Part IV — Bounded Cohomology
Definitions

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The cohomology of a group or a topological space in degree k is a real vector space which describes the "holes" bounded by k dimensional cycles and encodes their relations. Bounded cohomology is a refinement which provides these vector spaces with a (semi) norm and hence topological objects acquire mysterious numerical invariants. This theory, introduced in the beginning of the 80's by M. Gromov, has deep connections with the geometry of hyperbolic groups and negatively curved manifolds. For instance, hyperbolic groups can be completely characterized by the "size" of their bounded cohomology.

The aim of this course is to give an introduction to the bounded cohomology of groups, and treat more in detail one of its important applications to the study of groups acting by homeomorphisms on the circle. More precisely we will treat the following topics:

(i) Ordinary and bounded cohomology of groups: meaning of these objects in low degrees, that is, zero, one and two; relations with quasimorphisms. Proof that the bounded cohomology in degree two of a non abelian free group contains an isometric copy of the Banach space of bounded sequences of reals. Examples and meaning of bounded cohomology classes of geometric origin with non trivial coefficients.

(ii) Actions on the circle, the bounded Euler class: for a group acting by orientation preserving homeomorphisms of the circle, Ghys has introduced an invariant, the bounded Euler class of the action, and shown that it characterizes (minimal) actions up to conjugation. We will treat in some detail this work as it leads to important applications of bounded cohomology to the question of which groups can act non trivially on the circle: for instance $\mathrm{SL}(2, \mathbb{Z})$ can, while lattices in "higher rank Lie groups", like $\mathrm{SL}(n, \mathbb{Z})$ for n at least 3, can't.

(iii) Amenability and resolutions: we will set up the abstract machinery of resolutions and the notions of injective modules in ordinary as well as bounded cohomology; this will provide a powerful way to compute these objects in important cases. A fundamental role in this theory is played by various notions of amenability; the classical notion of amenability for a group, and amenability of a group action on a measure space, due to R. Zimmer. The goal is then to describe applications of this machinery to various rigidity questions, and in particular to the theorem due, independently to Ghys, and Burger–Monod, that lattices in higher rank groups don't act on the circle.

Pre-requisites

Prerequisites for this course are minimal: no prior knowledge of group cohomology of any form is needed; we'll develop everything we need from scratch. It is however an advantage to have a "zoo" of examples of infinite groups at one's disposal: for example free groups and surface groups. In the third part, we'll need basic measure theory; amenability and ergodic actions will play a role, but there again everything will be built up on elementary measure theory.

The basic reference for this course is R. Frigerio, "Bounded cohomology of discrete groups", arXiv:1611.08339, and for part 3, M. Burger & A. Iozzi, "A useful formula from bounded cohomology", available at: `https://people.math.ethz.ch/~iozzi/publications.html`.

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1 Quasi-homomorphisms

1.1 Quasi-homomorphisms

Definition (Quasi-homomorphism)**.** Let G be a group. A function $f\colon G \to A$ is a *quasi-homomorphism* if the function

$$\begin{aligned} \mathrm{d}f\colon G \times G &\to A \\ (x, y) &\mapsto f(x) + f(y) - f(xy) \end{aligned}$$

is *bounded*. We define the *defect* of f to be

$$D(f) = \sup_{x,y \in G} |\mathrm{d}f(x, y)|.$$

We write $\mathcal{QH}(G, A)$ for the A-module of quasi-homomorphisms.

Notation. We write

$$\ell^\infty(G, A) = \{f\colon G \to A : f \text{ is bounded}\}.$$

Definition (Homogeneous function)**.** A function $f\colon G \to \mathbb{R}$ is *homogeneous* if for all $n \in \mathbb{Z}$ and $g \in G$, we have $f(g^n) = nf(g)$.

Notation. We write $\mathcal{QH}_h(G, \mathbb{R})$ for the vector space of homogeneous quasi-homomorphisms $G \to \mathbb{R}$.

1.2 Relation to commutators

Definition (Commutator length)**.** Let $a \in [G, G]$. Then *commutator length* $\mathrm{cl}(a)$ is the word length with respect to the generators

$$\{[x, y] : x, y \in G\}.$$

In other words, it is the smallest n such that

$$a = [x_1, y_1][x_2, y_2] \cdots [x_n, y_n]$$

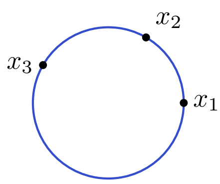
for some $x_i, y_i \in G$.

Definition (Stable commutator length)**.** The *stable commutator length* is defined by

$$\mathrm{scl}(a) = \lim_{n \to \infty} \frac{\mathrm{cl}(a^n)}{n}.$$

1.3 Poincare translation quasimorphism

Definition (Positively-oriented triple)**.** We say a triple of points $x_1, x_2, x_3 \in S^1$ is positively-oriented if they are distinct and ordered as follows:



More formally, recall that there is a natural covering map $\pi\colon \mathbb{R} \to S^1$ given by quotienting by $\mathbb{Z}$. Formally, we let $\tilde{x}_1 \in \mathbb{R}$ be any lift of x_1. Then let $\tilde{x}_2, \tilde{x}_3$ be the unique lifts of x_2 and x_3 respectively to $[\tilde{x}_1, \tilde{x}_1 + 1)$. Then we say x_1, x_2, x_3 are positively-oriented if $\tilde{x}_2 < \tilde{x}_3$.

Definition (Orientation-preserving map)**.** A map $S^1 \to S^1$ is orientation-preserving if it sends positively-oriented triples to positively-oriented triples.We write $\mathrm{Homeo}^+(S^1)$ for the group of orientation-preserving homeomorphisms of S^1.

Notation. We write Rot for the group of rotations in $\mathrm{Homeo}^+(S^1)$. This corresponds to the subgroup $T_\mathbb{R} \subseteq \mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$.

Definition (Poincare translation quasimorphism)**.** The *Poincare translation quasimorphism* $T\colon \mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R}) \to \mathbb{R}$ is the homogenization of F.

Definition (Rotation number)**.** The *rotation number* of $\varphi \in \mathrm{Homeo}^+(S^1)$ is $T(\tilde{\varphi}) \bmod \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, where $\tilde{\varphi}$ is a lift of φ to $\mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$.

2 Group cohomology and bounded cohomology

2.1 Group cohomology

Definition (Homogeneous k-cochain)**.** A *homogeneous k-cochain* with values in A is a function $f\colon \Gamma^{k+1} \to A$. The set $C(\Gamma^{k+1}, A)$ is an abelian group and Γ acts on it by automorphisms in the following way:

$$(\gamma_* f)(\gamma_0, \cdots, \gamma_m) = f(\gamma^{-1}\gamma_0, \cdots, \gamma^{-1}\gamma_k).$$

By convention, we set $C(\Gamma^0, A) \cong A$.

Definition (Differential $d^{(k)}$)**.** We define the differential $d^{(k)}\colon C(\Gamma^k, A) \to C(\Gamma^{k+1}, A)$ by

$$(d^{(k)} f)(\gamma_0, \cdots, \gamma_k) = \sum_{j=0}^{k} (-1)^j f(\gamma_0, \cdots, \hat{\gamma}_j, \cdots, \gamma_k).$$

In particular, we set $d^{(0)}(a)$ to be the function that is constantly a.

Definition (k-cocycle and k-coboundaries)**.**

- The k-cocycles are $\ker d^{(k+1)}$.
- The k-coboundaries are $\operatorname{im} d^{(k)}$.

Definition (Group cohomology $H^k(\Gamma, A)$)**.** We define the *kth cohomology group* to be

$$H^k = \frac{(\ker d^{(k+1)})^\Gamma}{d^{(k)}(C(\Gamma^k, A)^\Gamma)} = \frac{(d^{(k)}(C(\Gamma^k, A)))^\Gamma}{d^{(k)}(C(\Gamma^k, A)^\Gamma)}.$$

Definition (Central extension)**.** Let A be an abelian group, and Γ a group. Then a central extension of Γ by A is an exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0$$

such that the image of A is contained in the center of $\tilde{\Gamma}$.

Definition (Euler class)**.** The *Euler class* of the Γ-action by orientation-preserving homeomorphisms of S^1 is

$$h^*(e) \in H^2(\Gamma, \mathbb{Z}),$$

where $h\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ is the map defining the action.

2.2 Bounded cohomology of groups

Definition (Bounded cohomology)**.** The *k-th bounded cohomology group* of Γ with coefficients in A Is

$$H_b^k(\Gamma, A) = \frac{\ker(d^{(k+1)}\colon C_b(\Gamma^{k+1}, A)^\Gamma \to C_b(\Gamma^{k+2}, A)^\Gamma)}{d^{(k)}(C_b(\Gamma^k, A)^\Gamma)}.$$

Definition (Amenable group)**.** A discrete group Γ is *amenable* if there is a linear form $m\colon \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{R}$ such that

- $m(f) \geq 0$ if $f \geq 0$;
- $m(1) = 1$; and
- m is left-invariant, i.e. $m(\gamma_* f) = m(f)$, where $(\gamma_* f)(x) = f(\gamma^{-1}x)$.

3 Actions on S^1

3.1 The bounded Euler class

Definition (Bounded Euler class)**.** The *bounded Euler class*

$$e^b \in H_b^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$$

is the bounded cohomology class represented by the cocycle c.

Definition (Bounded Euler class of action)**.** The bounded Euler class of an action $h\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ is $h^*(e^b) \in H_b^2(\Gamma, \mathbb{Z})$.

Definition (Increasing map of degree 1)**.** A map $\varphi\colon S^1 \to S^1$ is *increasing of degree* 1 if there is some $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ lifting φ such that $\tilde{\varphi}$ is is monotonically increasing and

$$\tilde{\varphi}(x+1) = \tilde{\varphi}(x) + 1$$

for all $x \in \mathbb{R}$.

Definition (Semiconjugate action)**.** Two actions $h_1, h_2\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ are semi-conjugate if there are increasing maps of degree 1 $\varphi_1, \varphi_2\colon S^1 \to S^1$ such that

(i) $h_1(\gamma)\varphi_1 = \varphi_1 h_2(\gamma)$ for all $\gamma \in \Gamma$;

(ii) $h_2(\gamma)\varphi_2 = \varphi_2 h_1(\gamma)$ for all $\gamma \in \Gamma$.

Definition (Minimal action)**.** An action on S^1 is *minimal* if every orbit is dense.

3.2 The real bounded Euler class

Definition (Real bounded Euler class)**.** The *real bounded Euler class* is the class $e^b_{\mathbb{R}} \in H_b^2(\mathrm{Homeo}^+(S^1), \mathbb{R})$ obtained by change of coefficients from $\mathbb{Z} \to \mathbb{R}$.

The real bounded Euler class of an action $h\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ is the pullback

$$h^*(e^b_{\mathbb{R}}) \in H_b^2(\Gamma, \mathbb{R}).$$

Definition (Strongly proximal action)**.** A Γ-action by homeomorphisms on a compact metrizable space X is *strongly proximal* if for all probability measures μ on X, the weak-$*$ closure $\overline{\Gamma_* \mu}$ contains a Dirac mass.

Definition (Lattice)**.** A lattice in a locally compact group G is a discrete subgroup Γ such that on $\Gamma \backslash G$, there is a G-invariant probability measure.

4 The relative homological approach

4.1 Injective modules

Definition (Banach Γ module)**.** A Banach Γ-module is a Banach space V together with an action $\Gamma \times V \to V$ by linear isometries.

Definition (Admissible morphism)**.** An injective morphism $i\colon A \to B$ of Banach spaces is *admissible* if there exists $\sigma\colon B \to A$ with

– $\sigma i = \mathrm{id}_A$; and

– $\|\sigma\| \leq 1$.

Definition (Injective Banach Γ-module)**.** A Banach Γ-module is injective if for any diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \\ E & \end{tikzcd}$$

where i and α are morphisms of Γ-modules, and i is injective admissible, then there exists $\beta\colon B \to E$ a morphism of Γ-modules such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\beta"] \\ E & \end{tikzcd}$$

commutes and $\|\beta\| \leq \|\alpha\|$.

Definition (Injective resolution)**.** Let V be a Banach Γ-module. An *injective resolution* of V is an exact sequence

$$V \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

where each E_k is injective.

4.2 Amenable actions

Definition (Conditional expectation)**.** A *conditional expectation* on $G \times S$ is a linear map $M\colon L^\infty(G \times S) \to L^\infty(S)$ such that

(i) $M(1) = 1$;

(ii) If $M \geq 0$, then $M(f) \geq 0$; and

(iii) M is $L^\infty(S)$-linear.

Definition (Amenable action)**.** A G-action on S is amenable if there exists a G-equivariant conditional expectation.