Part IV — Topics in Number Theory
Theorems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The "Langlands programme" is a far-ranging series of conjectures describing the connections between automorphic forms on the one hand, and algebraic number theory and arithmetic algebraic geometry on the other. In these lectures we will give an introduction to some aspects of this programme.

Pre-requisites

The course will follow on naturally from the Michaelmas term courses *Algebraic Number Theory* and *Modular Forms and L-Functions*, and knowledge of them will be assumed. Some knowledge of algebraic geometry will be required in places.

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0 Introduction

1 Class field theory

1.1 Preliminaries

Theorem (Galois theory)**.** There are bijections

$$\left\{\begin{array}{c}\text{closed subgroups of}\\ \Gamma_K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subfields}\\ K \subseteq L \subseteq \bar{K}\end{array}\right\}$$
$$\left\{\begin{array}{c}\text{open subgroups of}\\ \Gamma_K\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite subfields}\\ K \subseteq L \subseteq \bar{K}\end{array}\right\}$$

Theorem. $\ker \hat{t} = P_F$.

1.2 Local class field theory

Theorem (Local class field theory)**.**

(i) Let F be a local field. Then there is a continuous homomorphism, the *local Artin map*

$$\mathrm{Art}_F : F^\times \to \Gamma_F^{\mathrm{ab}}$$

with dense image characterized by the properties

(a) The following diagram commutes:

$$\begin{array}{ccccc} F^\times & \xrightarrow{\mathrm{Art}_F} & \Gamma_F^{\mathrm{ab}} & \twoheadrightarrow & \Gamma_F/I_F \\ \downarrow{\scriptstyle v_F} & & & & \downarrow{\scriptstyle \sim} \\ \mathbb{Z} & & \hookrightarrow & & \hat{\mathbb{Z}} \end{array}$$

(b) If F'/F is finite, then the following diagram commutes:

$$\begin{array}{ccc} (F')^\times & \xrightarrow{\mathrm{Art}_{F'}} & \Gamma_{F'}^{\mathrm{ab}} = \mathrm{Gal}(F'^{\mathrm{ab}}/F') \\ \downarrow{\scriptstyle N_{F'/F}} & & \downarrow{\scriptstyle \text{restriction}} \\ F^\times & \xrightarrow{\mathrm{Art}_F} & \Gamma_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F) \end{array}$$

(ii) Moreover, the *existence theorem* says Art_F^{-1} induces a bijection

$$\left\{\begin{array}{c}\text{open finite index}\\ \text{subgroups of } F^\times\end{array}\right\} \longleftrightarrow \left\{\text{open subgroups of } \Gamma_F^{\mathrm{ab}}\right\}$$

Of course, open subgroups of Γ_F^{ab} further corresponds to finite abelian extensions of F.

(iii) Further, Art_F induces an isomorphism

$$\mathcal{O}_F^\times \xrightarrow{\sim} \mathrm{im}(I_F \to \Gamma_F^{\mathrm{ab}})$$

and this maps $(1 + \pi\mathcal{O}_F)^\times$ to the image of P_F. Of course, the quotient $\mathcal{O}_F^\times/(1 + \pi\mathcal{O}_F)^\times \cong k^\times = \mu_\infty(k)$.

(iv) Finally, this is functorial, namely if we have an isomorphism $\alpha : F \xrightarrow{\sim} F'$ and extend it to $\bar{\alpha} : \bar{F} \xrightarrow{\sim} \bar{F}'$, then this induces isomorphisms between the Galois groups $\alpha_* : \Gamma_F \xrightarrow{\sim} \Gamma_{F'}$ (up to conjugacy), and $\alpha_*^{\mathrm{ab}} \circ \mathrm{Art}_F = \mathrm{Art}_{F'} \circ \alpha_*^{\mathrm{ab}}$. □

Proposition. Art_F induces an *isomorphism* of topological groups

$$\mathrm{Art}_F^W : F^\times \to W_F^{\mathrm{ab}}.$$

This maps $\mathcal{O}_F^\times$ isomorphically onto the inertia subgroup of Γ_F^{ab}.

Lemma. There is a finite K/F' such that $K \cap E = F'$, so $\mathrm{Gal}(KE/K) \cong \mathrm{Gal}(E/F') = \langle g \rangle$. Moreover, KE/K is unramified.

1.3 Global class field theory

Proposition (Product formula)**.** If $x \in K^\times$, then

$$\prod_{v \in \Sigma_K} |x|_v = 1.$$

Theorem. The map $|\cdot|_{\mathbb{A}} : C_K \to \mathbb{R}_{>0}^\times$ has compact kernel.

Theorem (Artin reciprocity law)**.** $\mathrm{Art}_K(K^\times) = \{1\}$, so induces a map $C_K \to \Gamma_K^{\mathrm{ab}}$. Moreover,

(i) If $\mathrm{char}(K) = p > 0$, then Art_K is injective, and induces an isomorphism $\mathrm{Art}_K : C_k \xrightarrow{\sim} W_K^{\mathrm{ab}}$, where W_K is defined as follows: since K is a finite extension of $\mathbb{F}_q(T)$, and wlog assume $\bar{\mathbb{F}}_q \cap K = \mathbb{F}_q \equiv k$. Then W_K is defined as the pullback

$$\begin{array}{ccc} W_K & \hookrightarrow & \Gamma_K = \mathrm{Gal}(\bar{K}/K) \\ \downarrow & & \downarrow{\scriptstyle \text{restr.}} \\ \mathbb{Z} & \hookrightarrow & \hat{\mathbb{Z}} \cong \mathrm{Gal}(\bar{k}/k) \end{array}$$

(ii) If $\mathrm{char}(K) = 0$, we have an isomorphism

$$\mathrm{Art}_K : \pi_0(C_K) = \frac{C_K}{C_K^0} \xrightarrow{\sim} \Gamma_K^{\mathrm{ab}}.$$

Moreover, if L/K is finite, then we have a commutative diagram

$$\begin{array}{ccc} C_L & \xrightarrow{\mathrm{Art}_L} & \Gamma_L^{\mathrm{ab}} \\ \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \text{restr.}} \\ C_K & \xrightarrow{\mathrm{Art}_K} & \Gamma_K^{\mathrm{ab}} \end{array}$$

If this is in fact Galois, then this induces an isomorphism

$$\mathrm{Art}_{L/K} : \frac{J_K}{K^\times N_{L/K}(J_L)} \xrightarrow{\sim} \mathrm{Gal}(L/K)^{\mathrm{ab}}.$$

Finally, this is functorial, namely if $\sigma : K \xrightarrow{\sim} K'$ is an isomorphism, then we have a commutative square

$$\begin{array}{ccc} C_K & \xrightarrow{\mathrm{Art}_K} & \Gamma_K^{\mathrm{ab}} \\ \downarrow{\scriptstyle\sigma} & & \downarrow \\ C_{K'} & \xrightarrow{\mathrm{Art}_{K'}} & \Gamma_{K'}^{\mathrm{ab}} \end{array}$$

Proposition. If L/K is an abelian extension of global fields, which corresponds to the open subgroup $U \subseteq J_K$ under the Artin map, then L/K is unramified at a finite $v \nmid \infty$ iff $\mathcal{O}_v^\times \subseteq U$.

Proposition. If v is finite and unramified, then v splits completely iff $K_v^\times \subseteq U$.

Proposition.

$$C_K/C_K^0 = \frac{\{\pm 1\}^{r_1} \times \hat{K}^\times}{\overline{K^\times}}.$$

1.4 Ideal-theoretic description of global class field theory

Proposition. There is a canonical isomorphism

$$\frac{J_K}{K^\times U_{\mathfrak{m}}} \xrightarrow{\sim} \mathrm{Cl}_{\mathfrak{m}}(K)$$

such that for $v \notin S \cup \Sigma_{K,\infty}$, the composition

$$K_v^\times \hookrightarrow J_K \to \mathrm{Cl}_{\mathfrak{m}}(K)$$

sends $x \mapsto \mathfrak{p}_v^{-v(x)}$.

Thus, in particular, the Galois group $\mathrm{Gal}(L/K)$ of the ray class field modulo $\mathfrak{m}$ is $\mathrm{Cl}_{\mathfrak{m}}(K)$. Concretely, if $\mathfrak{p} \notin S$ is an ideal, then $[\mathfrak{p}] \in \mathrm{Cl}_{\mathfrak{m}}(K)$ corresponds to $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(L/K)$, the arithmetic Frobenius. This was Artin's original reciprocity law.

2 L-functions

2.1 Hecke characters

Proposition. Let G be a profinite group, and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ continuous. Then $\ker \rho$ is open.

Proposition. χ is unramified iff $\chi(x) = |x|_F^s$ for some $s \in \mathbb{C}$. □

Proposition. The set of continuous homomorphisms $\chi : J_K = \prod'_v K_v^\times \to \mathbb{C}^\times$ bijects with the set of all families $(\chi_v)_{v \in \Sigma_k}$, $\chi_v : K_v^\times \to \mathbb{C}^\times$ such that χ_v is unramified for almost all (i.e. all but finitely many) v, with the bijection given by $\chi \mapsto (\chi_v)$, $\chi_v = \chi|_{K_v^\times}$.

Lemma. Let χ be a Hecke character. Then the following are equivalent:

(i) χ has finite image.

(ii) $\chi_\infty(K_\infty^{\times,0}) = 1$.

(iii) $\chi_\infty^2 = 1$.

(iv) $\chi(C_K^0) = 1$.

(v) χ factors through $\mathrm{Cl}_{\mathfrak{m}}(K)$ for some modulus $\mathfrak{m}$.

In this case, we say χ is a *ray class character*.

Proposition. If χ is an algebraic Hecke character, then χ^∞ takes values in some number field. We write $E = E(\chi)$ for the smallest such field.

Lemma. Let K be a number field, $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ be an algebraic homomorphism, and suppose $E/\mathbb{Q}$ is Galois. Then φ factors as

$$K^\times \overset{\text{norm}}{\longrightarrow} (K \cap E)^\times \overset{\phi'}{\longrightarrow} E^\times.$$

Note that since E is Galois, the intersection $K \cap E$ makes perfect sense.

Proposition. Let $\varphi : K^\times \to \mathbb{C}^\times$ be an algebraic homomorphism. Then φ is the infinity type of an algebraic Hecke character χ iff $\varphi(\mathcal{O}_K^\times)$ is finite.

Theorem. Suppose K is arbitrary, and $\varphi : K^\times \to E^\times \subseteq \mathbb{C}^\times$ is algebraic, and we assume $E/\mathbb{Q}$ is Galois, containing the normal closure of K. Thus, we can write

$$\varphi(x) = \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{n(\sigma)}.$$

Then the following are equivalent:

(i) φ is of Serre type.

(ii) $\varphi = \psi \circ N_{K/F}$, where F is the maximal CM subfield and ψ is of Serre type.

(iii) For all $c' \in \mathrm{Gal}(E/\mathbb{Q})$ conjugate to complex conjugation c, the map $\sigma \mapsto n(\sigma) + n(c'\sigma)$ is constant.

(iv) (in the case $K \subseteq \mathbb{C}$ and $K/\mathbb{Q}$ is Galois with Galois group G) Let $\lambda = \sum n(\sigma)\sigma \in \mathbb{Z}[G]$. Then for all $\tau \in G$, we have

$$(\tau - 1)(c + 1)\lambda = 0 = (c + 1)(\tau - 1)\lambda.$$

2.2 Abelian L-functions

Theorem (Hecke–Tate).

(i) $\Lambda(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$, entire unless $\chi = |\cdot|_{\mathbb{A}}^t$ for some $t \in \mathbb{C}$, in which case there are simple poles at $s = 1 - t, -t$.

(ii) There is some function, the *global ε-factor*,
$$\varepsilon(\chi, s) = AB^s$$
for some $A \in \mathbb{C}^\times$ and $B \in \mathbb{R}_{>0}$ such that
$$\Lambda(\chi, s) = \varepsilon(\chi, s)\Lambda(\chi^{-1}, 1 - s).$$

(iii) There is a factorization
$$\varepsilon(\chi, s) = \prod_v \varepsilon_v(\chi_v, \mu_v, \psi_v, s),$$
where $\varepsilon_v = 1$ for almost all v, and ε_v depends only on χ_v and certain auxiliary data ψ_v, μ_v. These are the *local ε-factors*.

Proposition. Let $x \in K^\times$. Pick some invariant measure $\mathrm{d}u$ on $\Delta \otimes \mathbb{R}$. Then
$$\int_{\Delta\otimes\mathbb{R}} \frac{1}{\|ux\|^{2s}}\,\mathrm{d}u = \frac{\text{stuff}}{|N_{K/\mathbb{Q}}(x)|^{2s/n}},$$
where the stuff is some ratio of Γ factors and powers of π (and depends on s).

2.3 Non-abelian L-functions

Proposition.

(i) If
$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V') \to 0$$
is exact, then
$$L(\rho, s) = L(\rho', s) \cdot L(\rho'', s).$$

(ii) If E/F is finite separable, $\rho : W_E \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_{W_E}^{W_F} \rho : W_F \to \mathrm{GL}_{\mathbb{C}}(U)$, then
$$L(\rho, s) = L(\sigma, s).$$

Lemma. Let $G = \langle g\rangle \supseteq H = \langle h = g^d\rangle$, $\rho : H \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_H^G \rho$. Then
$$\det(1 - t^d\rho(h)) = \det(1 - t\sigma(g)).$$

Proposition.

(i) $L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s)$.

(ii) If L/K is finite separable and $\rho : \Gamma_L \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\sigma = \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(\rho)$, then
$$L(\rho, s) = L(\sigma, s).$$

The same are true for $\Lambda(\rho, s)$.

Theorem (Brauer induction theorem)**.** Suppose $\rho : G \to \mathrm{GL}_N(\mathbb{C})$ is a representation of a finite group. Then there exists subgroups $H_j \subseteq G$ and homomorphisms $\chi_j : H_j \to \mathbb{C}^\times$ and integers $m_i \in \mathbb{Z}$ such that

$$\operatorname{tr} \rho = \sum_j m_j \operatorname{tr} \operatorname{Ind}_{H_j}^G \chi_j.$$

Note that the m_j need not be non-negative. So we cannot quite state this as a statement about representations.

Corollary. Let $\rho : \Gamma_K \to \mathrm{GL}_N(\mathbb{C})$. Then there exists finite separable L_j/K and $\chi_j : \Gamma_{L_j} \to \mathbb{C}^\times$ of finite order and $m_j \in \mathbb{Z}$ such that

$$L(\rho, s) = \prod_j L(\chi_j, s)^{m_j}.$$

In particular, $L(\rho, s)$ has meromorphic continuation to $\mathbb{C}$ and has a functional equation

$$\Lambda(\rho, s) = L \cdot L_\infty = \varepsilon(\rho, s) L(\tilde{\rho}, 1 - s)$$

where

$$\varepsilon(\rho, s) = AB^s = \prod \varepsilon(\chi_j, s)^{m_j},$$

and $\tilde{\rho}(g) =^t \rho(g^{-1})$.

Theorem (Langlands–Deligne)**.** There exists a unique system of local constants $\varepsilon(\rho, \psi, \mu)$ for $\rho : W_F \to \mathrm{GL}_{\mathbb{C}}(V)$ such that

(i) ε is multiplicative in exact sequences, so it is well-defined for virtual representations.

(ii) $\varepsilon(\rho, \psi, b\mu) = b^{\dim V} \varepsilon(\rho, \psi, \mu)$.

(iii) If E/F is finite separable, and ρ is a virtual representation of W_F of degree 0 and $\sigma = \operatorname{Ind}_{W_E}^{W_F} \rho$, then

$$\varepsilon(\sigma, \psi, \mu) = \varepsilon(\rho, \psi \circ \operatorname{tr}_{E/F}, \mu').$$

Note that this is independent of the choice of μ and μ', since "$\dim V = 0$".

(iv) If $\dim \rho = 1$, then $\varepsilon(\rho)$ is the usual abelian $\varepsilon(\chi)$.

3 ℓ-adic representations

Theorem (Grothendieck's monodromy theorem)**.** Fix an isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$. In other words, fix a system (ζ_{ℓ^n}) such that $\zeta_{\ell^n}^\ell = \zeta_{\ell^{n-1}}$. We then view t_ℓ as a homomorphism $I_F \to \mathbb{Z}_\ell$ via this identification.

Let $\rho : W_F \to \mathrm{GL}(V)$ be an ℓ-adic representation over E. Then there exists an open subgroup $I' \subseteq I_F$ and a nilpotent $N \in \mathrm{End}_E V$ such that for all $\gamma \in I'$,

$$\rho(\gamma) = \exp(t_\ell(\gamma)N) = \sum_{j=0}^{\infty} \frac{(t_\ell(\gamma)N)^j}{j!}.$$

In particular, $\rho(I')$ unipotent and abelian.

Theorem. Let $E/\mathbb{Q}_\ell$ be finite (and $\ell \neq p$). Then there exists an equivalence of (symmetric monoidal) categories

$$\left\{\begin{array}{c}\ell\text{-adic representations}\\ \text{of } W_F \text{ over } E\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Weil–Deligne representations}\\ \text{of } W_F \text{ over } E\end{array}\right\}$$

Theorem. Let $\ell, \ell' \neq p$. Then the category of $\bar{\mathbb{Q}}_\ell$ representations of W_F is equivalent to the category of $\bar{\mathbb{Q}}_{\ell'}$ representations of W_F.

Proposition. Suppose ρ_ℓ is an ℓ-adic representation corresponding to a Weil–Deligne representation (ρ, N). Then the following are equivalent:

(i) $\rho_\ell(\Phi)$ is semi-simple (where Φ is a lift of Frob_q).

(ii) $\rho_\ell(\gamma)$ is semi-simple for all $\gamma \in W_F \setminus I_F$.

(iii) ρ is semi-simple.

(iv) $\rho(\Phi)$ is semi-simple.

In this case, we say ρ_ℓ and (ρ, N) are *F-semisimple* (where F refers to *Frobenius*).

Theorem (Jordan normal form)**.** If V is semi-simple, $N \in \mathrm{End}(V)$ is nilpotent with $N^{m+1} = 0$, then there exists subobjects $P_0, \ldots, P_m \subseteq V$ (not unique as subobjects, but unique up to isomorphism), such that $N^r : P_r \to N^r P_r$ is an isomorphism, and $N^{r+1} P_r = 0$, and

$$V = \bigoplus_{r=0}^{m} P_r \oplus NP_r \oplus \cdots \oplus N^r P_r = \bigoplus_{r=0}^{m} P_r \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[N]}{(N^{r+1})}.$$

Proposition. Let (ρ, N) be a Weil–Deligne representation.

(i) (ρ, N) is irreducible iff ρ is irreducible and $N = 0$.

(ii) (ρ, N) is indecomposable and F-semisimple iff

$$(\rho, N) = (\sigma, 0) \otimes \mathrm{sp}(n),$$

where σ is an irreducible representation of W_F and $\mathrm{sp}(n) \cong E^n$ is the representation

$$\rho = \mathrm{diag}(\omega^{n-1}, \ldots, \omega, 1), \quad N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Theorem. There exists a bijection between F-semisimple Weil–Deligne representations over $\mathbb{C}$ and semi-simple representations of $\mathcal{L}_F$, compatible with tensor products, duals, dimension, etc. In this correspondence:

- The representations ρ of $\mathcal{L}_F$ that factor through W_F correspond to the Weil–Deligne representations $(\rho, 0)$.
- More generally, simple $\mathcal{L}_F$ representations $\sigma \otimes (\operatorname{Sym}^{n-1} \mathbb{C}^2)$ correspond to the Weil–Deligne representation $(\sigma \otimes \omega^{(-1+n)/2}, 0) \otimes \operatorname{sp}(n)$. □

4 The Langlands correspondence

4.1 Representations of groups

Proposition. Let $G = \mathrm{GL}_n(F)$. If (π, V) is a smooth representation with $\dim V < \infty$, then

$$\pi = \sigma \circ \det$$

for some $\sigma : F^\times \to \mathrm{GL}_\mathbb{C}(V)$.

Lemma (Schur's lemma)**.** Let (π, V) be an irreducible representation. Then every endomorphism of V commuting with π is a scalar.

In particular, there exists $\omega_\pi : Z(G) \to \mathbb{C}^\times$ such that

$$\pi(zg) = \omega_\pi(z)\pi(g)$$

for all $z \in Z(G)$ and $g \in G$. This is called the *central character*. □

Theorem (Harris–Taylor, Henniart)**.** There is a bijection

$$\left\{ \begin{array}{c} \text{irreducible, admissible} \\ \text{representations of } \mathrm{GL}_n(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semi-simple } n\text{-dimensional} \\ \text{representations of } \mathcal{L}_F \end{array} \right\}.$$

4.2 Hecke algebras

Proposition. There is a bijection between isomorphism classes of irreducible admissible (π, V) with $V^K \neq 0$ and isomorphism classes of simple finite-dimensional $\mathcal{H}(G, K)$-modules, which sends (π, V) to V^K with the action we described.

4.3 The Langlands classification

Proposition.

(i) σ is admissible implies $\pi = \mathrm{Ind}_P^G \sigma$ is admissible.

(ii) σ is unitary implies π is unitary.

(iii) $\mathrm{Ind}_P^G(\tilde{\sigma}) = \tilde{\pi}$. □

Theorem. Let $n = mr$ with $m, r \geq 1$. Let σ be any supercuspidal representation of $\mathrm{GL}_m(F)$. Let

$$\sigma(x) = \sigma \otimes |\mathrm{det}_m|^x.$$

Write $\Delta = (\sigma, \sigma(1), \ldots, \sigma(r-1))$, a representation of $\mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$. Then $\mathrm{Ind}_P^G(\Delta)$ has a unique irreducible subquotient $Q(\Delta)$, which is essentially square integrable.

Moreover, $Q(\Delta)$ is square integrable iff the central character is unitary, iff $\sigma(\frac{r-1}{2})$ is square-integrable, and every essentially square integrable π is a $Q(\Delta)$ for a unique Δ.

Theorem. The tempered irreducible admissible representations of $\mathrm{GL}_n(F)$ are precisely the representations $\mathrm{Ind}_P^G \sigma$, where σ is irreducible square integrable. In particular, $\mathrm{Ind}_P^G \sigma$ are always irreducible when σ is square integrable.

Theorem. Let $n = n_1 + \cdots + n_r$ be a partition, and σ_i tempered representation of $\mathrm{GL}_{n_i}(F)$. Let $t_i \in \mathbb{R}$ with $t_1 > \cdots > t_r$. Then $\mathrm{Ind}_P^G(\sigma_1(t_1), \ldots, \sigma_r(t_r))$ has a unique irreducible quotient *Langlands quotient*, and every π is (uniquely) of this form.

4.4 Local Langlands correspondence

Theorem (Harris–Taylor, Henniart)**.** There is a bijection

$$\left\{ \begin{array}{c} \text{irreducible, admissible} \\ \text{representations of } \mathrm{GL}_n(F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semi-simple } n\text{-dimensional} \\ \text{representations of } \mathcal{L}_F \end{array} \right\}.$$

Moreover,

- For $n = 1$, this is the same as local class field theory.
- Under local class field theory, this corresponds between ω_π and $\det \sigma$.
- The supercuspidals correspond to the irreducible representations of W_F itself.
- If a supercuspidal π_0 corresponds to the representation σ_0 of W_F, then the essentially square integrable representation $\pi = Q(\pi_0(-\frac{r-1}{2}), \ldots, \pi_0(\frac{r-1}{2}))$ corresponds to $\sigma = \sigma_0 \otimes \mathrm{Sym}^{r-1} \mathbb{C}^2$.
- If π_i correspond to σ_i, where σ_i are irreducible and unitary, then the tempered representation $\mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_r)$ corresponds to $\sigma_1 \oplus \cdots \oplus \sigma_r$.
- For general representations, if π is the Langlands quotient of

  $$\mathrm{Ind}(\pi_1(t_1), \ldots, \pi_r(t_r))$$

  with each π_i tempered, and π_i corresponds to unitary representations σ_i of $\mathcal{L}_F$, then π corresponds to $\bigoplus \sigma_i \otimes |\mathrm{Art}_F^{-1}|_F^{t_i}$.

Proposition. Let $\sigma : W_F \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Then the following are equivalent:

(i) For some $g \in W_F \setminus I_F$, $\sigma(g)$ has an eigenvalue of absolute value 1.

(ii) $\mathrm{im}\, \sigma$ is relatively compact, i.e. has compact closure, i.e. is bounded.

(iii) σ is unitary.

5 Modular forms and representation theory

Theorem.

(i) The space V_f of adelic cusp forms generated by $f \in S_k(\Gamma_1(N))$ is irreducible iff f is a T_p eigenvector for all $p \nmid n$.

(ii) This gives a bijection between irreducible G-invariant spaces of adelic cusp forms and Atkin–Lehner newforms.

Theorem (Local Atkin–Lehner theorem)**.** If (π, V) is an irreducible representation of $\mathrm{GL}_2(F)$, where $F/\mathbb{Q}_p$ and $\dim V = \infty$, then there exists a unique $n_\pi > 0$ such that

$$V^{K_n} = \begin{cases} 0 & n < n_\pi \\ \text{one-dimensional} & n = n_\pi \end{cases}, \quad K_n = \left\{ g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \varpi^n \right\}.$$

Taking the product of these invariant vectors for $n = n_\pi$ over all p gives Atkin–Lehner newform.