Art_F induces an isomorphism

$$\mathcal{O}_F^\times \xrightarrow{\sim} \mathrm{im}(I_F \to \Gamma_F^{\mathrm{ab}})$$

and this maps $(1 + \pi\mathcal{O}_F)^\times$ to the image of P_F. Of course, the quotient $\mathcal{O}_F^\times/(1 + \pi\mathcal{O}_F)^\times \cong k^\times = \mu_\infty(k)$.

(iv) Finally, this is functorial, namely if we have an isomorphism $\alpha : F \xrightarrow{\sim} F'$ and extend it to $\bar{\alpha} : \bar{F} \xrightarrow{\sim} \bar{F}'$, then this induces isomorphisms between the Galois groups $\alpha_* : \Gamma_F \xrightarrow{\sim} \Gamma_{F'}$ (up to conjugacy), and $\alpha_*^{\text{ab}} \circ \text{Art}_F = \text{Art}_{F'} \circ \alpha_*^{\text{ab}}$. $\square$

Proposition. Art_F induces an *isomorphism* of topological groups

$$\text{Art}_F^W : F^\times \to W_F^{\text{ab}}.$$

This maps $\mathcal{O}_F^\times$ isomorphically onto the inertia subgroup of Γ_F^{ab}.

Lemma. There is a finite K/F' such that $K \cap E = F'$, so $\text{Gal}(KE/K) \cong \text{Gal}(E/F') = \langle g \rangle$. Moreover, KE/K is unramified.

1.3 Global class field theory

Proposition (Product formula)**.** If $x \in K^\times$, then

$$\prod_{v \in \Sigma_K} |x|_v = 1.$$

Theorem. The map $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}_{>0}^\times$ has compact kernel.

Theorem (Artin reciprocity law)**.** $\text{Art}_K(K^\times) = \{1\}$, so induces a map $C_K \to \Gamma_K^{\text{ab}}$. Moreover,

(i) If $\text{char}(K) = p > 0$, then Art_K is injective, and induces an isomorphism $\text{Art}_K : C_k \xrightarrow{\sim} W_K^{\text{ab}}$, where W_K is defined as follows: since K is a finite extension of $\mathbb{F}_q(T)$, and wlog assume $\bar{\mathbb{F}}_q \cap K = \mathbb{F}_q \equiv k$. Then W_K is defined as the pullback

$$\begin{array}{ccc} W_K & \hookrightarrow & \Gamma_K = \text{Gal}(\bar{K}/K) \\ \downarrow & & \downarrow \text{restr.} \\ \mathbb{Z} & \hookrightarrow & \hat{\mathbb{Z}} \cong \text{Gal}(\bar{k}/k) \end{array}$$

(ii) If $\text{char}(K) = 0$, we have an isomorphism

$$\text{Art}_K : \pi_0(C_K) = \frac{C_K}{C_K^0} \xrightarrow{\sim} \Gamma_K^{\text{ab}}.$$

Moreover, if L/K is finite, then we have a commutative diagram

$$\begin{array}{ccc} C_L & \xrightarrow{\text{Art}_L} & \Gamma_L^{\text{ab}} \\ \downarrow N_{L/K} & & \downarrow \text{restr.} \\ C_K & \xrightarrow{\text{Art}_K} & \Gamma_K^{\text{ab}} \end{array}$$

If this is in fact Galois, then this induces an isomorphism

$$\text{Art}_{L/K} : \frac{J_K}{K^\times N_{L/K}(J_L)} \xrightarrow{\sim} \text{Gal}(L/K)^{\text{ab}}.$$

Finally, this is functorial, namely if $\sigma : K \xrightarrow{\sim} K'$ is an isomorphism, then we have a commutative square

$$\begin{array}{ccc} C_K & \xrightarrow{\mathrm{Art}_K} & \Gamma_K^{\mathrm{ab}} \\ \downarrow{\scriptstyle\sigma} & & \downarrow \\ C_{K'} & \xrightarrow{\mathrm{Art}_{K'}} & \Gamma_{K'}^{\mathrm{ab}} \end{array}$$

Proposition. If L/K is an abelian extension of global fields, which corresponds to the open subgroup $U \subseteq J_K$ under the Artin map, then L/K is unramified at a finite $v \nmid \infty$ iff $\mathcal{O}_v^\times \subseteq U$.

Proposition. If v is finite and unramified, then v splits completely iff $K_v^\times \subseteq U$.

Proof. v splits completely iff $L_w = K_v$ for all $w \mid v$, iff $\mathrm{Art}_{L_w/K_v}(K_v^\times) = \{1\}$. □

Proposition.

$$C_K/C_K^0 = \frac{\{\pm 1\}^{r_1} \times \hat{K}^\times}{\overline{K^\times}}.$$

1.4 Ideal-theoretic description of global class field theory

Proposition. There is a canonical isomorphism

$$\frac{J_K}{K^\times U_\mathfrak{m}} \xrightarrow{\sim} \mathrm{Cl}_\mathfrak{m}(K)$$

such that for $v \notin S \cup \Sigma_{K,\infty}$, the composition

$$K_v^\times \hookrightarrow J_K \to \mathrm{Cl}_\mathfrak{m}(K)$$

sends $x \mapsto \mathfrak{p}_v^{-v(x)}$.

Thus, in particular, the Galois group $\mathrm{Gal}(L/K)$ of the ray class field modulo $\mathfrak{m}$ is $\mathrm{Cl}_\mathfrak{m}(K)$. Concretely, if $\mathfrak{p} \notin S$ is an ideal, then $[\mathfrak{p}] \in \mathrm{Cl}_\mathfrak{m}(K)$ corresponds to $\sigma_\mathfrak{p} \in \mathrm{Gal}(L/K)$, the arithmetic Frobenius. This was Artin's original reciprocity law.

Proof sketch. Let $J_K(S) \subseteq J_K$ be given by

$$J_K(S) = \prod_{v \notin S \cup \Sigma_{K,\infty}} K_v^\times.$$

Here we do have the inverse of the content map

$$\begin{aligned} c^{-1} : J_K(S) &\twoheadrightarrow I_S \\ (x_v) &\mapsto \prod \mathfrak{p}_v^{-v(x_v)} \end{aligned}$$

We want to extend it to an isomorphism. Observe that

$$J_K(S) \cap U_\mathfrak{m} = \prod_{v \notin S \cup \Sigma_{K,\infty}} \mathcal{O}_v^\times,$$

which is precisely the kernel of the map c^{-1}. So c^{-1} extends uniquely to a homomorphism

$$\frac{J_K(S)U_\mathfrak{m}}{U_\mathfrak{m}} \cong \frac{J_K(S)}{J_K(S) \cap U_\mathfrak{m}} \to I_S.$$

We then use that $K^\times J_K(S)U_\mathfrak{m} = J_K$ (weak approximation), and

$$K^\times \cap V_\mathfrak{m} = \{x \equiv 1 \bmod \mathfrak{m},\ x \in K^*\},$$

where

$$V_\mathfrak{m} = J_K(S)U_\mathfrak{m} = \{(x_v) \in J_K \mid \text{ for all } v \text{ with } m_v > 0,\, x_v \in U_\mathfrak{m}\}. \qquad \square$$

2 L-functions

2.1 Hecke characters

Proposition. Let G be a profinite group, and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ continuous. Then $\ker \rho$ is open.

Proof. It suffices to show that $\ker \rho$ contains an open subgroup. We use the fact that $\mathrm{GL}_n(\mathbb{C})$ has "no small subgroups", i.e. there is an open neighbourhood U of $1 \in \mathrm{GL}_n(\mathbb{C})$ such that U contains no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ (exercise!). For example, if $n = 1$, then we can take U to be the right half plane.

Then for such U, we know $\rho^{-1}(U)$ is open. So it contains an open subgroup V. Then $\rho(V)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ contained in U, hence is trivial. So $V \subseteq \ker(\rho)$. □

Proposition. χ is unramified iff $\chi(x) = |x|_F^s$ for some $s \in \mathbb{C}$. □

Proposition. The set of continuous homomorphisms $\chi : J_K = \prod_v' K_v^\times \to \mathbb{C}^\times$ bijects with the set of all families $(\chi_v)_{v \in \Sigma_k}$, $\chi_v : K_v^\times \to \mathbb{C}^\times$ such that χ_v is unramified for almost all (i.e. all but finitely many) v, with the bijection given by $\chi \mapsto (\chi_v)$, $\chi_v = \chi|_{K_v^\times}$.

Proof. Let $\chi : J_K \to \mathbb{C}^\times$ be a character. Since $\hat{\mathcal{O}}_K \subseteq J_K$ is profinite, we know $\ker \chi|_{\hat{\mathcal{O}}_K^\times}$ is an open subgroup. Thus, it contains $\mathcal{O}_v^\times$ for all but finitely many v. So we have a map from the LHS to the RHS.

In the other direction, suppose we are given a family $(\chi_v)_v$. We attempt to define a character $\chi : J_K \to \mathbb{C}^\times$ by

$$\chi(x_v) = \prod \chi_v(x_v).$$

By assumption, $\chi_v(x_v) = 1$ for all but finitely many v. So this is well-defined. These two operations are clearly inverses to each other. □

Lemma. Let χ be a Hecke character. Then the following are equivalent:

(i) χ has finite image.

(ii) $\chi_\infty(K_\infty^{\times,0}) = 1$.

(iii) $\chi_\infty^2 = 1$.

(iv) $\chi(C_K^0) = 1$.

(v) χ factors through $\mathrm{Cl}_\mathfrak{m}(K)$ for some modulus $\mathfrak{m}$.

In this case, we say χ is a *ray class character*.

Proof. Since $\chi_\infty(K_\infty^{\times,0})$ is either 1 or infinite, we know (i) ⇒ (ii). It is clear that (ii) ⇔ (iii), and these easily imply (iv). Since C_K/C_K^0 is profinite, if (iii) holds, then χ factors through C_K/C_K^0 and has open kernel, hence the kernel contains $U_\mathfrak{m}$ for some modulus $\mathfrak{m}$. So χ factors through $\mathrm{Cl}_\mathfrak{m}(K)$. Finally, since $\mathrm{Cl}_\mathfrak{m}(K)$ is finite, (v) ⇒ (i) is clear. □

Proposition. If χ is an algebraic Hecke character, then χ^∞ takes values in some number field. We write $E = E(\chi)$ for the smallest such field.

Proof. Observe that $\chi^\infty(\hat{\mathcal{O}}_K^\times)$ is finite subgroup, so is μ_n for some n. Let $x \in K^\times$, totally positive. Then

$$\chi^\infty(x) = \chi_\infty(x)^{-1} = \varphi(x)^{-1} \in K^{cl},$$

where K^{cl} is the Galois closure. Then since $K_{>0}^\times \times \hat{\mathcal{O}}_K^\times \to \hat{K}^\times$ has finite cokernel (by the finiteness of the class group), so

$$\chi^\infty(\hat{K}^\times) = \coprod_{i=1}^d z_i \chi^\infty(K_{>0}^\times \hat{\mathcal{O}}_K^\times),$$

where $z_i^d \in \chi^\infty(K_{>0}^\times \hat{\mathcal{O}}_K^\times)$, and is therefore contained inside a finite extension of the image of $K_{>0}^\times \times \hat{\mathcal{O}}_K^\times$. □

Lemma. Let K be a number field, $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ be an algebraic homomorphism, and suppose $E/\mathbb{Q}$ is Galois. Then φ factors as

$$K^\times \overset{\text{norm}}{\longrightarrow} (K \cap E)^\times \overset{\phi'}{\longrightarrow} E^\times.$$

Note that since E is Galois, the intersection $K \cap E$ makes perfect sense.

Proof. By definition, we can write

$$\varphi(x) = \prod_{\sigma: K \hookrightarrow \mathbb{C}} \sigma(x)^{n(\sigma)}.$$

Then since $\varphi(x) \in E$, for all $x \in K^\times$ and $\tau \in \Gamma_E$, we have

$$\prod \tau\sigma(x)^{n(\sigma)} = \prod \sigma(x)^{n(\sigma)}.$$

In other words, we have

$$\prod_\sigma \sigma(x)^{n(\tau^{-1}\sigma)} = \prod_\sigma \sigma(x)^{n(\sigma)}.$$

Since the homomorphisms σ are independent, we must have $n(\tau\sigma) = n(\sigma)$ for all embeddings $\sigma : K \hookrightarrow \bar{\mathbb{Q}}$ and $\tau \in \Gamma_E$. This implies the theorem. □

Proposition. Let $\varphi : K^\times \to \mathbb{C}^\times$ be an algebraic homomorphism. Then φ is the infinity type of an algebraic Hecke character χ iff $\varphi(\mathcal{O}_K^\times)$ is finite.

Proof. To prove the ($\Rightarrow$) direction, suppose $\chi = \chi_\infty\chi^\infty$ is an algebraic Hecke character with infinity type φ. Then $\chi^\infty(U_\mathfrak{m}^\infty) = 1$ for some $\mathfrak{m}$. Let $E_\mathfrak{m} = K^\times \cap U_\mathfrak{m} \subseteq \mathcal{O}_K^\times$, a subgroup of finite index. As $\chi^\infty(E_\mathfrak{m}) = 1 = \chi(E_\mathfrak{m})$, we know $\chi_\infty(E_\mathfrak{m}) = 1$. So $\varphi(\mathcal{O}_K^\times)$ is finite.

To prove ($\Leftarrow$), given φ with $\varphi(\mathcal{O}_K^\times)$ finite, we can find some $\mathfrak{m}$ such that $\varphi(E_\mathfrak{m}) = 1$. Then $(\varphi, 1) : K_\infty^\times \times U_\mathfrak{m}^\infty \to \mathbb{C}^\times$ is trivial on $E_\mathfrak{m}$. So we can extend this to a homomorphism

$$\frac{K_\infty^\times U_\mathfrak{m} K^\times}{K^\times} \cong \frac{K_\infty^\times U_\mathfrak{m}}{E_\mathfrak{m}} \to \mathbb{C}^\times,$$

since $E_\mathfrak{m} = K^\times \cap U_\mathfrak{m}$. But the LHS is a finite index subgroup of C_K. So the map extends to some χ. □

Theorem. Suppose K is arbitrary, and $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ is algebraic, and we assume $E/\mathbb{Q}$ is Galois, containing the normal closure of K. Thus, we can write

$$\varphi(x) = \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{n(\sigma)}.$$

Then the following are equivalent:

(i) φ is of Serre type.

(ii) $\varphi = \psi \circ N_{K/F}$, where F is the maximal CM subfield and ψ is of Serre type.

(iii) For all $c' \in \mathrm{Gal}(E/\mathbb{Q})$ conjugate to complex conjugation c, the map $\sigma \mapsto n(\sigma) + n(c'\sigma)$ is constant.

(iv) (in the case $K \subseteq \mathbb{C}$ and $K/\mathbb{Q}$ is Galois with Galois group G) Let $\lambda = \sum n(\sigma)\sigma \in \mathbb{Z}[G]$. Then for all $\tau \in G$, we have

$$(\tau - 1)(c + 1)\lambda = 0 = (c + 1)(\tau - 1)\lambda.$$

Proof.

- (iii) $\Leftrightarrow$ (iv): This is just some formal symbol manipulation.
- (ii) $\Rightarrow$ (i): The norm takes units to units.
- (i) $\Rightarrow$ (iii): By the previous lecture, we know that if φ is of Serre type, then

$$|\varphi(K_\infty^{\times,1})| = 1.$$

  Now if $(x_v) \in K_\infty^\times$, we have

$$|\varphi((x_v))| = \prod_{\text{real } v} |x_v|^{n(\sigma_v)} \prod_{\text{complex } v} |x_v|^{n(\sigma_v)+n(\bar{\sigma}_v)} = \prod_v |x_v|_v^{\frac{1}{2}(n(\sigma_v)+n(\bar{\sigma}_v))}.$$

  Here the modulus without the subscript is the usual modulus. Then $|\varphi(K_\infty^{\times,1})| = 1$ implies $n(\sigma_v) + n(\bar{\sigma}_v)$ is constant. In other words, $n(\sigma) + n(c\sigma) = m$ is constant.

  But if $\tau \in \mathrm{Gal}(E/\mathbb{Q})$, and $\varphi' = \tau \circ \varphi$, $n'(\sigma) = n(\tau^{-1}\sigma)$, then this is also of Serre type. So

$$m = n'(\sigma) + n'(c\sigma) = n(\tau^{-1}\sigma) + n(\tau^{-1}c\sigma) = n(\tau^{-1}\sigma) + n((\tau^{-1}c\tau)\tau^{-1}\sigma).$$

- (iii) $\Rightarrow$ (ii): Suppose $n(\sigma) + n(c'\sigma) = m$ for all σ and all $c' = \tau c \tau^{-1}$. Then we must have

$$n(c'\sigma) = n(c\sigma)$$

  for all σ. So

$$n(\sigma) = n(c\tau c\tau^{-1}\sigma)$$

  So n is invariant under $H = [c, \mathrm{Gal}(E/\mathbb{Q})] \leq \mathrm{Gal}(E/\mathbb{Q})$, noting that c has order 2. So φ takes values in the fixed field $E^H = E \cap \mathbb{Q}^{\mathrm{CM}}$. By the proposition last time, this implies φ factors through $N_{K/F}$, where $F = E^H \cap K = K \cap \mathbb{Q}^{\mathrm{CM}}$. □

2.2 Abelian L-functions

Theorem (Hecke–Tate)**.**

(i) $\Lambda(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$, entire unless $\chi = |\cdot|_{\mathbb{A}}^t$ for some $t \in \mathbb{C}$, in which case there are simple poles at $s = 1-t, -t$.

(ii) There is some function, the *global ε-factor*,

$$\varepsilon(\chi, s) = AB^s$$

for some $A \in \mathbb{C}^\times$ and $B \in \mathbb{R}_{>0}$ such that

$$\Lambda(\chi, s) = \varepsilon(\chi, s)\Lambda(\chi^{-1}, 1-s).$$

(iii) There is a factorization

$$\varepsilon(\chi, s) = \prod_v \varepsilon_v(\chi_v, \mu_v, \psi_v, s),$$

where $\varepsilon_v = 1$ for almost all v, and ε_v depends only on χ_v and certain auxiliary data ψ_v, μ_v. These are the *local ε-factors*.

Proposition. Let $x \in K^\times$. Pick some invariant measure $\mathrm{d}u$ on $\Delta \otimes \mathbb{R}$. Then

$$\int_{\Delta\otimes\mathbb{R}} \frac{1}{\|ux\|^{2s}}\,\mathrm{d}u = \frac{\text{stuff}}{|N_{K/\mathbb{Q}}(x)|^{2s/n}},$$

where the stuff is some ratio of Γ factors and powers of π (and depends on s).

2.3 Non-abelian L-functions

Proposition.

(i) If

$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V') \to 0$$

is exact, then

$$L(\rho, s) = L(\rho', s) \cdot L(\rho'', s).$$

(ii) If E/F is finite separable, $\rho : W_E \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_{W_E}^{W_F} \rho : W_F \to \mathrm{GL}_{\mathbb{C}}(U)$, then

$$L(\rho, s) = L(\sigma, s).$$

Proof.

(i) Since ρ has open kernel, we know $\rho(I_F)$ is *finite.* So

$$0 \to (V')^{I_F} \to V^{I_F} \to (V')^{I_F} \to 0$$

is exact. Then the result follows from the multiplicativity of det.

(ii) We can write

$$U = \{\varphi : W_F \to V : \varphi(gx) = \rho(g)\varphi(x) \text{ for all } g \in W_E, x \in W_F\}.$$

where W_F acts by

$$\sigma(g)\varphi(x) = \varphi(xg).$$

Then we have

$$U^{I_F} = \{\varphi : W_F/I_F \to V : \cdots\}.$$

Then whenever $\varphi \in U^{I_F}$ and $g \in I_E$, then

$$\sigma(g)\varphi(x) = \varphi(xg) = \varphi((xgx^{-1})x) = \varphi(x).$$

So in fact φ takes values in V^{I_E}. Therefore

$$U^{I_F} = \mathrm{Ind}_{W_E/I_E}^{W_F/I_F} V^{I_E}.$$

Of course, $W_F/I_F \cong \mathbb{Z}$, which contains W_E/I_E as a subgroup. Moreover,

$$\mathrm{Frob}_F^d = \mathrm{Frob}_E,$$

where $d = [k_E : k_F]$. We note the following lemma:

Lemma. Let $G = \langle g \rangle \supseteq H = \langle h = g^d \rangle$, $\rho : H \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_H^G \rho$. Then

$$\det(1 - t^d \rho(h)) = \det(1 - t\sigma(g)).$$

Proof. Both sides are multiplicative for exact sequences of representations of H. So we can reduce to the case of $\dim V = 1$, where $\rho(h) = \lambda \in \mathbb{C}^\times$. We then check it explicitly. □

To complete the proof of (ii), take $g = \mathrm{Frob}_F$ and $t = q_F^{-s}$ so that $t^d = q_E^{-s}$. □

Proposition.

(i) $L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s)$.

(ii) If L/K is finite separable and $\rho : \Gamma_L \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(\rho)$, then

$$L(\rho, s) = L(\sigma, s).$$

The same are true for $\Lambda(\rho, s)$.

Proof. (i) is clear. For (ii), we saw that if $w \in \Sigma_L$ over $v \in \Sigma_K$ and consider the local extension L_w/K_v, then

$$L(\rho_w, s) = L(\mathrm{Ind}_{\Gamma_{L_w}}^{\Gamma_{K_v}} \rho_w).$$

In the global world, we have to take care of the splitting of primes. This boils down to the fact that

$$\left(\mathrm{Ind}_{\Gamma_L}^{\Gamma_K} \rho\right)\Big|_{\Gamma_{K_v}} = \bigoplus_{w|v} \mathrm{Ind}_{\Gamma_{L_w}}^{\Gamma_{K_v}}(\rho|_{\Gamma_{L_w}}). \qquad (*)$$

We fix a valuation $\bar{v}$ of $\bar{K}$ over v. Write $\Gamma_{\bar{v}/v}$ for the decomposition group in Γ_K. Write $\bar{S}$ for the places of $\bar{K}$ over v, and S the places of L over v.

The Galois group acts transitively on $\bar{S}$, and we have

$$\bar{S} \cong \Gamma_K/\Gamma_{\bar{v}/v}.$$

We then have

$$S \cong \Gamma_L \backslash \Gamma_K/\Gamma_{\bar{v}/v},$$

which is compatible with the obvious map $\bar{S} \to S$.

For $\bar{w} = g\bar{v}$, we have

$$\Gamma_{\bar{w}/v} = g\Gamma_{\bar{v}/v}g^{-1}.$$

Conjugating by g^{-1}, we can identify this with $\Gamma_{\bar{v}/v}$. Similarly, if $w = \bar{w}|_L$, then this contains

$$\Gamma_{\bar{w}/w} = g\Gamma_{\bar{v}/v}g^{-1} \cap \Gamma_L,$$

and we can identify this with $\Gamma_{\bar{v}/v} \cap g^{-1}\Gamma_L g$.

There is a theorem, usually called Mackey's formula, which says if $H, K \subseteq G$ are two subgroups of finite index, and $\rho : H \to \mathrm{GL}_{\mathbb{C}}(V)$ is a representation of H. Then

$$(\mathrm{Ind}_H^G V)|_K \cong \bigoplus_{g \in H\backslash G/K} \mathrm{Ind}^K_{K \cap g^{-1}Hg}({}^{g^{-1}}V),$$

where ${}^{g^{-1}}V$ is the $K \cap g^{-1}Hg$-representation where $g^{-1}xg$ acts by $\rho(x)$. We then apply this to $G = \Gamma_K, H = \Gamma_L, K = \Gamma_{\bar{v}/v}$. □

Theorem (Brauer induction theorem)**.** Suppose $\rho : G \to \mathrm{GL}_N(\mathbb{C})$ is a representation of a finite group. Then there exists subgroups $H_j \subseteq G$ and homomorphisms $\chi_j : H_j \to \mathbb{C}^\times$ and integers $m_i \in \mathbb{Z}$ such that

$$\mathrm{tr}\,\rho = \sum_j m_j \,\mathrm{tr}\,\mathrm{Ind}_{H_j}^G \chi_j.$$

Note that the m_j need not be non-negative. So we cannot quite state this as a statement about representations.

Corollary. Let $\rho : \Gamma_K \to \mathrm{GL}_N(\mathbb{C})$. Then there exists finite separable L_j/K and $\chi_j : \Gamma_{L_j} \to \mathbb{C}^\times$ of finite order and $m_j \in \mathbb{Z}$ such that

$$L(\rho, s) = \prod_j L(\chi_j, s)^{m_j}.$$

In particular, $L(\rho, s)$ has meromorphic continuation to $\mathbb{C}$ and has a functional equation

$$\Lambda(\rho, s) = L \cdot L_\infty = \varepsilon(\rho, s) L(\tilde{\rho}, 1 - s)$$

where

$$\varepsilon(\rho, s) = AB^s = \prod \varepsilon(\chi_j, s)^{m_j},$$

and $\tilde{\rho}(g) =^t \rho(g^{-1})$.

Theorem (Langlands–Deligne)**.** There exists a unique system of local constants $\varepsilon(\rho, \psi, \mu)$ for $\rho : W_F \to \mathrm{GL}_{\mathbb{C}}(V)$ such that

(i) ε is multiplicative in exact sequences, so it is well-defined for virtual representations.

(ii) $\varepsilon(\rho, \psi, b\mu) = b^{\dim V}\varepsilon(\rho, \psi, \mu)$.

(iii) If E/F is finite separable, and ρ is a virtual representation of W_F of degree 0 and $\sigma = \operatorname{Ind}_{W_E}^{W_F} \rho$, then

$$\varepsilon(\sigma, \psi, \mu) = \varepsilon(\rho, \psi \circ \operatorname{tr}_{E/F}, \mu').$$

Note that this is independent of the choice of μ and μ', since "$\dim V = 0$".

(iv) If $\dim \rho = 1$, then $\varepsilon(\rho)$ is the usual abelian $\varepsilon(\chi)$.

3 ℓ-adic representations

Theorem (Grothendieck's monodromy theorem)**.** Fix an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$. In other words, fix a system (ζ_{ℓ^n}) such that $\zeta_{\ell^n}^\ell = \zeta_{\ell^{n-1}}$. We then view t_ℓ as a homomorphism $I_F \to \mathbb{Z}_\ell$ via this identification.

Let $\rho : W_F \to \mathrm{GL}(V)$ be an ℓ-adic representation over E. Then there exists an open subgroup $I' \subseteq I_F$ and a nilpotent $N \in \mathrm{End}_E V$ such that for all $\gamma \in I'$,

$$\rho(\gamma) = \exp(t_\ell(\gamma)N) = \sum_{j=0}^{\infty} \frac{(t_\ell(\gamma)N)^j}{j!}.$$

In particular, $\rho(I')$ unipotent and abelian.

Proof. If $\rho(I_F)$ is finite, let $I' = \ker \rho \cap I_F$ and $N = 0$, and we are done.

Otherwise, first observe that G is any compact group and $\rho : G \to \mathrm{GL}(V)$ is an ℓ-adic representation, then V contains a G-invariant lattice, i.e. a finitely-generated $\mathcal{O}_E$-submodule of maximal rank. To see this, pick any lattice $L_0 \subseteq V$. Then $\rho(G)L_0$ is compact, so generates a lattice which is G-invariant.

Thus, pick a basis of an I_F-invariant lattice. Then $\rho : W_F \to \mathrm{GL}_n(E)$ restricts to a map $I_F \to \mathrm{GL}_n(\mathcal{O}_E)$.

We seek to understand this group $\mathrm{GL}_n(\mathcal{O}_E)$ better. We define a filtration on $\mathrm{GL}_n(\mathcal{O}_E)$ by

$$G_k = \{g \in \mathrm{GL}_n(\mathcal{O}_E) : g \equiv I \bmod \ell^k\},$$

which is an open subgroup of $\mathrm{GL}_n(\mathcal{O}_E)$. Note that for $k \geq 1$, there is an isomorphism

$$G_k/G_{k+1} \to M_n(\mathcal{O}_E/\ell\mathcal{O}_E),$$

sending $1 + \ell^k g$ to g. Since the latter is an ℓ-group, we know G_1 is a pro-ℓ group. Also, by definition, $(G_k)^\ell \subseteq G_{k+1}$.

Since $\rho^{-1}(G_2)$ is open, we can pick an open subgroup $I' \subseteq I_F$ such that $\rho(I') \subseteq G_2$. Recall that $t_\ell(I_F)$ is the maximal pro-ℓ quotient of I_F, because the tame characters give an isomorphism

$$I_F/P_F \cong \prod_{\ell \nmid p} \mathbb{Z}_\ell(1).$$

So $\rho|_{I'} : I' \to G_2$ factors as

$$I' \overset{t_\ell}{\twoheadrightarrow} t_\ell(I') = \ell^s \mathbb{Z}_\ell \xrightarrow{\nu} G_2\ ,$$

using the assumption that $\rho(I_F)$ is infinite.

Now for $r \geq s$, let $T_r = \nu(\ell^r) = T_s^{r-s} \in G_{r+2-s}$. For r sufficiently large,

$$N_r = \log(T_r) = \sum_{m \geq 1} (-1)^{m-1} \frac{(T_r - 1)^m}{m}$$

converges ℓ-locally, and then $T_r = \exp N_r$.

We claim that N_r is nilpotent. To see this, if we enlarge E, we may assume that all the eigenvalues of N_r are in E. For $\delta \in W_F$ and $\gamma \in I_F$, we know

$$t_\ell(\delta\gamma\delta^{-1}) = \omega(\delta) t_\ell(\gamma).$$

So

$$\rho(\delta\gamma\delta^{-1}) = \rho(\gamma)^{w(\sigma)}$$

for all $\gamma \in I'$. So

$$\rho(\sigma)N_r\rho(\delta^{-1}) = \omega(\delta)N_r.$$

Choose δ lifting φ_q, $w(\delta) = q$. Then if v is an eigenvector for N_r with eigenvalue λ, then $\rho(\delta)v$ is an eigenvector of eigenvalue $q^{-1}\lambda$. Since N_r has finitely many eigenvalues, but we can do this as many times as we like, it must be the case that $\lambda = 0$.

Then take

$$N = \frac{1}{\ell^r}N_r$$

for r sufficiently large, and this works. □

Theorem. Let $E/\mathbb{Q}_\ell$ be finite (and $\ell \neq p$). Then there exists an equivalence of (symmetric monoidal) categories

$$\left\{\begin{array}{c}\ell\text{-adic representations}\\ \text{of } W_F \text{ over } E\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Weil–Deligne representations}\\ \text{of } W_F \text{ over } E\end{array}\right\}$$

Proof. We have already fixed an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$. We also pick a lift $\Phi \in W_F$ of the geometric Frobenius. In other words, we are picking a splitting

$$W_F = \langle\Phi\rangle \ltimes I_F.$$

The equivalence will take an ℓ-adic representation ρ_ℓ to the Weil–Deligne representation (ρ, N) on the same vector space such that

$$\rho_\ell(\Phi^m\gamma) = \rho(\Phi^m\gamma)\exp t_\ell(\gamma)N \tag{$*$}$$

for all $m \in \mathbb{Z}$ and $\gamma \in I_F$.

To check that this "works", we first look at the right-to-left direction. Suppose we have a Weil–Deligne representation (ρ, N) on V. We then define $\rho_\ell : W_F \to \mathrm{Aut}_E(V)$ by $(*)$. Since ρ has open kernel, it is continuous. Since t_ℓ is also continuous, we know ρ_ℓ is continuous. To see that ρ_ℓ is a homomorphism, suppose

$$\Phi^m\gamma \cdot \Phi^m\delta = \Phi^{m+n}\gamma'\delta$$

where $\gamma, \delta \in I_F$ and

$$\gamma' = \Phi^{-n}\gamma\Phi^n.$$

Then

$$\begin{aligned}\exp t_\ell(\gamma)N \cdot \rho(\Phi^n\delta) &= \sum_{j\geq 0}\frac{1}{j!}t_\ell(\gamma)^j N^j\rho(\Phi^n\delta)\\ &= \sum_{j\geq 0}\frac{1}{j!}t_\ell(\gamma)q^{nj}\rho(\Phi^n\delta)N^j\\ &= \rho(\Phi^n\delta)\exp(q^n t_\ell(\gamma)).\end{aligned}$$

But

$$t_\ell(\gamma') = t_\ell(\Phi^{-n}\gamma\Phi^n) = \omega(\Phi^{-n})t_\ell(\gamma) = q^n t_\ell(\gamma).$$

So we know that

$$\rho_\ell(\Phi^m\gamma)\rho_\ell(\Phi^n\delta) = \rho_\ell(\Phi^{m+n}\gamma'\delta).$$

Notice that if $\gamma \in I_F \cap \ker\rho$, then $\rho_\ell(\gamma) = \exp t_\ell(\gamma)N$. So N is the nilpotent endomorphism occurring in the Grothendieck theorem.

Conversely, given an ℓ-adic representation ρ_ℓ, let $N \in \operatorname{End}_E V$ be given by the monodromy theorem. We then define ρ by $(*)$. Then the same calculation shows that (ρ, N) is a Weil–Deligne representation, and if $I' \subseteq I_F$ is the open subgroup occurring in the theorem, then $\rho_\ell(\gamma) = \exp t_\ell(\gamma)N$ for all $\gamma \in I'$. So by $(*)$, we know $\rho(I') = \{1\}$, and so ρ has open kernel. □

Theorem. Let $\ell, \ell' \neq p$. Then the category of $\bar{\mathbb{Q}}_\ell$ representations of W_F is equivalent to the category of $\bar{\mathbb{Q}}_{\ell'}$ representations of W_F.

Proposition. Suppose ρ_ℓ is an ℓ-adic representation corresponding to a Weil–Deligne representation (ρ, N). Then the following are equivalent:

(i) $\rho_\ell(\Phi)$ is semi-simple (where Φ is a lift of Frob_q).

(ii) $\rho_\ell(\gamma)$ is semi-simple for all $\gamma \in W_F \setminus I_F$.

(iii) ρ is semi-simple.

(iv) $\rho(\Phi)$ is semi-simple.

In this case, we say ρ_ℓ and (ρ, N) are *F-semisimple* (where F refers to *Frobenius*).

Proof. Recall that $W_F \cong \mathbb{Z} \rtimes I_F$, and $\rho(I_F)$ is finite. So that part is always semisimple, and thus (iii) and (iv) are equivalent.

Moreover, since $\rho_\ell(\Phi) = \rho(\Phi)$, we know (i) and (iii) are equivalent. Finally, $\rho_\ell(\Phi)$ is semi-simple iff $\rho_\ell(\Phi^n)$ is semi-simple for all Φ. Then this is equivalent to (ii) since the equivalence before does not depend on the choice of Φ. □

Theorem (Jordan normal form)**.** If V is semi-simple, $N \in \operatorname{End}(V)$ is nilpotent with $N^{m+1} = 0$, then there exists subobjects $P_0, \ldots, P_m \subseteq V$ (not unique as subobjects, but unique up to isomorphism), such that $N^r : P_r \to N^r P_r$ is an isomorphism, and $N^{r+1}P_r = 0$, and

$$V = \bigoplus_{r=0}^m P_r \oplus NP_r \oplus \cdots \oplus N^r P_r = \bigoplus_{r=0}^m P_r \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[N]}{(N^{r+1})}.$$

Proof. If we had the desired decomposition, then heuristically, we want to set P_0 to be the things killed by N but not in the image of N. Thus, using semisimplicity, we pick P_0 to be a splitting

$$\ker N = (\ker N \cap \operatorname{im} N) \oplus P_0.$$

Similarly, we can pick P_1 by

$$\ker N^2 = (\ker N + (\operatorname{im} N \cap \ker N^2)) \oplus P_1.$$

One then checks that this works. □

Proposition. Let (ρ, N) be a Weil–Deligne representation.

(i) (ρ, N) is irreducible iff ρ is irreducible and $N = 0$.

(ii) (ρ, N) is indecomposable and F-semisimple iff

$$(\rho, N) = (\sigma, 0) \otimes \mathrm{sp}(n),$$

where σ is an irreducible representation of W_F and $\mathrm{sp}(n) \cong E^n$ is the representation

$$\rho = \mathrm{diag}(\omega^{n-1}, \ldots, \omega, 1), \quad N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Proof. (i) is obvious.

For (ii), we first prove ($\Leftarrow$). If $(\rho, N) = (\sigma, 0) \otimes \mathrm{sp}(n)$, then F-semisimplicity is clear, and we have to check that it is indecomposable. Observe that the kernel of N is still a representation of W_F. Writing $V^{N=0}$ for the kernel of N in V, we note that $V^{N=0} = \sigma \otimes \omega^{n-1}$, which is irreducible. Suppose that

$$(\rho, N) = U_1 \oplus U_2.$$

Then for each i, we must have $U_i^{N=0} = 0$ or $V^{N=0}$. We may wlog assume $U_1^{N=0} = 0$. Then this forces $U_1 = 0$. So we are done.

Conversely, if (ρ, N, V) is F-semisimple and indecomposable, then V is a representation of W_F which is semi-simple and $N : V \to V \otimes \omega^{-1}$. By Jordan normal form, we must have

$$V = U \oplus NU \oplus \cdots \oplus N^r U$$

with $N^{r+1} = 0$, and U is irreducible. So $V = (\sigma, 0) \otimes \mathrm{sp}(r+1)$. □

Theorem. There exists a bijection between F-semisimple Weil–Deligne representations over $\mathbb{C}$ and semi-simple representations of $\mathcal{L}_F$, compatible with tensor products, duals, dimension, etc. In this correspondence:

- The representations ρ of $\mathcal{L}_F$ that factor through W_F correspond to the Weil–Deligne representations $(\rho, 0)$.
- More generally, simple $\mathcal{L}_F$ representations $\sigma \otimes (\mathrm{Sym}^{n-1} \mathbb{C}^2)$ correspond to the Weil–Deligne representation $(\sigma \otimes \omega^{(-1+n)/2}, 0) \otimes \mathrm{sp}(n)$. □

4 The Langlands correspondence

4.1 Representations of groups

Proposition. Let $G = \mathrm{GL}_n(F)$. If (π, V) is a smooth representation with $\dim V < \infty$, then
$$\pi = \sigma \circ \det$$
for some $\sigma : F^\times \to \mathrm{GL}_{\mathbb{C}}(V)$.

Proof. If $V = \bigoplus_{i=1}^d Ce_i$, then
$$\ker \pi = \bigcap_{i=1}^{d} (\text{stabilizers of } e_i)$$
is open. It is also a normal subgroup, so
$$\ker \pi \supseteq K_m = \{g \in \mathrm{GL}_n(\mathcal{O}) : g \equiv I \bmod \varpi^m\}$$
for some m, where ϖ is a uniformizer of F. In particular, $\ker \pi$ contains $E_{ij}(x)$ for some $i \neq j$ and x, which is the matrix that is the identity except at entry (i, j), where it is x.

But since $\ker \pi$ is normal, conjugation by diagonal matrices shows that it contains all $E_{ij}(x)$ for all $x \in F$ and $i \neq j$. For any field, these matrices generate $\mathrm{SL}_n(F)$. So we are done. □

Lemma (Schur's lemma). Let (π, V) be an irreducible representation. Then every endomorphism of V commuting with π is a scalar.

In particular, there exists $\omega_\pi : Z(G) \to \mathbb{C}^\times$ such that
$$\pi(zg) = \omega_\pi(z)\pi(g)$$
for all $z \in Z(G)$ and $g \in G$. This is called the *central character*. □

Theorem (Harris–Taylor, Henniart). There is a bijection
$$\left\{\begin{array}{c}\text{irreducible, admissible}\\ \text{representations of } \mathrm{GL}_n(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{semi-simple } n\text{-dimensional}\\ \text{representations of } \mathcal{L}_F\end{array}\right\}.$$

4.2 Hecke algebras

Proposition. There is a bijection between isomorphism classes of irreducible admissible (π, V) with $V^K \neq 0$ and isomorphism classes of simple finite-dimensional $\mathcal{H}(G, K)$-modules, which sends (π, V) to V^K with the action we described.

4.3 The Langlands classification

Proposition.

(i) σ is admissible implies $\pi = \mathrm{Ind}_P^G \sigma$ is admissible.

(ii) σ is unitary implies π is unitary.

(iii) $\mathrm{Ind}_P^G(\tilde{\sigma}) = \tilde{\pi}$. □

Theorem. Let $n = mr$ with $m, r \geq 1$. Let σ be any supercuspidal representation of $\mathrm{GL}_m(F)$. Let

$$\sigma(x) = \sigma \otimes |\det_m|^x.$$

Write $\Delta = (\sigma, \sigma(1), \ldots, \sigma(r-1))$, a representation of $\mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$. Then $\mathrm{Ind}_P^G(\Delta)$ has a unique irreducible subquotient $Q(\Delta)$, which is essentially square integrable.

Moreover, $Q(\Delta)$ is square integrable iff the central character is unitary, iff $\sigma(\frac{r-1}{2})$ is square-integrable, and every essentially square integrable π is a $Q(\Delta)$ for a unique Δ.

Theorem. The tempered irreducible admissible representations of $\mathrm{GL}_n(F)$ are precisely the representations $\mathrm{Ind}_P^G \sigma$, where σ is irreducible square integrable. In particular, $\mathrm{Ind}_P^G \sigma$ are always irreducible when σ is square integrable.

Theorem. Let $n = n_1 + \cdots + n_r$ be a partition, and σ_i tempered representation of $\mathrm{GL}_{n_i}(F)$. Let $t_i \in \mathbb{R}$ with $t_1 > \cdots > t_r$. Then $\mathrm{Ind}_P^G(\sigma_1(t_1), \ldots, \sigma_r(t_r))$ has a unique irreducible quotient *Langlands quotient*, and every π is (uniquely) of this form.

4.4 Local Langlands correspondence

Theorem (Harris–Taylor, Henniart)**.** There is a bijection

$$\left\{\begin{array}{c}\text{irreducible, admissible}\\ \text{representations of } \mathrm{GL}_n(F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{semi-simple } n\text{-dimensional}\\ \text{representations of } \mathcal{L}_F\end{array}\right\}.$$

Moreover,

- For $n = 1$, this is the same as local class field theory.
- Under local class field theory, this corresponds between ω_π and $\det \sigma$.
- The supercuspidals correspond to the irreducible representations of W_F itself.
- If a supercuspidal π_0 corresponds to the representation σ_0 of W_F, then the essentially square integrable representation $\pi = Q(\pi_0(-\frac{r-1}{2}), \ldots, \pi_0(\frac{r-1}{2}))$ corresponds to $\sigma = \sigma_0 \otimes \mathrm{Sym}^{r-1} \mathbb{C}^2$.
- If π_i correspond to σ_i, where σ_i are irreducible and unitary, then the tempered representation $\mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$ corresponds to $\sigma_1 \oplus \cdots \oplus \sigma_r$.
- For general representations, if π is the Langlands quotient of

  $$\mathrm{Ind}(\pi_1(t_1), \ldots, \pi_r(t_r))$$

  with each π_i tempered, and π_i corresponds to unitary representations σ_i of $\mathcal{L}_F$, then π corresponds to $\bigoplus \sigma_i \otimes |\mathrm{Art}_F^{-1}|_F^{t_i}$.

Proposition. Let $\sigma : W_F \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Then the following are equivalent:

(i) For some $g \in W_F \setminus I_F$, $\sigma(g)$ has an eigenvalue of absolute value 1.

(ii) $\operatorname{im}\sigma$ is relatively compact, i.e. has compact closure, i.e. is bounded.

(iii) σ is unitary.

Proof. The only non-trivial part is (i) $\Rightarrow$ (ii). We know

$$\operatorname{im}\sigma = \langle \sigma(\Phi), \sigma(I_F) = H\rangle,$$

where Φ is some lift of the Frobenius and H is a finite group. Moreover, I_F is normal in W_F. So for some $n \geq 1$, $\sigma(\Phi^n)$ commutes with H. Thus, replacing g and Φ^n with suitable non-zero powers, we can assume $\sigma(g) = \sigma(\Phi^n)h$ for some $h \in \mathbb{H}$. Since H is finite, and $\sigma(\Phi^n)$ commutes with h, we may in fact assume $\sigma(g) = \sigma(\Phi)^n$. So we know $\sigma(\Phi)$ has eigenvalue with absolute value 1.

Let $V_1 \subseteq V = \mathbb{C}^n$ be a sum of eigenspaces for $\sigma(\Phi)^n$ with all eigenvalues having absolute value 1. Since $\sigma(\Phi^n)$ is central, we know V_1 is invariant, and hence V is irreducible. So $V_1 = V$. So all eigenvalues of $\sigma(\Phi)$ have eigenvalue 1. Since V is irreducible, we know it is F-semisimple. So $\sigma(\Phi)$ is semisimple. So $\langle\sigma(\Phi)\rangle$ is bounded. So $\operatorname{im}\sigma$ is bounded. □

5 Modular forms and representation theory

Theorem.

(i) The space V_f of adelic cusp forms generated by $f \in S_k(\Gamma_1(N))$ is irreducible iff f is a T_p eigenvector for all $p \nmid n$.

(ii) This gives a bijection between irreducible G-invariant spaces of adelic cusp forms and Atkin–Lehner newforms.

Theorem (Local Atkin–Lehner theorem)**.** If (π, V) is an irreducible representation of $\mathrm{GL}_2(F)$, where $F/\mathbb{Q}_p$ and $\dim V = \infty$, then there exists a unique $n_\pi > 0$ such that

$$V^{K_n} = \begin{cases} 0 & n < n_\pi \\ \text{one-dimensional} & n = n_\pi \end{cases}, \quad K_n = \left\{ g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \varpi^n \right\}.$$

Taking the product of these invariant vectors for $n = n_\pi$ over all p gives Atkin–Lehner newform.