Part IV — Bounded Cohomology

Theorems with proof

Based on lectures by M. Burger

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The cohomology of a group or a topological space in degree k is a real vector space which describes the "holes" bounded by k dimensional cycles and encodes their relations. Bounded cohomology is a refinement which provides these vector spaces with a (semi) norm and hence topological objects acquire mysterious numerical invariants. This theory, introduced in the beginning of the 80's by M. Gromov, has deep connections with the geometry of hyperbolic groups and negatively curved manifolds. For instance, hyperbolic groups can be completely characterized by the "size" of their bounded cohomology.

The aim of this course is to give an introduction to the bounded cohomology of groups, and treat more in detail one of its important applications to the study of groups acting by homeomorphisms on the circle. More precisely we will treat the following topics:

(i) Ordinary and bounded cohomology of groups: meaning of these objects in low degrees, that is, zero, one and two; relations with quasimorphisms. Proof that the bounded cohomology in degree two of a non abelian free group contains an isometric copy of the Banach space of bounded sequences of reals. Examples and meaning of bounded cohomology classes of geometric origin with non trivial coefficients.

(ii) Actions on the circle, the bounded Euler class: for a group acting by orientation preserving homeomorphisms of the circle, Ghys has introduced an invariant, the bounded Euler class of the action, and shown that it characterizes (minimal) actions up to conjugation. We will treat in some detail this work as it leads to important applications of bounded cohomology to the question of which groups can act non trivially on the circle: for instance $\mathrm{SL}(2, \mathbb{Z})$ can, while lattices in "higher rank Lie groups", like $\mathrm{SL}(n, \mathbb{Z})$ for n at least 3, can't.

(iii) Amenability and resolutions: we will set up the abstract machinery of resolutions and the notions of injective modules in ordinary as well as bounded cohomology; this will provide a powerful way to compute these objects in important cases. A fundamental role in this theory is played by various notions of amenability; the classical notion of amenability for a group, and amenability of a group action on a measure space, due to R. Zimmer. The goal is then to describe applications of this machinery to various rigidity questions, and in particular to the theorem due, independently to Ghys, and Burger–Monod, that lattices in higher rank groups don't act on the circle.

Pre-requisites

Prerequisites for this course are minimal: no prior knowledge of group cohomology of any form is needed; we'll develop everything we need from scratch. It is however an advantage to have a "zoo" of examples of infinite groups at one's disposal: for example free groups and surface groups. In the third part, we'll need basic measure theory; amenability and ergodic actions will play a role, but there again everything will be built up on elementary measure theory.

The basic reference for this course is R. Frigerio, "Bounded cohomology of discrete groups", arXiv:1611.08339, and for part 3, M. Burger & A. Iozzi, "A useful formula from bounded cohomology", available at: `https://people.math.ethz.ch/~iozzi/publications.html`.

Contents

1 Quasi-homomorphisms

1.1 Quasi-homomorphisms

Lemma. Let $f \in \mathcal{QH}(G, A)$. Then for every $g \in G$, the limit

$$Hf(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}$$

exists in $\mathbb{R}$. Moreover,

(i) $Hf : G \to \mathbb{R}$ is a homogeneous quasi-homomorphism.

(ii) $f - Hf \in \ell^\infty(G, \mathbb{R})$.

Proof. We iterate the quasi-homomorphism property

$$|f(ab) - f(a) - f(b)| \leq D(f).$$

Then, viewing $g^{mn} = g^m \cdots g^m$, we obtain

$$|f(g^{mn}) - nf(g^m)| \leq (n-1)D(f).$$

Similarly, we also have

$$|f(g^{mn}) - mf(g^n)| \leq (m-1)D(f).$$

Thus, dividing by nm, we find

$$\left|\frac{f(g^{mn})}{nm} - \frac{f(g^m)}{m}\right| \leq \frac{1}{m}D(f)$$
$$\left|\frac{f(g^{mn})}{nm} - \frac{f(g^n)}{n}\right| \leq \frac{1}{n}D(f).$$

So we find that

$$\left|\frac{f(g^n)}{n} - \frac{f(g^m)}{m}\right| \leq \left(\frac{1}{m} + \frac{1}{n}\right) D(f). \qquad (*)$$

Hence the sequence $\frac{f(g^n)}{n}$ is Cauchy, and the limit exists.

The fact that Hf is a quasi-homomorphism follows from the second assertion. To prove the second assertion, we can just take $n = 1$ in $(*)$ and take $m \to \infty$. Then we find

$$|f(g) - Hf(g)| \leq D(f).$$

So this shows that $f - Hf$ is bounded, hence Hf is a quasi-homomorphism.

The homogeneity is left as an easy exercise. □

Corollary. We have

$$\mathcal{QH}(G, \mathbb{R}) = \mathcal{QH}_h(G, \mathbb{R}) \oplus \ell^\infty(G, \mathbb{R})$$

Proof. Indeed, observe that a bounded homogeneous quasi-homomorphism must be identically zero. □

Lemma. Let $f : G \to \mathbb{R}$ be a homogeneous quasi-homomorphism.

(i) We have $f(xyx^{-1}) = f(y)$ for all $x, y \in G$.

(ii) If G is abelian, then f is in fact a homomorphism. Thus

$$\mathcal{QH}_h(G, \mathbb{R}) = \mathrm{Hom}(G, \mathbb{R}).$$

Proof.

(i) Note that for any x, the function

$$y \mapsto f(xyx^{-1})$$

is a homogeneous quasi-homomorphism. It suffices to show that the function

$$y \mapsto f(xyx^{-1}) - f(y)$$

is a bounded homogeneous quasi-homomorphism, since all such functions must be zero. Homogeneity is clear, and the quasi-homomorphism property follows from the computation

$$|f(xyx^{-1}) - f(y)| \leq |f(x) + f(y) + f(x^{-1}) - f(y)| + 2D(f) = 2D(f),$$

using the fact that $f(x^{-1}) = -f(x)$ by homogeneity.

(ii) If x and y commute, then $(xy)^n = x^n y^n$. So we can use homogeneity to write

$$\begin{aligned} |f(xy) - f(x) - f(y)| &= \frac{1}{n}|f((xy)^n) - f(x^n) - f(y^n)| \\ &= \frac{1}{n}|f(x^n y^n) - f(x^n) - f(y^n)| \\ &\leq \frac{1}{n} D(f). \end{aligned}$$

Since n is arbitrary, the difference must vanish. □

Theorem (P. Rolli, 2009)**.** The function $f_{\alpha,\beta}$ is a quasi-homomorphism, and the map

$$\ell^\infty_{\mathrm{odd}}(\mathbb{Z}) \oplus \ell^\infty_{\mathrm{odd}}(\mathbb{Z}) \to \frac{\mathcal{QH}(F_2, \mathbb{R})}{\ell^\infty(F_2, \mathbb{R}) + \mathrm{Hom}(F_2, \mathbb{R})}$$

is injective.

Proof. Let $\alpha, \beta \in \ell^\infty_{\mathrm{odd}}(\mathbb{Z}, \mathbb{R})$, and define $f_{\alpha,\beta}$ as before. By staring at it long enough, we find that

$$|f(xy) - f(x) - f(y)| \leq 3 \max(\|\alpha\|_\infty, \|\beta\|_\infty),$$

and so it is a quasi-homomorphism. The main idea is that

$$f(b^n) + f(b^{-n}) = f(a^n) + f(a^{-n}) = 0$$

by oddness of α and β. So when we do the word reduction in the product, the amount of error we can introduce is at most $3 \max(\|\alpha\|_\infty, \|\beta\|_\infty)$.

To show that the map is injective, suppose

$$f_{\alpha,\beta} = \varphi + h,$$

where $\varphi\colon F_2 \to \mathbb{R}$ is bounded and $h\colon F_2 \to \mathbb{R}$ is a homomorphism. Then we must have

$$h(a^\ell) = f(a^\ell) - \varphi(a^\ell) = \alpha(\ell) - \psi(a^\ell),$$

which is bounded. So the map $\ell \mapsto h(a^\ell) = \ell h(a)$ is bounded, and so $h(a) = 0$. Similarly, $h(b) = 0$. So $h \equiv 0$. In other words, $f_{\alpha,\beta}$ is bounded.

Finally,

$$f((a^{\ell_1} b^{\ell_2})^k) = k(\alpha(\ell_1) + \beta(\ell_2)) = 0.$$

Since this is bounded, we must have $\alpha(\ell_1) + \beta(\ell_2) = 0$ for all $\ell_1, \ell_2 \neq 0$. Using the fact that α and β are odd, this easily implies that $\alpha(\ell_1) = \beta(\ell_2) = 0$ for all ℓ_1 and ℓ_2.

□

Theorem (Hull–Osin 2013)**.** The space

$$\frac{\mathcal{QH}(G, \mathbb{R})}{\ell^\infty(G, \mathbb{R}) + \mathrm{Hom}(G, \mathbb{R})}$$

is infinite-dimensional if G is acylindrically hyperbolic.

1.2 Relation to commutators

Lemma. If f is a homogeneous quasi-homomorphism and $x, y \in G$, then

$$|f([x, y])| \leq D(f).$$

Proof. By definition of $D(f)$, we have

$$|f([x, y]) - f(xyx^{-1}) - f(y^{-1})| \leq D(f).$$

But since f is homogeneous, we have $f(xyx^{-1}) = f(y) = -f(y^{-1})$. So we are done. □

Lemma (Bavard, 1992)**.** If f is a homogeneous quasi-homomorphism, then

$$\sup_{x,y} |f([x, y])| = D(f).$$

Lemma. For $a \in [G, G]$, we have

$$|f(a)| \leq 2D(f)\,\mathrm{cl}(a).$$

Proposition.

$$|f(a)| \leq 2D(f)\mathrm{scl}(a).$$

Theorem (Bavard, 1992)**.** For all $a \in [G, G]$, we have

$$\mathrm{scl}(a) = \frac{1}{2} \sup_{\phi \in \mathcal{QH}_h(G, \mathbb{R})} \frac{|\phi(a)|}{|D(\phi)|},$$

where, of course, we skip over those $\phi \in \mathrm{Hom}(G, \mathbb{R})$ in the supremum to avoid division by zero.

Corollary. The stable commutator length vanishes identically iff every homogeneous quasi-homomorphism is a homomorphism.

Theorem (Carder–Keller 1983)**.** For $n \geq 3$, we have

$$\mathrm{SL}(n, \mathbb{Z}) = [\mathrm{SL}(n, \mathbb{Z}), \mathrm{SL}(n, \mathbb{Z})],$$

and the commutator length is bounded.

Theorem (D. Witte Morris, 2007)**.** Let $\mathcal{O}$ be the ring of integers of some number field. Then $\mathrm{cl}\colon [\mathrm{SL}(n, \mathcal{O}), \mathrm{SL}(n, \mathcal{O})] \to \mathbb{R}$ is bounded iff $n \geq 3$ or $n = 2$ and $\mathcal{O}^\times$ is infinite.

Theorem (Burger–Monod, 2002)**.** Let $\Gamma < G$ be an irreducible lattice in a connected semisimple group G with finite center and rank $G \geq 2$. Then every homogeneous quasimorphism $\Gamma \to \mathbb{R}$ is $\equiv 0$.

Theorem (Burger–Monod, 2009)**.** Let Γ be a finitely-generated group and let μ be a symmetric probability measure on Γ whose support generates Γ. Then every class in $\mathcal{QH}(\Gamma, \mathbb{R})/\ell^\infty(\Gamma, \mathbb{R})$ has a unique μ-harmonic representative. In addition, this harmonic representative f satisfies the following:

$$\|\mathrm{d}f\|_\infty \leq \|\mathrm{d}g\|_\infty$$

for any $g \in f + \ell^\infty(\Gamma, R)$.

1.3 Poincare translation quasimorphism

Proposition. Every lift $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ of an orientation preserving homeomorphism $\varphi\colon S^1 \to S^1$ is a monotone increasing homeomorphism of $\mathbb{R}$, commuting with translation by $\mathbb{Z}$, i.e.

$$\tilde{\varphi} \circ T_m = T_m \circ \tilde{\varphi}$$

for all $m \in \mathbb{Z}$.

Conversely, any such map is a lift of an orientation-preserving homeomorphism.

Lemma. The function $F\colon \mathrm{Homeo}_{\mathbb{Z}}^+(\mathbb{R}) \to \mathbb{R}$ given by $\varphi \mapsto \varphi(0)$ is a quasi-homomorphism.

Proof. The commutation property of φ reads as follows:

$$\varphi(x + m) = \varphi(x) + m.$$

For a real number $x \in \mathbb{R}$, we write

$$x = \{x\} + [x],$$

where $0 \leq \{x\} < 1$ and $[x] = 1$. Then we have

$$\begin{aligned} F(\varphi_1\varphi_2) &= \varphi_1(\varphi_2(0)) \\ &= \varphi_1(\varphi_2(0)) \\ &= \varphi_1(\{\varphi_2(0)\} + [\varphi_2(0)]) \\ &= \varphi_1(\{\varphi_2(0)\}) + [\varphi_2(0)] \\ &= \varphi_1(\{\varphi_2(0)\}) + \varphi_2(0) - \{\varphi_2(0)\}. \end{aligned}$$

Since $0 \leq \{\varphi_2(0)\} < 1$, we know that

$$\varphi_1(0) \leq \varphi_1(\{\varphi_2(0)\}) < \varphi_1(1) = \varphi_1(0) + 1.$$

Then we have

$$\varphi_1(0) + \varphi_2(0) - \{\varphi_2(0)\} \leq F(\varphi_1\varphi_2) < \varphi_1(0) + 1 + \varphi_2(0) - \{\varphi_2(0)\}.$$

So subtracting, we find that

$$-1 \leq -\{\varphi_2(0)\} \leq F(\varphi_1\varphi_2) - F(\varphi_1) - F(\varphi_2) < 1 - \{\varphi_2(0)\} \leq 1.$$

So we find that

$$D(f) \leq 1. \qquad \square$$

2 Group cohomology and bounded cohomology

2.1 Group cohomology

Lemma.

(i) $d^{(k)}$ is a Γ-equivariant group homomorphism.

(ii) $d^{(k+1)} \circ d^{(k)} = 0$. So $\operatorname{im} d^{(k)} \subseteq \ker d^{(k+1)}$.

(iii) In fact, we have $\operatorname{im} d^{(k)} = \ker d^{(k+1)}$.

Proof.

(i) This is clear.

(ii) You just expand it out and see it is zero.

(iii) If $f \in \ker d^{(k)}$, then setting $\gamma_k = e$, we have

$$\begin{aligned} 0 = d^{(k)} f(\gamma_0, \cdots, \gamma_{k-1}, e) = (-1)^k f(\gamma_0, \cdots, \gamma_{k-1}) \\ + \sum_{j=0}^{k-1} (-1)^j f(\gamma_0, \cdots, \hat{\gamma}_j, \cdots, \gamma_{k-1}, e). \end{aligned}$$

Now define the following $(k-1)$-cochain

$$h(\gamma_0, \cdots, \gamma_{k-2}) = (-1)^k f(\gamma_0, \cdots, \gamma_{k-2}, e).$$

Then the above reads

$$f = d^{(k-1)} h.$$ □

Lemma. A homomorphism $f : \Gamma \to \Gamma'$ of groups induces a natural map $f^* : H^k(\Gamma', \mathbb{Z}) \to H^k(\Gamma, \mathbb{Z})$ for all k. Moreover, if $g : \Gamma' \to \Gamma''$ is another group homomorphism, then $f^* \circ g^* = (gf)^*$.

Proposition. $H^0(\Gamma, A) \cong A$.

Proof. The relevant part of the cochain is

$$0 \longrightarrow A \xrightarrow{d^1 = 0} C(\Gamma, A) .$$ □

Proposition. $H^1(\Gamma, A) = \operatorname{Hom}(\Gamma, A)$.

Proof. The relevant part of the complex is

$$A \xrightarrow{d^1 = 0} C(\Gamma, A) \xrightarrow{d^2} C(\Gamma^2, A) ,$$

and we have

$$(d^2 f)(\gamma_1, \gamma_2) = f(\gamma_1) - f(\gamma_1 \gamma_2) + f(\gamma_2).$$ □

Proposition. $H^2(\Gamma, A)$ parametrizes the set of isomorphism classes of central extensions of Γ by A.

Proof sketch. Consider a central extension

$$0 \longrightarrow A \xrightarrow{\ i\ } G \xrightarrow{\ p\ } \Gamma \longrightarrow 0\ .$$

Arbitrarily choose a section $s\colon \Gamma \to G$ of p, as a function of sets. Then we know there is a unique $\alpha(\gamma_1, \gamma_2)$ such that

$$s(\gamma_1\gamma_2)\alpha(\gamma_1, \gamma_2) = s(\gamma_1)s(\gamma_2).$$

We then check that α is a (normalized) 2-cocycle, i.e. $\alpha(\gamma_1, e) = \gamma(e, \gamma_2) = 0$.

One then verifies that different choices of s give cohomologous choices of α, i.e. they represent the same class in $H^2(\Gamma, A)$.

Conversely, given a 2-cocycle β, we can show that it is cohomologous to a normalized 2-cocycle α. This gives rise to a central extension $G = \Gamma \times_\alpha A$ as constructed before (and also a canonical section $s(\gamma) = (\gamma, 0)$).

One then checks this is a bijection. □

Lemma. We have $c(f_1, f_2) \in \{0, 1\}$.

Proof. We have $\overline{f_1 f_2}(0) \in [0, 1)$, while $\bar{f}_2(0) \in [0, 1)$. So we find that

$$\bar{f}_1(\bar{f}_2(0)) \in [\bar{f}_1(0), \bar{f}_1(1)) = [\bar{f}_1(0), \bar{f}_1(0) + 1) \subseteq [0, 2).$$

But we also know that $c(f_1, f_2)$ is an integer. So $c(f_1, f_2) \in \{0, 1\}$. □

Theorem (Milnor–Wood)**.** If $h\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$, then $|h^*(e)| \leq 2g - 2$.

Theorem (Gauss–Bonnet)**.** If $h\colon \Gamma_g \to \mathrm{PSL}(2, \mathbb{R}) \subseteq \mathrm{Homeo}^+(S^1)$ is the holonomy representation of a hyperbolic structure, then

$$h^*(e) = \pm(2g - 2).$$

Theorem (Matsumoko, 1986)**.** If h defines a minimal action of Γ_g on S^1 and $|h^*(e)| = 2g - 2$, then h is conjugate to a hyperbolization.

2.2 Bounded cohomology of groups

Proposition. The map $\bar{d}^2$ induces an isomorphism

$$\frac{\mathcal{QH}(\Gamma, A)}{\ell^\infty(\Gamma, A) + \mathrm{Hom}(\Gamma, A)} \cong \ker c_2.$$

Proof. We know that $\bar{d}^2$ is surjective. So it suffices to show that the kernel is $\ell^\infty(\Gamma, A) + \mathrm{Hom}(\Gamma, A)$.

Suppose $f \in \mathcal{QH}(\Gamma, A)$ is such that $\bar{d}^2 f \in H_b^2(\Gamma, A) = 0$. Then there exists some $g \in C_b(\Gamma, A)$ such that

$$d^2 f = d^2 g.$$

So it follows that $d^2(f - g) = 0$. That is, $f - g \in \mathrm{Hom}(\Gamma, A)$. Hence it follows that

$$\ker \bar{d}^2 \subseteq \ell^\infty(\Gamma, A) + \mathrm{Hom}(\Gamma, A).$$

The other inclusion is clear. □

Theorem. Assume Γ is finitely-generated. Let G_α be the central extension of Γ by $\mathbb{Z}$, defined by a class in $H^2(\Gamma, \mathbb{Z})$ which admits a bounded representative. Then with any word metric, Γ_α is quasi-isometric to $\Gamma \times \mathbb{Z}$ via the "identity map".

Proposition. Let Γ be an amenable group. Then $H_b^k(\Gamma, \mathbb{R}) = 0$ for $k \geq 1$.

Proof. Let $k \geq 1$ and $f\colon \Gamma^{k+1} \to \mathbb{R}$ a Γ-invariant bounded cocycle. In other words,

$$\begin{aligned} d^{(k+1)} f &= 0 \\ f(\gamma\gamma_0, \cdots, \gamma\gamma_k) &= f(\gamma_0, \cdots, \gamma_k). \end{aligned}$$

We have to find $\varphi\colon \Gamma^k \to \mathbb{R}$ bounded such that

$$\begin{aligned} d^{(k)}\varphi &= f \\ \varphi(\gamma\gamma_0, \cdots, \gamma\gamma_{k-1}) &= \varphi(\gamma_0, \cdots, \gamma_{k-1}). \end{aligned}$$

Recall that for $\eta \in \Gamma$, we can define

$$h_\eta(\gamma_0, \cdots, \gamma_{k-1}) = (-1)^{k+1} f(\gamma_0, \cdots, \gamma_{k+1}, \eta),$$

and then

$$d^{(k+1)} f = 0 \Longleftrightarrow f = d^{(k)}(h_\eta).$$

However, h_η need not be invariant. Instead, we have

$$h_\eta(\gamma\gamma_0, \cdots, \gamma\gamma_{k-1}) = h_{\gamma^{-1}\eta}(\gamma_0, \cdots, \gamma_{k-1}).$$

To fix this, let $m\colon \ell^\infty(\Gamma) \to \mathbb{R}$ be a left-invariant mean. We notice that the map

$$\eta \mapsto h_\eta(\gamma_0, \cdots, \gamma_{k-1})$$

is bounded by $\|f\|_\infty$. So we can define

$$\varphi(\gamma_0, \cdots, \gamma_{k-1}) = m\Big\{\eta \mapsto h_\eta(\gamma_0, \cdots, \gamma_{k-1})\Big\}.$$

Then this is the φ we want. Indeed, we have

$$\varphi(\gamma\gamma_0, \cdots, \gamma\gamma_{k-1}) = m\Big\{\eta \mapsto h_{\gamma^{-1}\eta}(\gamma_0, \cdots, \gamma_{k-1})\Big\}.$$

But this is just the mean of a left translation of the original function. So this is just $\varphi(\gamma_0, \cdots, \gamma_{k-1})$. Also, by properties of the mean, we know $\|\varphi\|_\infty \leq \|f\|_\infty$.

Finally, by linearity, we have

$$\begin{aligned} d^{(k)}\varphi(\gamma_0, \cdots, \gamma_k) &= m\Big\{\eta \mapsto d^{(k)} h_\eta(\gamma_0, \cdots, \gamma_k)\Big\} \\ &= m\Big\{f(\gamma_0, \cdots, \gamma_k) \cdot \mathbf{1}_\Gamma\Big\} \\ &= f(\gamma_0, \cdots, \gamma_k) m(\mathbf{1}_\Gamma) \\ &= f(\gamma_0, \cdots, \gamma_k). \end{aligned}$$

□

3 Actions on S^1

3.1 The bounded Euler class

Lemma. If h_1 and h_2 are minimal actions that are semiconjugate via φ_1 and φ_2, then φ_1 and φ_2 are homeomorphisms and are inverses of each other.

Proof. The condition (i) tells us that

$$h_1(\gamma)(\varphi_1(x)) = \varphi_1(h_2(\gamma)(x)).$$

for all $x \in S^1$ and $\gamma \in \Gamma$. This means $\operatorname{im} \varphi_1$ is $h_1(\Gamma)$-invariant, hence dense in S^1. Thus, we know that $\operatorname{im} \tilde{\varphi}_1$ is dense in $\mathbb{R}$. But $\tilde{\varphi}$ is increasing. So $\tilde{\varphi}_1$ must be continuous. Indeed, we can look at the two limits

$$\lim_{x \nearrow y} \tilde{\varphi}_1(x) \leq \lim_{x \searrow y} \tilde{\varphi}_1(x).$$

But since $\tilde{\varphi}_1$ is increasing, if $\tilde{\varphi}_1$ were discontinuous at $y \in \mathbb{R}$, then the inequality would be strict, and hence the image misses a non-trivial interval. So $\tilde{\varphi}_1$ is continuous.

We next claim that $\tilde{\varphi}_1$ is injective. Suppose not. Say $\varphi(x_1) = \varphi(x_2)$. Then by looking at the lift, we deduce that $\varphi((x_1, x_2)) = \{x\}$ for some x. Then by minimality, it follows that φ is locally constant, hence constant, which is absurd.

We can continue on and then decide that φ_1, φ_2 are homeomorphisms. □

Theorem (F. Ghys, 1984)**.** Two actions h_1 and h_2 are semiconjugate iff $h_1^*(e^b) = h_2^*(e^b)$.

Proof. We shall only prove one direction, that if the bounded Euler classes agree, then the actions are semi-conjugate.

Let $h_1, h_2 \colon \Gamma \to \mathrm{Homeo}^+(S^1)$. Recall that $c(f, g) \in \{0, 1\}$ refers to the (normalized) cocycle defining the bounded Euler class. Therefore

$$\begin{aligned} c_1(\gamma, \eta) &= c(h_1(\gamma), h_1(\eta)) \\ c_2(\gamma, \eta) &= c(h_2(\gamma), h_2(\eta)). \end{aligned}$$

are representative cocycles of $h_1^*(e^b), h_2^*(e^b) \in H_b^2(\Gamma, \mathbb{Z})$.

By the hypothesis, there exists $u \colon \Gamma \to \mathbb{Z}$ bounded such that

$$c_2(\gamma, \eta) = c_1(\gamma, \eta) + u(\gamma) - u(\gamma\eta) + u(\eta)$$

for all $\gamma, \eta \in \Gamma$.

Let $\bar{\Gamma} = \Gamma \times_{c_1} \mathbb{Z}$ be constructed with c_1, with group law

$$(\gamma, n)(\eta, m) = (\gamma\eta, c_1(\gamma, \eta) + n + m)$$

We have a section

$$\begin{aligned} s_1 \colon \Gamma &\to \bar{\Gamma} \\ \gamma &\mapsto (\gamma, 0). \end{aligned}$$

We also write $\delta = (e, 1) \in \bar{\Gamma}$, which generates the copy of $\mathbb{Z}$ in $\bar{\Gamma}$. Then we have

$$s_1(\gamma\eta)\delta^{c_1(\gamma,\eta)} = s_1(\gamma)s_2(\eta).$$

Likewise, we can define a section by

$$s_2(\gamma) = s_1(\gamma)\delta^{u(\gamma)}.$$

Then we have

$$\begin{aligned} s_2(\gamma\eta) &= s_1(\gamma\eta)\delta^{u(\gamma\eta)} \\ &= \delta^{-c_1(\gamma,\eta)} s_1(\gamma)s_1(\eta)\delta^{u(\gamma\eta)} \\ &= \delta^{-c_1(\gamma,\eta)}\delta^{-u(\gamma)} s_2(\gamma)\delta^{-u(\eta)} s_2(\eta)\delta^{u(\gamma\eta)} \\ &= \delta^{-c_1(\gamma,\eta)-u(\gamma)+u(\gamma\eta)-u(\eta)} s_2(\gamma)s_2(\eta) \\ &= \delta^{-c_2(\gamma,\eta)} s_2(\gamma)s_2(\eta). \end{aligned}$$

Now every element in $\bar{\Gamma}$ can be uniuely written as a product $s_1(\gamma)\delta^n$, and the same holds for $s_2(\gamma)\delta^m$.

Recall that for $f \in \text{Homeo}^+(S^1)$, we write $\bar{f}$ for the unique lift with $\bar{f}(0) \in [0, 1)$. We define

$$\Phi_i(s_i(\gamma)\delta^n) = \overline{h_i(\gamma)} \cdot T_n.$$

We claim that this is a homomorphism! We simply compute

$$\begin{aligned} \Phi_i(s_i(\gamma)\delta^n s_i(\eta)\delta^m) &= \Phi_i(s_i(\gamma)s_i(\eta)\delta^{n+m}) \\ &= \Phi_i(s_i(\gamma\eta)\delta^{c_i(\gamma,\eta)+n+m}) \\ &= \overline{h_i(\gamma\eta)} T_{c_i(\gamma,\eta)} T_{n+m} \\ &= \overline{h}_i(\gamma)\overline{h}_i(\eta)T_{n+m} \\ &= \overline{h}_i(\gamma)T_n\overline{h}_i(\eta)T_m \\ &= \Phi_i(s_i(\gamma)\delta^n)\Phi_i(s_i(\eta)\delta^m). \end{aligned}$$

So we get group homomorphisms $\Phi_i \colon \bar{\Gamma} \to \text{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

Claim. For any $x \in \mathbb{R}$, the map

$$\begin{aligned} &\bar{\Gamma} \to \mathbb{R} \\ &g \mapsto \Phi_1(g)^{-1}\Phi_2(g)(x) \end{aligned}$$

is bounded.

Proof. We define

$$v(g, x) = \Phi_1(g)^{-1}\Phi(g)x.$$

We notice that

$$\begin{aligned} v(g\delta^m, x) &= \Phi_1(g\delta^m)^{-1}\Phi_2(g\delta^m)(x) \\ &= \Phi_1(g)^{-1}T_{-m}T_m\Phi_2(g) \\ &= v(g, x). \end{aligned}$$

Also, for all g, the map $x \mapsto v(g, x)$ is in $\text{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

Hence it is sufficient to show that

$$\gamma \mapsto v(s_2(\gamma), 0)$$

is bounded. Indeed, we just have

$$\begin{aligned} v(s_2(\gamma), 0) &= \Phi_1(s_2(\gamma)^{-1}\Phi_2(s_2(\gamma))(0) \\ &= \Phi_1(s_1(\gamma)\delta^{u(\gamma)})^{-1}\Phi_2(s_2(\gamma))(0) \\ &= \delta^{-u(\gamma)}\overline{h_1(\gamma)}^{-1}\overline{h_2(\gamma)}(0) \\ &= -u(\gamma) + \overline{h_1(\gamma)}^{-1}(\overline{h_2}(\gamma)(0)). \end{aligned}$$

But u is bounded, and also

$$\overline{h_1(\gamma)}^{-1}(\overline{h_2(\gamma)}(0)) \in (-1, 1).$$

So we are done. □

Finally, we can write down our two quasi-conjugations. We define

$$\tilde{\varphi}(x) = \sup_{g \in \bar{\Gamma}} v(g, x).$$

Then we verify that

$$\tilde{\varphi}(\Phi_2(h)x) = \Phi_1(h)(\varphi(x)).$$

Reducing everything modulo $\mathbb{Z}$, we find that

$$\varphi h_2(\gamma) = h_1(\gamma)\varphi.$$

The other direction is symmetric. □

3.2 The real bounded Euler class

Corollary. An action h is semi-conjugate to an action by rotations iff $h^*(e^b_{\mathbb{R}}) = 0$.

Theorem. Let $h\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ be an action. Then one of the following holds:

(i) There is a finite orbit, and all finite orbits have the same cardinality.

(ii) The action is minimal.

(iii) There is a closed, minimal, invariant, infinite, proper subset $K \subsetneq S^1$ such that any $x \in S^1$, the closure of the orbit $\overline{h(\Gamma)x}$ contains K.

Proof sketch. By compactness and Zorn's lemma, we can find a minimal, non-empty, closed, invariant subset $K \subseteq S^1$. Let $\partial K = K \setminus \mathring{K}$, and let K' be the set of all accumulation points of K (i.e. the set of all points x such that every neighbourhood of x contains infinitely many points of K). Clearly K' and ∂K are closed and invariant as well, and are contained in K. By minimality, they must be K or empty.

(i) If $K' = \emptyset$, then K is finite. It is an exercise to show that all orbits have the same size.

(ii) If $K' = K$, and $\partial K = \emptyset$, then $K = \mathring{K}$, and hence is open. Since S^1 is connected, $K = S^1$, and the action is minimal.

(iii) If $K' = K = \partial K$, then K is *perfect*, i.e. every point is an accumulation point, and K is totally disconnected. We definitely have $K \neq S^1$ and K is infinite. It is also minimal and invariant.

Let $x \in S^1$. We want to show that the closure of its orbit contains K. Since K is minimal, it suffices to show that $\overline{h(\Gamma)x}$ contains a point in K. If $x \in K$, then we are done. Otherwise, the complement of K is open, hence a disjoint union of open intervals.

For the sake of explicitness, we define an interval of a circle as follows — if $a, b \in S^1$ and $a \neq b$, then

$$(a, b) = \{z \in S^1 : (a, z, b) \text{ is positively oriented}\}.$$

Now let (a, b) be the connected component of $S^1 \setminus K$ containing x. Then we know $a \in K$.

We observe that $S^1 \setminus K$ has to be the union of *countably* many intervals, and moreover $h(\Gamma)a$ consists of end points of these open intervals. So $h(\Gamma)a$ is a countable set. On the other hand, since K is perfect, we know K is uncountable. The point is that this allows us to pick some element $y \in K \setminus h(\Gamma)a$.

Since $a \in K$, minimality tells us there exists a sequence $(\gamma_n)_{n\geq 1}$ such that $h(\gamma_n)a \to y$. But since $y \notin h(\Gamma)a$, we may wlog assume that all the points $\{h(\gamma_n)a : n \geq 1\}$ are distinct. Hence $\{h(\gamma_n)(a, b)\}_{n\geq 1}$ is a collection of disjoint intervals in S^1. This forces their lengths tend to 0. We are now done, because then $h(\gamma_n)x$ gets arbitrarily close to $h(\gamma_n)a$ as well. □

Corollary. Let $h: \Gamma \to S^1$ be an action. Then one of the following is true:

(i) $h^*(e^b_{\mathbb{R}}) = 0$ and h is semi-conjugate to an action by rotations.

(ii) $h^*(e^b_{\mathbb{R}}) \neq 0$, and then h is semi-conjugate to a minimal *unbounded* action, i.e. $\{h(\gamma) : \gamma \in \Gamma\}$ is not equicontinuous.

Lemma. A minimal compact subgroup $U \subsetneq \text{Homeo}^+(S^1)$ is conjugate to a subgroup of Rot.

Proof. By Kakutani fixed point theorem, we can pick an U-invariant probability measure on S^1, say μ, such that $\mu(S^1) = 2\pi$.

We parametrize the circle by $p: [0, 2\pi) \to S^1$. We define $\varphi \in \text{Homeo}^+(S^1)$ by

$$\varphi(p(t)) = p(s),$$

where $s \in [0, 2\pi)$ is unique with the property that

$$\mu(p([0, s)) = t.$$

One then verifies that φ is a homeomorphism, and $\varphi U \varphi^{-1} \subseteq \text{Rot}$. □

Proof of corollary. Suppose $h^*(e^b_{\mathbb{R}}) \neq 0$. Thus we are in case (ii) or (iii) of the previous trichotomy.

We first show how to reduce (iii) to (ii). Let $K \subsetneq S^1$ be the minimal $h(\Gamma)$-invariant closed set given by the trichotomy theorem. The idea is that this K misses a lot of open intervals, and we want to collapse those intervals.

We define the equivalence relation on S^1 by $x \sim y$ if $\{x, y\} \subseteq \bar{I}$ for some connected component I of $S^1 \setminus K$. Then $\sim$ is an equivalence relation that is $h(\Gamma)$-invariant, and the quotient map is homeomorphic to S^1 (exercise!). Write $i\colon S^1/\sim \to S^1$ for the isomorphism.

In this way, we obtain an action of $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ which is minimal, and the map

$$\varphi\colon\ S^1 \xrightarrow{\mathrm{pr}} S^1/\sim \xrightarrow{i} S^1$$

intertwines the two actions, i.e.

$$\varphi h(\gamma) = \rho(\gamma)\varphi.$$

Then one shows that φ is increasing of degree 1. Then we would need to find $\psi\colon S^1 \to S^1$ which is increasing of degree 1 with

$$\psi\rho(\gamma) = h(\gamma)\psi.$$

But φ is surjective, and picking an appropriate section of this would give the ψ desired.

So h is semi-conjugate to ρ, and $0 \neq h^*(e^b_{\mathbb{R}}) = \rho^*(e^b_{\mathbb{R}})$.

Thus we are left with ρ minimal, with $\rho^*(e^b_{\mathbb{R}}) \neq 0$. We have to show that ρ is not equicontinuous. But if it were, then $\rho(\Gamma)$ would be contained in a compact subgroup of $\mathrm{Homeo}^+(S^1)$, and hence by the previous lemma, would be conjugate to an action by rotation. □

Theorem (Ghys, Margulis)**.** If $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ is an action which is minimal and unbounded. Then the centralizer $C_{\mathrm{Homeo}^+(S^1)}(\rho(\Gamma))$ is finite cyclic, say $\langle\varphi\rangle$, and the factor action ρ_0 on $S^1/\langle\varphi\rangle \cong S^1$ is minimal and strongly proximal. We call this action the *strongly proximal quotient* of ρ.

Proof of theorem. Let ψ commute with all $\rho(\gamma)$ for all $\gamma \in \Gamma$, and assume $\psi \neq \mathrm{id}$.

Claim. ψ has no fixed points.

Proof. Otherwise, if $\psi(p) = p$, then

$$\psi(\rho(\gamma)p) = \rho(\gamma)\psi(p) = \rho(\gamma)(p).$$

Then by density of $\{\rho(\gamma)p : \gamma \in \Gamma\}$, we have $\psi = \mathrm{id}$. □

Hence we can find $\varepsilon > 0$ such that $\mathrm{length}([x, \psi(x)]) \geq \varepsilon$ for all x by compactness. Observe that

$$\rho(\gamma)[x, \psi(x)] = [\rho(\gamma)x, \rho(\gamma)\psi(x)] = [\rho(\gamma)x, \psi(\rho(\gamma)x)].$$

This is just an element of the above kind. So $\mathrm{length}(\rho(\gamma)[x, \psi(x)]) \geq \varepsilon$.

Now assume $\rho(\Gamma)$ is minimal and not equicontinuous.

Claim. Every point $x \in S^1$ has a neighbourhood that can be contracted.

Proof. Indeed, since $\rho(\Gamma)$ is not equicontinuous, there exists $\varepsilon > 0$, a sequence $(\gamma_n)_{n\geq 1}$ and intervals I_k such that $\mathrm{length}(I_k) \searrow 0$ and $\mathrm{length}(\rho(\gamma_n)I_n) \geq \varepsilon$.

Since we are on a compact space, after passing to a subsequence, we may assume that for n large enough, we can find some interval J such that $\mathrm{length}(J) \geq \frac{\varepsilon}{2}$ and $J \subseteq \rho(\gamma_n)I_n$.

But this means

$$\rho(\gamma_n)^{-1} J \subseteq I_n.$$

So J can be contracted. Since the action is minimal,

$$\bigcup_{\gamma \in \Gamma} \rho(\gamma) J = S^1.$$

So every point in S^1 is contained in some interval that can be contracted. □

We shall now write down what the homeomorphism that generates the centralizer. Fix $x \in S^1$. Then the set

$$\mathcal{C}_x = \{[x, y) \in S^1 : [x, y) \text{ can be contracted}\}$$

is totally ordered ordered by inclusion. Define

$$\varphi(x) \sup \mathcal{C}_x.$$

Then

$$[x, \varphi(x)) = \bigcup \mathcal{C}_x.$$

This gives a well-defined map φ that commutes with the action of γ. It is then an interesting exercise to verify all the desired properties.

- To show φ is homeomorphism, we show φ is increasing of degree 1, and since it commutes with a minimal action, it is a homeomorphism.
- If φ is not periodic, then there is some n such that $\varphi^n(x)$ is between x and $\varphi(x)$. But since φ commutes with the action of Γ, this implies $[x, \varphi^n(x)]$ cannot be contracted, which is a contradiction. □

Theorem (Burger, 2007)**.** Let G be a second-countable locally compact group, and $\Gamma < G$ be a lattice, and $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ a minimal unbounded action. Then the following are equivalent:

- $\rho^*(e^b_{\mathbb{R}})$ is in the image of the restriction map $H^2_{bc}(G, \mathbb{R}) \to H^2_b(\Gamma, \mathbb{R})$
- The strongly proximal quotient $\rho_{ss}\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ extends continuously to G.

Theorem (Burger–Monod, 2002)**.** The restriction map $H^2_{bc}(G) \to H^2_b(\Gamma, \mathbb{R})$ is an isomorphism in the following cases:

(i) $G = G_1 \times \cdots \times G_n$ is a cartesian product of locally compact groups and Γ has dense projections on each individual factor.

(ii) G is a connected semisimple Lie group with finite center and rank $G \geq 2$, and Γ is irreducible.

4 The relative homological approach

4.1 Injective modules

Theorem. Let $E^\bullet$ be an injective resolution of $\mathbb{R}$ Then

$$H^\bullet(E^{\bullet\Gamma}) \cong H_b^\bullet(\Gamma, \mathbb{R})$$

as topological vector spaces.

In case $E^\bullet$ admits contracting homotopies, this isomorphism is semi-norm decreasing.

Lemma.

- $\ell^\infty(\Gamma^n)$ for $n \geq 1$ are all injective Banach Γ-modules.
- $\ell^\infty_{\mathrm{alt}}(\Gamma^n)$ for $n \geq 1$ are injective Banach Γ-modules as well.

Proposition. The trivial Γ-module $\mathbb{R}$ is injective iff Γ is amenable.

Proof.

($\Rightarrow$) Suppose A is injective. Consider the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{i} & \ell^\infty(\Gamma) \\ \| & & \\ \mathbb{R} & & \end{array},$$

where $i(t)$ is the constant function t. We need to verify that i is an admissible injection. Then we see that $\sigma(f) = f(e)$ is a left inverse to i and $\|\sigma\| \leq 1$. Then there exists a morphism $\beta\colon \ell^\infty(\Gamma) \to \mathbb{R}$ filling in the diagram with $\|\beta\| \leq \|\operatorname{id}_{\mathbb{R}}\| = 1$ and in particular

$$\beta(\mathbf{1}_\Gamma) = 1$$

Since the action of Γ on $\mathbb{R}$ is trivial, this β is an invariant linear form on Γ, and we see that this is an invariant mean.

($\Leftarrow$) Assume Γ is amenable, and let $m\colon \ell^\infty(\Gamma) \to \mathbb{R}$ be an invariant mean. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow{\scriptstyle\alpha} & & \\ \mathbb{R} & & \end{array}$$

as in the definition of injectivity. Since i is an admissible, it has a left inverse $\sigma\colon B \to A$. Then we can define

$$\beta(v) = m\{\gamma \mapsto \alpha(\sigma(\gamma_* v))\}.$$

Then this is an injective map $B \to \mathbb{R}$ and one can verify this works. $\square$

4.2 Amenable actions

Theorem (Burger–Monod, 2002)**.** Let $G \times S \to S$ be a non-singular action. Then the following are equivalent:

(i) The G action is amenable.

(ii) $L^\infty(S)$ is an injective G-module.

(iii) $L^\infty(S^n)$ for all $n \geq 1$ is injective.

Corollary. If (S, μ) is an amenable G-space, then we have an isometric isomorphism $H^\bullet(L^\infty(S^n, \mu)^G, d_n) \cong H^\bullet(L^\infty_{\mathrm{alt}}(S^n, \mu)^G, d_n) \cong H_b(G, \mathbb{R})$.