Part II — Probability and Measure
Theorems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Analysis II is essential

Measure spaces, σ-algebras, π-systems and uniqueness of extension, statement *and proof* of Carathéodory's extension theorem. Construction of Lebesgue measure on $\mathbb{R}$. The Borel σ-algebra of $\mathbb{R}$. Existence of non-measurable subsets of $\mathbb{R}$. Lebesgue-Stieltjes measures and probability distribution functions. Independence of events, independence of σ-algebras. The Borel–Cantelli lemmas. Kolmogorov's zero-one law. [6]

Measurable functions, random variables, independence of random variables. Construction of the integral, expectation. Convergence in measure and convergence almost everywhere. Fatou's lemma, monotone and dominated convergence, differentiation under the integral sign. Discussion of product measure and statement of Fubini's theorem. [6]

Chebyshev's inequality, tail estimates. Jensen's inequality. Completeness of L^p for $1 \leq p \leq \infty$. The Hölder and Minkowski inequalities, uniform integrability. [4]

L^2 as a Hilbert space. Orthogonal projection, relation with elementary conditional probability. Variance and covariance. Gaussian random variables, the multivariate normal distribution. [2]

The strong law of large numbers, proof for independent random variables with bounded fourth moments. Measure preserving transformations, Bernoulli shifts. Statements *and proofs* of maximal ergodic theorem and Birkhoff's almost everywhere ergodic theorem, proof of the strong law. [4]

The Fourier transform of a finite measure, characteristic functions, uniqueness and inversion. Weak convergence, statement of Lévy's convergence theorem for characteristic functions. The central limit theorem. [2]

Contents

0 Introduction

1 Measures

1.1 Measures

Proposition. A collection $\mathcal{A}$ is a σ-algebra if and only if it is both a π-system and a d-system.

Lemma (Dynkin's π-system lemma)**.** Let $\mathcal{A}$ be a π-system. Then any d-system which contains $\mathcal{A}$ contains $\sigma(\mathcal{A})$.

Theorem (Caratheodory extension theorem)**.** Let $\mathcal{A}$ be a ring on E, and μ a countably additive set function on $\mathcal{A}$. Then μ extends to a measure on the σ-algebra generated by $\mathcal{A}$.

Theorem. Suppose that μ_1, μ_2 are measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. If $\mathcal{A}$ is a π-system with $\sigma(A) = \mathcal{E}$, and μ_1 agrees with μ_2 on $\mathcal{A}$, then $\mu_1 = \mu_2$.

Theorem. There exists a unique Borel measure μ on $\mathbb{R}$ with $\mu([a, b]) = b - a$.

Proposition. The Lebesgue measure is *translation invariant*, i.e.

$$\mu(A + x) = \mu(A)$$

for all $A \in \mathcal{B}$ and $x \in \mathbb{R}$, where

$$A + x = \{y + x, y \in A\}.$$

Proposition. Let $\tilde{\mu}$ be a Borel measure on $\mathbb{R}$ that is translation invariant and $\mu([0, 1]) = 1$. Then $\tilde{\mu}$ is the Lebesgue measure.

1.2 Probability measures

Proposition. Events (A_n) are independent iff the σ-algebras $\sigma(\mathcal{A}_n)$ are independent.

Theorem. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are π-systems in $\mathcal{F}$. If

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\mathbb{P}[A_2]$$

for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, then $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

Lemma (Borel–Cantelli lemma)**.** If

$$\sum_n \mathbb{P}[A_n] < \infty,$$

then

$$\mathbb{P}[A_n \text{ i.o.}] = 0.$$

Lemma (Borel–Cantelli lemma II)**.** Let (A_n) be independent events. If

$$\sum_n \mathbb{P}[A_n] = \infty,$$

then

$$\mathbb{P}[A_n \text{ i.o.}] = 1.$$

2 Measurable functions and random variables

2.1 Measurable functions

Lemma. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, and $\mathcal{G} = \sigma(\mathcal{Q})$ for some $\mathcal{Q}$. If $f^{-1}(A) \in \mathcal{E}$ for all $A \in \mathcal{Q}$, then f is measurable.

Proposition. Let $f_i : E \to F_i$ be functions. Then $\{f_i\}$ are all measurable iff $(f_i) : E \to \prod F_i$ is measurable, where the function (f_i) is defined by setting the ith component of $(f_i)(x)$ to be $f_i(x)$.

Proposition. Let $(E, \mathcal{E})$ be a measurable space. Let $(f_n : n \in \mathbb{N})$ be a sequence of non-negative measurable functions on E. Then the following are measurable:

$$\begin{gathered} f_1 + f_2, \quad f_1 f_2, \quad \max\{f_1, f_2\}, \quad \min\{f_1, f_2\}, \\ \inf_n f_n, \quad \sup_n f_n, \quad \liminf_n f_n, \quad \limsup_n f_n. \end{gathered}$$

The same is true with "real" replaced with "non-negative", provided the new functions are real (i.e. not infinity).

Theorem (Monotone class theorem)**.** Let $(E, \mathcal{E})$ be a measurable space, and $\mathcal{A} \subseteq \mathcal{E}$ be a π-system with $\sigma(\mathcal{A}) = \mathcal{E}$. Let $\mathcal{V}$ be a vector space of functions such that

(i) The constant function $1 = \mathbf{1}_E$ is in $\mathcal{V}$.

(ii) The indicator functions $\mathbf{1}_A \in \mathcal{V}$ for all $A \in \mathcal{A}$

(iii) $\mathcal{V}$ is closed under bounded, monotone limits.

More explicitly, if (f_n) is a bounded non-negative sequence in $\mathcal{V}$, $f_n \nearrow f$ (pointwise) and f is also bounded, then $f \in \mathcal{V}$.

Then $\mathcal{V}$ contains all bounded measurable functions.

2.2 Constructing new measures

Lemma. Let $g : \mathbb{R} \to \mathbb{R}$ be non-constant, non-decreasing and right continuous. We set

$$g(\pm\infty) = \lim_{x \to \pm\infty} g(x).$$

We set $I = (g(-\infty), g(\infty))$. Since g is non-constant, this is non-empty.

Then there is a non-decreasing, left continuous function $f : I \to \mathbb{R}$ such that for all $x \in I$ and $y \in \mathbb{R}$, we have

$$x \leq g(y) \Leftrightarrow f(x) \leq y.$$

Thus, taking the negation of this, we have

$$x > g(y) \Leftrightarrow f(x) > y.$$

Explicitly, for $x \in I$, we define

$$f(x) = \inf\{y \in \mathbb{R} : x \leq g(y)\}.$$

Theorem. Let $g : \mathbb{R} \to \mathbb{R}$ be non-constant, non-decreasing and right continuous. Then there exists a unique Radon measure $\mathrm{d}g$ on $\mathcal{B}$ such that

$$\mathrm{d}g((a, b]) = g(b) - g(a).$$

Moreover, we obtain all non-zero Radon measures on $\mathbb{R}$ in this way.

2.3 Random variables

Proposition. We have

$$F_X(x) \to \begin{cases} 0 & x \to -\infty \\ 1 & x \to +\infty \end{cases}.$$

Also, $F_X(x)$ is non-decreasing and right-continuous.

Proposition. Let F be any distribution function. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X such that $F_X = F$.

Proposition. Two real-valued random variables X, Y are independent iff

$$\mathbb{P}[X \leq x, Y \leq y] = \mathbb{P}[X \leq x]\mathbb{P}[Y \leq y].$$

More generally, if (X_n) is a sequence of real-valued random variables, then they are independent iff

$$\mathbb{P}[x_1 \leq x_1, \cdots, x_n \leq x_n] = \prod_{j=1}^n \mathbb{P}[X_j \leq x_j]$$

for all n and x_j.

Proposition. Let

$$(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1), \mathcal{B}(0, 1), \text{Lebesgue}).$$

be our probability space. Then there exists as sequence R_n of independent Bernoulli(1/2) random variables.

Proposition. Let

$$(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1), \mathcal{B}(0, 1), \text{Lebesgue}).$$

Given any sequence (F_n) of distribution functions, there is a sequence (X_n) of independent random variables with $F_{X_n} = F_n$ for all n.

2.4 Convergence of measurable functions

Theorem.

(i) If $\mu(E) < \infty$, then $f_n \to f$ a.e. implies $f_n \to f$ in measure.

(ii) For any E, if $f_n \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e.

Theorem (Skorokhod representation theorem of weak convergence)**.**

(i) If $(X_n), X$ are defined on the same probability space, and $X_n \to X$ in probability. Then $X_n \to X$ in distribution.

(ii) If $X_n \to X$ in distribution, then there exists random variables $(\tilde{X}_n)$ and $\tilde{X}$ defined on a common probability space with $F_{\tilde{X}_n} = F_{X_n}$ and $F_{\tilde{X}} = F_X$ such that $\tilde{X}_n \to \tilde{X}$ a.s.

2.5 Tail events

Theorem (Kolmogorov 0-1 law)**.** Let (X_n) be a sequence of independent (real-valued) random variables. If $A \in \mathcal{T}$, then $\mathbb{P}[A] = 0$ or 1.

Moreover, if X is a $\mathcal{T}$-measurable random variable, then there exists a constant c such that
$$\mathbb{P}[X = c] = 1.$$

3 Integration

3.1 Definition and basic properties

Proposition. A function is simple iff it is measurable, non-negative, and takes on only finitely-many values.

Proposition. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Then it is also Lebesgue integrable, and the two integrals agree.

Theorem (Monotone convergence theorem)**.** Suppose that $(f_n), f$ are non-negative measurable with $f_n \nearrow f$. Then $\mu(f_n) \nearrow \mu(f)$.

Theorem. Let f, g be non-negative measurable, and $\alpha, \beta \geq 0$. We have that

(i) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$.

(ii) $f \leq g$ implies $\mu(f) \leq \mu(g)$.

(iii) $f = 0$ a.e. iff $\mu(f) = 0$.

Theorem. Let f, g be integrable, and $\alpha, \beta \geq 0$. We have that

(i) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$.

(ii) $f \leq g$ implies $\mu(f) \leq \mu(g)$.

(iii) $f = 0$ a.e. implies $\mu(f) = 0$.

Proposition. If $\mathcal{A}$ is a π-system with $E \in \mathcal{A}$ and $\sigma(\mathcal{A}) = \mathcal{E}$, and f is an integrable function that

$$\mu(f\mathbf{1}_A) = 0$$

for all $A \in \mathcal{A}$. Then $\mu(f) = 0$ a.e.

Proposition. Suppose that (g_n) is a sequence of non-negative measurable functions. Then we have

$$\mu\left(\sum_{n=1}^{\infty} g_n\right) = \sum_{n=1}^{\infty} \mu(g_n).$$

3.2 Integrals and limits

Theorem (Fatou's lemma)**.** Let (f_n) be a sequence of non-negative measurable functions. Then

$$\mu(\liminf f_n) \leq \liminf \mu(f_n).$$

Theorem (Dominated convergence theorem)**.** Let $(f_n), f$ be measurable with $f_n(x) \to f(x)$ for all $x \in E$. Suppose that there is an integrable function g such that

$$|f_n| \leq g$$

for all n, then we have

$$\mu(f_n) \to \mu(f)$$

as $n \to \infty$.

3.3 New measures from old

Lemma. For $(E, \mathcal{E}, \mu)$ a measure space and $A \in \mathcal{E}$, the restriction to A is a measure space. □

Proposition. Let $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \mu')$ be measure spaces and $A \in \mathcal{E}$. Let $f : E \to F$ be a measurable function. Then $f|_A$ is $\mathcal{E}_A$-measurable.

Proposition. If f is integrable, then $f|_A$ is μ_A-integrable and $\mu_A(f|_A) = \mu(f\mathbf{1}_A)$. □

Proposition. If g is a non-negative measurable function on G, then

$$\nu(g) = \mu(g \circ f).$$

Proposition. The ν defined above is indeed a measure.

3.4 Integration and differentiation

Proposition (Change of variables formula)**.** Let $\phi : [a, b] \to \mathbb{R}$ be continuously differentiable and increasing. Then for any bounded Borel function g, we have

$$\int_{\phi(a)}^{\phi(b)} g(y)\,\mathrm{d}y = \int_a^b g(\phi(x))\phi'(x)\,\mathrm{d}x. \tag{$*$}$$

Theorem (Differentiation under the integral sign)**.** Let $(E, \mathcal{E}, \mu)$ be a space, and $U \subseteq \mathbb{R}$ be an open set, and $f : U \times E \to \mathbb{R}$. We assume that

(i) For any $t \in U$ fixed, the map $x \mapsto f(t, x)$ is integrable;

(ii) For any $x \in E$ fixed, the map $t \mapsto f(t, x)$ is differentiable;

(iii) There exists an integrable function g such that

$$\left|\frac{\partial f}{\partial t}(t, x)\right| \leq g(x)$$

for all $x \in E$ and $t \in U$.

Then the map

$$x \mapsto \frac{\partial f}{\partial t}(t, x)$$

is integrable for all t, and also the function

$$F(t) = \int_E f(t, x)\mathrm{d}\mu$$

is differentiable, and

$$F'(t) = \int_E \frac{\partial f}{\partial t}(t, x)\,\mathrm{d}\mu.$$

3.5 Product measures and Fubini's theorem

Lemma. Let $E = E_1 \times E_2$ be a product of σ-algebras. Suppose $f : E \to \mathbb{R}$ is $\mathcal{E}$-measurable function. Then

(i) For each $x_2 \in E_2$, the function $x_1 \mapsto f(x_1, x_2)$ is $\mathcal{E}_1$-measurable.

(ii) If f is bounded or non-negative measurable, then

$$f_2(x_2) = \int_{E_1} f(x_1, x_2)\ \mu_1(\mathrm{d}x_1)$$

is $\mathcal{E}_2$-measurable.

Theorem. There exists a unique measurable function $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{E}$ such that

$$\mu(A_1 \times A_2) = \mu(A_1)\mu(A_2)$$

for all $A_1 \times A_2 \in \mathcal{A}$.

Theorem (Fubini's theorem)**.**

(i) If f is non-negative measurable, then

$$\mu(f) = \int_{E_1} \left(\int_{E_2} f(x_1, x_2)\ \mu_2(\mathrm{d}x_2)\right) \mu_1(\mathrm{d}x_1). \tag{$*$}$$

In particular, we have

$$\int_{E_1}\left(\int_{E_2} f(x_1, x_2)\ \mu_2(\mathrm{d}x_2)\right) \mu_1(\mathrm{d}x_1) = \int_{E_2}\left(\int_{E_1} f(x_1, x_2)\ \mu_1(\mathrm{d}x_1)\right) \mu_2(\mathrm{d}x_2).$$

This is sometimes known as *Tonelli's theorem*.

(ii) If f is integrable, and

$$A = \left\{x_1 \in E : \int_{E_2} |f(x_1, x_2)| \mu_2(\mathrm{d}x_2) < \infty\right\}.$$

then

$$\mu_1(E_1 \setminus A) = 0.$$

If we set

$$f_1(x_1) = \begin{cases} \int_{E_2} f(x_1, x_2)\ \mu_2(\mathrm{d}x_2) & x_1 \in A \\ 0 & x_1 \not\in A \end{cases},$$

then f_1 is a μ_1 integrable function and

$$\mu_1(f_1) = \mu(f).$$

Proposition. Let $X_1, \cdots, X_n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E_1, \mathcal{E}_1), \cdots, (E_n, \mathcal{E}_n)$ respectively. We define

$$E = E_1 \times \cdots \times E_n, \quad \mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n.$$

Then $X = (X_1, \cdots, X_n)$ is $\mathcal{E}$-measurable and the following are equivalent:

(i) $X_1, \cdots, X_n$ are independent.

(ii) $\mu_X = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$.

(iii) For any $f_1, \cdots, f_n$ bounded and measurable, we have

$$\mathbb{E}\left[\prod_{k=1}^{n} f_k(X_k)\right] = \prod_{k=1}^{n} \mathbb{E}[f_k(X_k)].$$

4 Inequalities and L^p spaces

4.1 Four inequalities

Proposition (Chebyshev's/Markov's inequality)**.** Let f be non-negative measurable and $\lambda > 0$. Then

$$\mu(\{f \geq \lambda\}) \leq \frac{1}{\lambda}\mu(f).$$

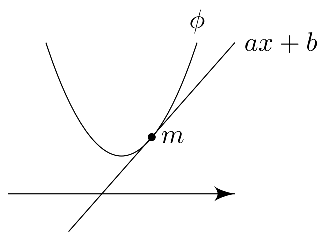
Proposition (Jensen's inequality)**.** Let X be an integrable random variable with values in I. If $c : I \to \mathbb{R}$ is convex, then we have

$$\mathbb{E}[c(X)] \geq c(E[X]).$$

Lemma. If $c : I \to \mathbb{R}$ is a convex function and m is in the interior of I, then there exists real numbers a, b such that

$$c(x) \geq ax + b$$

for all $x \in I$, with equality at $x = m$.



Proposition (Hölder's inequality)**.** Let $p, q \in (1, \infty)$ be conjugate. Then for f, g measurable, we have

$$\mu(|fg|) = \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

When $p = q = 2$, then this is the Cauchy-Schwarz inequality.

Lemma. Let $a, b \geq 0$ and $p \geq 1$. Then

$$(a + b)^p \leq 2^p(a^p + b^p).$$

Theorem (Minkowski inequality)**.** Let $p \in [1, \infty]$ and f, g measurable. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

4.2 L^p spaces

Theorem. Let $1 \leq p \leq \infty$. Then $\mathcal{L}^p$ is a Banach space. In other words, if (f_n) is a sequence in L^p, with the property that $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$, then there is some $f \in L^p$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$.

4.3 Orthogonal projection in $\mathcal{L}^2$

Theorem. Let V be a closed subspace of L^2. Then each $f \in L^2$ has an *orthogonal decomposition*

$$f = u + v,$$

where $v \in V$ and $u \in V^\perp$. Moreover,

$$\|f - v\|_2 \leq \|f - g\|_2$$

for all $g \in V$ with equality iff $g \sim v$.

Lemma (Pythagoras identity)**.**

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2 + 2\langle f, g\rangle.$$

Lemma (Parallelogram law)**.**

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

Proposition. The conditional expectation of X given $\mathcal{G}$ is the projection of X onto the subspace $L^2(\mathcal{G}, \mathbb{P})$ of $\mathcal{G}$-measurable L^2 random variables in the ambient space $L^2(\mathbb{P})$.

4.4 Convergence in $L^1(\mathbb{P})$ and uniform integrability

Theorem (Bounded convegence theorem)**.** Suppose $X, (X_n)$ are random variables. Assume that there exists a (non-random) constant $C > 0$ such that $|X_n| \leq C$. If $X_n \to X$ in probability, then $X_n \to X$ in L^1.

Proposition. Finite unions of uniformly integrable sets are uniformly integrable.

Proposition. Let $\mathcal{X}$ be an L^p-bounded family for some $p > 1$. Then $\mathcal{X}$ is uniformly integrable.

Lemma. Let $\mathcal{X}$ be a family of random variables. Then $\mathcal{X}$ is uniformly integrable if and only if

$$\sup\{\mathbb{E}[|X|\mathbf{1}_{|X|>k}] : X \in \mathcal{X}\} \to 0$$

as $k \to \infty$.

Corollary. Let $\mathcal{X} = \{X\}$, where $X \in L^1(\mathbb{P})$. Then $\mathcal{X}$ is uniformly integrable.

Hence, a finite collection of L^1 functions is uniformly integrable.

Theorem. Let $X, (X_n)$ be random variables. Then the following are equivalent:

(i) $X_n, X \in L^1$ for all n and $X_n \to X$ in L^1.

(ii) $\{X_n\}$ is uniformly integrable and $X_n \to X$ in probability.

5 Fourier transform

5.1 The Fourier transform

Proposition.
$$\|\hat{f}\|_\infty \leq \|f\|_1, \quad \|\hat{\mu}\|_\infty \leq \mu(\mathbb{R}^d).$$

Proposition. The functions $\hat{f}, \hat{\mu}$ are continuous.

5.2 Convolutions

Proposition. For any $f \in L^p$ and ν a probability measure, we have
$$\|f * \nu\|_p \leq \|f\|_p.$$

Proposition.
$$\widehat{f * \nu}(u) = \hat{f}(u)\hat{\nu}(u).$$

Proposition. Let μ, ν be probability measures, and X, Y be independent variables with laws μ, ν respectively. Then
$$\widehat{\mu * \nu}(u) = \hat{\mu}(u)\hat{\nu}(u).$$

5.3 Fourier inversion formula

Theorem (Fourier inversion formula)**.** Let $f, \hat{f} \in L^1$. Then
$$f(x) = \frac{1}{(2\pi)^d} \int \hat{f}(u) e^{-i(u, x)}\;\mathrm{d}u \text{ a.e.}$$

Proposition. Let $Z \sim N(0, 1)$. Then
$$\phi_Z(a) = e^{-u^2/2}.$$

Proposition. Let $Z = (Z_1, \cdots, Z_d)$ with $Z_j \sim N(0, 1)$ independent. Then $\sqrt{t}Z$ has density
$$g_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)}.$$
with
$$\hat{g}_t(u) = e^{-|u|^2 t/2}.$$

Lemma. The Fourier inversion formula holds for the Gaussian density function.

Proposition.
$$\|f * g_t\|_1 \leq \|f\|_1.$$

Proposition.
$$\|f * g_t\|_\infty \leq (2\pi t)^{-d/2}\|f\|_1.$$

Proposition.
$$\|\widehat{f * g_t}\|_1 = \|\hat{f}\hat{g}_t\|_1 \leq (2\pi)^{d/2} t^{-d/2} \|\hat{f}\|_1,$$
and
$$\|\widehat{f * g_t}\|_\infty \leq \|\hat{f}\|_\infty.$$

Lemma. The Fourier inversion formula holds for Gaussian convolutions.

Theorem (Fourier inversion formula)**.** Let $f \in L^1$ and

$$f_t(x) = (2\pi)^{-d} \int \hat{f}(u) e^{-|u|^2 t/2} e^{-i(u,x)}\, \mathrm{d}u = (2\pi)^{-d} \int \widehat{f * g_t}(u) e^{-i(u,x)}\, \mathrm{d}u.$$

Then $\|f_t - f\|_1 \to 0$, as $t \to 0$, and the Fourier inversion holds whenever $f, \hat{f} \in L^1$.

Lemma. Suppose that $f \in L^p$ with $p \in [1, \infty)$. Then $\|f * g_t - f\|_p \to 0$ as $t \to 0$.

5.4 Fourier transform in $\mathcal{L}^2$

Theorem (Plancherel identity)**.** For any function $f \in L^1 \cap L^2$, the *Plancherel identity* holds:

$$\|\hat{f}\|_2 = (2\pi)^{d/2} \|f\|_2.$$

Theorem. There exists a unique Hilbert space automorphism $F : \mathcal{L}^2 \to \mathcal{L}^2$ such that

$$F([f]) = [(2\pi)^{-d/2} \hat{f}]$$

whenever $f \in L^1 \cap L^2$.

Here $[f]$ denotes the equivalence class of f in $\mathcal{L}^2$, and we say $F : \mathcal{L}^2 \to \mathcal{L}^2$ is a Hilbert space automorphism if it is a linear bijection that preserves the inner product.

5.5 Properties of characteristic functions

Theorem. The characteristic function ϕ_X of a distribution μ_X of a random variable X determines μ_X. In other words, if X and $\tilde{X}$ are random variables and $\phi_X = \phi_{\tilde{X}}$, then $\mu_X = \mu_{\tilde{X}}$

Theorem. If ϕ_X is integrable, then μ_X has a bounded, continuous density function

$$f_X(x) = (2\pi)^{-d} \int \phi_X(u) e^{-i(u,x)}\, \mathrm{d}u.$$

Theorem. Let $X, (X_n)$ be random variables with values in $\mathbb{R}^d$. If $\phi_{X_n}(u) \to \phi_X(u)$ for each $u \in \mathbb{R}^d$, then $\mu_{X_n} \to \mu_X$ weakly.

5.6 Gaussian random variables

Proposition. Let $X \sim N(\mu, \sigma^2)$. Then

$$\mathbb{E}[X] = \mu, \quad \mathrm{var}(X) = \sigma^2.$$

Also, for any $a, b \in \mathbb{R}$, we have

$$aX + b \sim N(a\mu + b, a^2\sigma^2).$$

Lastly, we have

$$\phi_X(u) = e^{-i\mu u - u^2\sigma^2/2}.$$

Theorem. Let X be Gaussian on $\mathbb{R}^n$, and le tA be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then

(i) $AX + b$ is Gaussian on $\mathbb{R}^m$.

(ii) $X \in L^2$ and its law μ_X is determined by $\mu = \mathbb{E}[X]$ and $V = \mathrm{var}(X)$, the covariance matrix.

(iii) We have
$$\phi_X(u) = e^{i(u,\mu) - (u,Vu)/2}.$$

(iv) If V is invertible, then X has a density of
$$f_X(x) = (2\pi)^{-n/2}(\det V)^{-1/2} \exp\left(-\frac{1}{2}(x - \mu, V^{-1}(x - \mu))\right).$$

(v) If $X = (X_1, X_2)$ where $X_i \in \mathbb{R}^{n_i}$, then $\mathrm{cov}(X_1, X_2) = 0$ iff X_1 and X_2 are independent.

6 Ergodic theory

Proposition. If f is integrable and Θ is measure-preserving. Then $f \circ \Theta$ is integrable and

$$\int f \circ \Theta \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

Proposition. If Θ is ergodic and f is invariant, then there exists a constant c such that $f = c$ a.e.

Theorem. The shift map Θ is an ergodic, measure preserving transformation.

6.1 Ergodic theorems

Lemma (Maximal ergodic lemma)**.** Let f be integrable, and

$$S^* = \sup_{n \geq 0} S_n(f) \geq 0,$$

where $S_0(f) = 0$ by convention. Then

$$\int_{\{S^* > 0\}} f \; \mathrm{d}\mu \geq 0.$$

Theorem (Birkhoff's ergodic theorem)**.** Let $(E, \mathcal{E}, \mu)$ be σ-finite and f be integrable. There exists an invariant function $\bar{f}$ such that

$$\mu(|\bar{f}|) \leq \mu(|f|),$$

and

$$\frac{S_n(f)}{n} \to \bar{f} \text{ a.e.}$$

If Θ is ergodic, then $\bar{f}$ is a constant.

Theorem (von Neumann's ergodic theorem)**.** Let $(E, \mathcal{E}, \mu)$ be a *finite* measure space. Let $p \in [1, \infty)$ and assume that $f \in L^p$. Then there is some function $\bar{f} \in L^p$ such that

$$\frac{S_n(f)}{n} \to \bar{f} \text{ in } L^p.$$

7 Big theorems

7.1 The strong law of large numbers

Theorem (Strong law of large numbers assuming finite fourth moments)**.** Let (X_n) be a sequence of independent random variables such that there exists $\mu \in \mathbb{R}$ and $M > 0$ such that

$$\mathbb{E}[X_n] = \mu, \quad \mathbb{E}[X_n^4] \leq M$$

for all n. With $S_n = X_1 + \cdots + X_n$, we have that

$$\frac{S_n}{n} \to \mu \text{ a.s. as } n \to \infty.$$

Theorem (Strong law of large numbers)**.** Let (Y_n) be an iid sequence of integrable random variables with mean ν. With $S_n = Y_1 + \cdots + Y_n$, we have

$$\frac{S_n}{n} \to \nu \text{ a.s.}$$

7.2 Central limit theorem

Theorem. Let (X_n) be a sequence of iid random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_1^2] = 1$. Then if we set

$$S_n = X_1 + \cdots + X_n,$$

then for all $x \in \mathbb{R}$, we have

$$\mathbb{P}\left[\frac{S_n}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^x \frac{e^{-y^2/2}}{\sqrt{2\pi}}\;\mathrm{d}y = \mathbb{P}[N(0,1) \leq x]$$

as $n \to \infty$.