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Probability and Measure 1

1.1. Let E be a set and let $\mathcal{S}$ be a set of σ-algebras on E. Define

$$\mathcal{E}^* = \{A \subseteq E : A \in \mathcal{E} \quad \text{for all} \quad \mathcal{E} \in \mathcal{S}\}.$$

Show that $\mathcal{E}^*$ is a σ-algebra on E. Show, on the other hand, by example, that the union of two σ-algebras on the same set need not be a σ-algebra.

1.2. Show that the following sets of subsets of $\mathbb{R}$ all generate the same σ-algebra:

(a) $\{(a, b) : a < b\}$, (b) $\{(a, b] : a < b\}$, (c) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.3. Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4. Show that a π-system which is also a d-system is a σ-algebra.

1.5. Let μ be a finite-valued additive set function on a ring $\mathcal{A}$. Show that μ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $\mathcal{A}$, with $\cap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6. Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$,

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events A_n are independent if and only if the σ-algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8. Let B be a Borel subset of the interval $[0, 1]$. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \triangle B$ $(= (A^c \cap B) \cup (A \cap B^c))$ is less than ε. Show further that this remains true for every Borel set in $\mathbb{R}$ of finite Lebesgue measure.

1.9. Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ-algebra and show that μ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the *completion of* $\mathcal{E}$ *with respect to* μ. Suppose now that E is σ-finite and write μ^* for the outer measure associated to μ, as in the proof of Carathéodory's Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of μ^*-measurable sets.

2.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on a measurable space $(E, \mathcal{E})$. Show that the following functions are also measurable: $f_1 + f_2$, $f_1 f_2$, $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show also that $\{x \in E : f_n(x) \text{ converges as } n \to \infty\} \in \mathcal{E}$.

2.2. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, let μ be a measure on $\mathcal{E}$, and let $f : E \to G$ be a measurable function. Show that we can define a measure ν on $\mathcal{G}$ by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$.

2.3. Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$ for all $x, y \in \mathbb{R}$.

2.4. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

2.5. The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

2.6. Let $(X_n : n \in \mathbb{N})$ be independent $N(0,1)$ random variables. Prove that

$$\limsup_n \left(X_n/\sqrt{2\log n}\right) = 1 \quad \text{a.s.}$$

2.7. Let C_n denote the nth approximation to the Cantor set C: thus $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n. Show that

(a) C is uncountable and has Lebesgue measure 0,
(b) for all $x \in [0,1]$, the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists,
(c) the function F is continuous on $[0,1]$, with $F(0) = 0$ and $F(1) = 1$,
(d) for almost all $x \in [0,1]$, F is differentiable at x with $F'(x) = 0$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.

Probability and Measure 2

3.1. Suppose that a simple function f has two representations

$$f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{j=1}^{n} b_k 1_{B_k}.$$

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0,1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \dots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly B_δ for $\delta \in \{0,1\}^n$. Then set $f_{\varepsilon,\delta} = \sum_{k=1}^m \varepsilon_k a_k$ if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon,\delta} = 0$ otherwise. Show that, for any measure μ,

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon,\delta} f_{\varepsilon,\delta} \mu(A_\varepsilon \cap B_\delta)$$

and deduce that

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j).$$

3.2. Let μ and ν be finite Borel measures on $\mathbb{R}$. Let f be a continuous bounded function on $\mathbb{R}$. Show that f is integrable with respect to μ and ν. Show further that, if $\mu(f) = \nu(f)$ for all such f, then $\mu = \nu$.

3.3. Let f be an integrable function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that, for some π-system $\mathcal{A}$ containing E and generating $\mathcal{E}$, we have $\mu(f 1_A) = 0$ for all $A \in \mathcal{A}$. Show that $f = 0$ a.e.

3.4. Let X be a non-negative integer-valued random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Deduce that, if $\mathbb{E}(X) = \infty$ and $X_1, X_2, \dots$ is a sequence of independent random variables with the same distribution as X, then, almost surely, $\limsup_n (X_n/n) \geq 1$, and moreover $\limsup_n (X_n/n) = \infty$.

Now suppose that $Y_1, Y_2, \dots$ is any sequence of independent identically distributed random variables with $\mathbb{E}|Y_1| = \infty$. Show that, almost surely, $\limsup_n (|Y_n|/n) = \infty$, and moreover $\limsup_n (|Y_1 + \cdots + Y_n|/n) = \infty$.

3.5. For $\alpha \in (0, \infty)$ and $x \in (0, \infty)$, define $f_\alpha(x) = x^{-\alpha}$. Show that f_α is integrable with respect to Lebesgue measure on $(0, 1]$ if and only if $\alpha < 1$. Show also that f_α is integrable with respect to Lebesgue measure on $[1, \infty)$ if and only if $\alpha > 1$.

3.6. Show that the function $\sin x/x$ is not Lebesgue integrable over $[1,\infty)$ but that integral $\int_1^N (\sin x/x)dx$ converges as $N \to \infty$.

3.7. Show that, as $n \to \infty$,

$$\int_0^\infty \sin(e^x)/(1+nx^2)dx \to 0 \quad \text{and} \quad \int_0^1 (n\cos x)/(1+n^2x^{\frac{3}{2}})dx \to 0.$$

3.8. Let u and v be differentiable functions on $\mathbb{R}$ with continuous derivatives u' and v'. Suppose that uv' and $u'v$ are integrable on $\mathbb{R}$ and $u(x)v(x) \to 0$ as $|x| \to \infty$. Show that

$$\int_{\mathbb{R}} u(x)v'(x)dx = -\int_{\mathbb{R}} u'(x)v(x)dx.$$

3.9. Let $(E,\mathcal{E})$ and $(G,\mathcal{G})$ be measurable spaces and let $f : E \to G$ be a measurable function. Given a measure μ on $(E,\mathcal{E})$, consider the image measure $\nu = \mu \circ f^{-1}$ on $(G,\mathcal{G})$. Show that $\nu(g) = \mu(g \circ f)$ for all non-negative measurable functions g on G.

3.10. The moment generating function ϕ of a real-valued random variable X is defined by $\phi(\theta) = \mathbb{E}(e^{\theta X}), \quad \theta \in \mathbb{R}$.

Suppose that ϕ is finite on an open interval containing 0. Show that ϕ has derivatives of all orders at 0 and that X has finite moments of all orders given by

$$\mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n \bigg|_{\theta=0} \phi(\theta).$$

3.11. Let $X_1,\dots,X_n$ be random variables with density functions $f_1,\dots,f_n$ respectively. Suppose that the $\mathbb{R}^n$-valued random variable $X = (X_1,\dots,X_n)$ also has a density function f. Show that $X_1,\dots,X_n$ are independent if and only if

$$f(x_1,\dots,x_n) = f_1(x_1)\dots f_n(x_n) \qquad \text{a.e.}$$

3.12. Show that, for all non-negative measurable functions f on $[0,\infty)$, the function $(x,y) \mapsto f(|(x,y)|)$ is measurable on $\mathbb{R}^2$ and (without using the Jacobian formula)

$$\int_{\mathbb{R}^2} f(|(x,y)|)dxdy = 2\pi \int_0^\infty rf(r)dr.$$

Hence show that $(2\pi)^{-1/2}e^{-x^2/2}$ is a probability density function.

3.13. Let μ and ν be probability measures on $(E,\mathcal{E})$ and let $f : E \to [0,R]$ be a measurable function. Suppose that $\nu(A) = \mu(f1_A)$ for all $A \in \mathcal{E}$. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables in E with law μ and let $(U_n : n \in \mathbb{N})$ be a sequence of independent $U[0,1]$ random variables. Set

$$T = \min\{n \in \mathbb{N} : RU_n \le f(X_n)\}, \qquad Y = X_T.$$

Show that Y has law ν. (This justifies simulation by rejection sampling.)

Probability and Measure 3

4.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of integrable functions and suppose that $f_n \to f$ a.e. for some integrable function f. Show that, if $\|f_n\|_1 \to \|f\|_1$, then $\|f_n - f\|_1 \to 0$.

4.2. Let X be a random variable and let $1 \le p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \ge \lambda) = O(\lambda^{-p})$ as $\lambda \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \ge \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \ge \lambda) = O(\lambda^{-q})$ as $\lambda \to \infty$, then $X \in L^p(\mathbb{P})$.

4.3. Give a simple proof of Schwarz' inequality $\|fg\|_1 \le \|f\|_2\|g\|_2$ for measurable functions f and g.

4.4. Show that $\|XY\|_1 = \|X\|_1\|Y\|_1$ for independent random variables X and Y. Show further that, if X and Y are also integrable, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

4.5. A *stepfunction* $f : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$: that is, for all $f \in L^p(\mathbb{R})$ and all $\varepsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \varepsilon$. Deduce that the set of continuous functions of compact support is also dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

4.6. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2}\max_{k \le n}|X_k| \to 0$ in probability.

5.1. Let $(E, \mathcal{E}, \mu)$ be a measure space and let $V_1 \le V_2 \le \dots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E, \mathcal{E}, \mu)$ for $f \in L^2$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in L^2.

5.2. Let $X = (X_1, \dots, X_n)$ be a random variable, with all components in $L^2(\mathbb{P})$. The covariance matrix $\mathrm{var}(X) = (c_{ij} : 1 \le i, j \le n)$ of X is defined by $c_{ij} = \mathrm{cov}(X_i, X_j)$. Show that $\mathrm{var}(X)$ is a non-negative definite matrix.

6.1. Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

6.2. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that

$$\mathbb{E}\big(\max_{k \le n}|X_k|\big)/\sqrt{n} \to 0 \qquad \text{as} \quad n \to \infty.$$

7.1. Let $u, v \in L^1(\mathbb{R}^d)$ and define $f : \mathbb{R}^d \to \mathbb{C}$ by $f(x) = u(x) + iv(x)$. Set

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} u(x)dx + i \int_{\mathbb{R}^d} v(x)dx.$$

Show that, for all $y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} f(x-y)dx = \int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(-x)dx$$

and show that

$$\left| \int_{\mathbb{R}^d} f(x)dx \right| \le \int_{\mathbb{R}^d} |f(x)|dx.$$

7.2. Show that the Fourier transform of a finite Borel measure on $\mathbb{R}^d$ is a bounded continuous function.

7.3. Determine which of the following distributions on $\mathbb{R}$ have an integrable characteristic function: $\mathrm{N}(\mu, \sigma^2)$, $\mathrm{Bin}(N, p)$, $\mathrm{Poisson}(\lambda)$, $\mathrm{U}[0, 1]$.

7.4. For a finite Borel measure μ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a kth continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).$$

7.5. Define a function ψ on $\mathbb{R}$ by setting $\psi(x) = C \exp\{-(1 - x^2)^{-1}\}$ for $|x| < 1$ and $\psi(x) = 0$ otherwise, where C is a constant chosen so that $\int_{\mathbb{R}} \psi(x)dx = 1$. For $f \in L^1(\mathbb{R})$ of compact support, show that $f * \psi$ is C^∞ and of compact support.

7.6. (i) Show that for any real numbers a, b one has $\int_a^b e^{itx} dx \to 0$ as $|t| \to \infty$.

(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$$

tends to 0 as $|t| \to \infty$. This is the *Riemann–Lebesgue Lemma.*

7.7. Say that $f \in L^2(\mathbb{R})$ is L^2-differentiable with L^2-derivative Df if

$$\|\tau_h f - f - hDf\|_2 / h \to 0 \quad \text{as} \quad h \to 0,$$

where $\tau_h f(x) = f(x + h)$. Show that the function $f(x) = \max(1 - |x|, 0)$ is L^2-differentiable and find its L^2-derivative.

Suppose that $f \in L^1 \cap L^2$ is L^2-differentiable. Show that $u\hat{f}(u) \in L^2$. Deduce that f has a continuous version and that $\|f\|_\infty \le C\|(1 + |u|)\hat{f}(u)\|_2$ for some absolute constant $C < \infty$, to be determined. This is a simple example of a *Sobolev inequality.*

Probability and Measure 4

7.8. Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables in $\mathbb{R}$ and let X be another such random variable. Show that $X_n \to X$ weakly if and only if $X_n \to X$ in distribution.

7.9. Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $(\mu_n : n \in \mathbb{N})$ be a sequence of such measures. Suppose that $\mu_n(f) \to \mu(f)$ for all C^∞ functions on $\mathbb{R}^d$ of compact support. Show that μ_n converges weakly to μ on $\mathbb{R}^d$.

8.1. Let $X = (X_1, \ldots, X_n)$ be a Gaussian random variable in $\mathbb{R}^n$ with mean μ and covariance matrix V. Assume that V is invertible write $V^{-1/2}$ for the positive-definite square root of V^{-1}. Set $Y = (Y_1, \ldots, Y_n) = V^{-1/2}(X - \mu)$. Show that $Y_1, \ldots, Y_n$ are independent $N(0, 1)$ random variables. Show further that we can write X_2 in the form $X_2 = aX_1 + Z$ where Z is independent of X_1 and determine the distribution of Z.

8.2. Let $X_1, \ldots, X_n$ be independent $N(0, 1)$ random variables. Show that

$$\left(\overline{X}, \sum_{m=1}^{n} (X_m - \overline{X})^2\right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2\right)$$

have the same distribution, where $\overline{X} = (X_1 + \cdots + X_n)/n$.

9.1. Let $(E, \mathcal{E}, \mu)$ be a measure space and $\tau : E \to E$ a measure-preserving transformation. Show that $\mathcal{E}_\tau := \{A \in \mathcal{E} : \tau^{-1}(A) = A\}$ is a σ-algebra, and that a measurable function f is $\mathcal{E}_\tau$-measurable if and only if it is *invariant*, that is $f \circ \tau = f$.

9.2. Show that, if θ is an ergodic measure-preserving transformation and f is a θ-invariant function, then there exists a constant $c \in \mathbb{R}$ such that $f = c$ a.e..

9.3. For $x \in [0, 1)$, set $\tau(x) = 2x \bmod 1$. Show that τ is a measure-preserving transformation of $([0, 1), \mathcal{B}([0, 1)), dx)$, and that τ is ergodic. Identify the invariant function $\overline{f}$ corresponding to each integrable function f.

9.4. Fix $a \in [0, 1)$ and define, for $x \in [0, 1)$, $\tau(x) = x + a \bmod 1$. Show that τ is also a measure-preserving transformation of $([0, 1), \mathcal{B}([0, 1)), dx)$. Determine for which values of a the transformation τ is ergodic. *Hint: you may use the fact that any integrable function f on $[0, 1)$ whose Fourier coefficients all vanish must itself vanish a.e..* Identify, for all values of a, the invariant function $\overline{f}$ corresponding to an integrable function f.

9.5. Call a sequence of random variables $(X_n : n \in \mathbb{N})$ on a probability space
$(\Omega, \mathcal{F}, \mathbb{P})$
stationary if for each $n, k \in \mathbb{N}$ the random vectors $(X_1, \dots, X_n)$ and $(X_{k+1}, \dots, X_{k+n})$ have the same distribution: for $A_1, \dots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \dots, X_{k+n} \in A_n).$$

Show that, if $(X_n : n \in \mathbb{N})$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to X \qquad \text{a.s. and in } L^p,$$

for some random variable $X \in L^p$ and find $\mathbb{E}(X)$.

10.1. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $\mathbb{E}(X_n) = \mu$ and $\mathbb{E}(X_n^4) \leq M$ for all n, for some constants $\mu \in \mathbb{R}$ and $M < \infty$. Set $P_n = X_1X_2 + X_2X_3 + \dots + X_{n-1}X_n$. Show that P_n/n converges a.s. as $n \to \infty$ and identify the limit.

10.2. The Cauchy distribution has density function $f(x) = \pi^{-1}(1+x^2)^{-1}$ for $x \in \mathbb{R}$. Show that the corresponding characteristic function is given by $\varphi(u) = e^{-|u|}$. Show also that, if $X_1, \dots, X_n$ are independent Cauchy random variables, then the random variable $(X_1 + \dots + X_n)/n$ is also Cauchy.

10.3. Let f be a bounded continuous function on $(0, \infty)$, having Laplace transform

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx, \quad \lambda \in (0, \infty).$$

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent exponential random variables, of parameter λ. Show that $\hat{f}$ has derivatives of all orders on $(0, \infty)$ and that, for all $n \in \mathbb{N}$, for some $C(\lambda, n) \neq 0$ independent of f, we have

$$(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n) \mathbb{E}\big(f(S_n)\big)$$

where $S_n = X_1 + \dots + X_n$. Deduce that if $\hat{f} \equiv 0$ then also $f \equiv 0$.

10.4. For each $n \in \mathbb{N}$, there is a unique probability measure μ_n on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ such that $\mu_n(A) = \mu_n(UA)$ for all Borel sets A and all orthogonal $n \times n$ matrices U. Fix $k \in \mathbb{N}$ and, for $n \geq k$, let γ_n denote the probability measure on $\mathbb{R}^k$ which is the law of $\sqrt{n}(x^1, \dots, x^k)$ under μ_n. Show

(a) if $X \sim N(0, I_n)$ then $X/|X| \sim \mu_n$,

(b) if $(X_n : n \in \mathbb{N})$ is a sequence of independent $N(0,1)$ random variables and if $R_n = \sqrt{X_1^2 + \dots + X_n^2}$ then $R_n/\sqrt{n} \to 1$ a.s.,

(c) γ_n converges weakly to the standard Gaussian distribution on $\mathbb{R}^k$ as $n \to \infty$.