Part II — Linear Analysis
Theorems

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Part IB Linear Algebra, Analysis II and Metric and Topological Spaces are essential

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tiezte's extension theorem. Spaces of continuous functions. The Stone-Weierstrass theorem and applications. Equicontinuity: the Ascoli-Arzelà theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz-Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Contents

0 Introduction

1 Normed vector spaces

Proposition. Addition $+ : V \times V \to V$, and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ are continuous with respect to the topology induced by the norm (and the usual product topology).

Proposition. If $(V, \|\cdot\|)$ is a normed vector space, then $B(t) = B(\mathbf{0}, t) = \{\mathbf{v} : \|\mathbf{v}\| < t\}$ is absolutely convex.

Proposition. A topological vector space $(V, \mathcal{U})$ is normable if and only if there exists an absolutely convex, bounded open neighbourhood of $\mathbf{0}$.

1.1 Bounded linear maps

Proposition. Let X, Y be normed vector spaces, $T : X \to Y$ a linear map. Then the following are equivalent:

(i) T is continuous.

(ii) T is continuous at $\mathbf{0}$.

(iii) T is bounded.

1.2 Dual spaces

Proposition. Let V be a normed vector space. Then V^* is a Banach space.

1.3 Adjoint

Proposition. T^* is bounded.

1.4 The double dual

Proposition. Let $\phi : V \to V^{**}$ be defined by $\phi(\mathbf{v})(g) = g(\mathbf{v})$. Then ϕ is a bounded linear map and $\|\phi\|_{\mathcal{B}(V,V^*)} \leq 1$

1.5 Isomorphism

1.6 Finite-dimensional normed vector spaces

Proposition. Let V be an n-dimensional vector space. Then all norms on V are equivalent to the norm $\|\cdot\|_{\ell_1^n}$.

Corollary. All norms on a finite-dimensional vector space are equivalent.

Proposition. Let V be a finite-dimensional normed vector space. Then the closed unit ball

$$\bar{B}(1) = \{\mathbf{v} \in V : \|\mathbf{v}\| \leq 1\}$$

is compact.

Proposition. Let V be a finite-dimensional normed vector space. Then V is a Banach space.

Proposition. Let V, W be normed vector spaces, V be finite-dimensional. Also, let $T : V \to W$ be a linear map. Then T is bounded.

Proposition. Let V be a normed vector space. Suppose that the closed unit ball $\bar{B}(1)$ is compact. Then V is finite dimensional.

1.7 Hahn–Banach Theorem

Proposition. Let V be a real normed vector space, and $W \subseteq V$ has co-dimension 1. Assume we have the following two items:

- $p : V \to \mathbb{R}$ (not necessarily linear), which is positive homogeneous, i.e.

  $$p(\lambda \mathbf{v}) = \lambda p(\mathbf{v})$$

  for all $\mathbf{v} \in V, \lambda > 0$, and subadditive, i.e.

  $$p(\mathbf{v}_1 + \mathbf{v}_2) \leq p(\mathbf{v}_1) + p(\mathbf{v}_2)$$

  for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. We can think of something like a norm, but more general.

- $f : W \to \mathbb{R}$ a linear map such that $f(\mathbf{w}) \leq p(\mathbf{w})$ for all $\mathbf{w} \in W$.

Then there exists an extension $\tilde{f} : V \to \mathbb{R}$ which is linear such that $\tilde{f}|_W = f$ and $\tilde{f}(\mathbf{v}) \leq p(\mathbf{v})$ for all $\mathbf{v} \in V$.

Lemma (Zorn's lemma)**.** Let $(S, \leq)$ be a non-empty partially ordered set such that every totally-ordered subset S' has an upper bound in S. Then S has a maximal element.

Theorem (Hahn–Banach theorem*)**.** Let V be a real normed vector space, and $W \subseteq V$ a subspace. Assume we have the following two items:

- $p : V \to \mathbb{R}$ (not necessarily linear), which is positive homogeneous and subadditive;
- $f : W \to \mathbb{R}$ a linear map such that $f(\mathbf{w}) \leq p(\mathbf{w})$ for all $\mathbf{w} \in W$.

Then there exists an extension $\tilde{f} : V \to \mathbb{R}$ which is linear such that $\tilde{f}|_W = f$ and $\tilde{f}(\mathbf{v}) \leq p(\mathbf{v})$ for all $\mathbf{v} \in V$.

Corollary (Hahn-Banach theorem 2.0)**.** Let $W \subseteq V$ be real normed vector spaces. Given $f \in W^*$, there exists a $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$ and $\|\tilde{f}\|_{V^*} = \|f\|_{W^*}$.

Proposition. Let V be a real normed vector space. For every $\mathbf{v} \in V \setminus \{0\}$, there is some $f_{\mathbf{v}} \in V^*$ such that $f_{\mathbf{v}}(\mathbf{v}) = \|\mathbf{v}\|_V$ and $\|f_{\mathbf{v}}\|_{V^*} = 1$.

Corollary. Let V be a real normed vector space. Then $\mathbf{v} = \mathbf{0}$ if and only if $f(\mathbf{v}) = 0$ for all $f \in V^*$.

Corollary. Let V be a non-trivial real normed vector space, $\mathbf{v}, \mathbf{w} \in V$ with $\mathbf{v} \neq \mathbf{w}$. Then there is some $f \in V^*$ such that $f(\mathbf{v}) \neq f(\mathbf{w})$.

Corollary. If V is a non-trivial real normed vector space, then V^* is non-trivial.

Proposition. The map $\phi : V \to V^{**}$ is an isometry, i.e. $\|\phi(\mathbf{v})\|_{V^{**}} = \|\mathbf{v}\|_V$.

Proposition.

$$\|T^*\|_{\mathcal{B}(W^*, V^*)} = \|T\|_{\mathcal{B}(V, W)}.$$

2 Baire category theorem

2.1 The Baire category theorem

Theorem (Baire category theorem)**.** Let X be a complete metric space. Then X is of second category.

2.2 Some applications

Proposition. $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$, i.e. there is an irrational number.

Proposition. Let $\hat{\ell}_1$ be a normed vector space defined by the vector space

$$V = \{(x_1, x_2, \cdots) : x_i \in \mathbb{R}, \exists I \in \mathbb{N} \text{ such that } i > I \Rightarrow x_i = 0\},$$

with componentwise addition and scalar multiplication. This is the space of all sequences that are eventually zero.

We define the norm by

$$\|x\|_{\hat{\ell}_1} = \sum_{i=1}^{\infty} |x_i|.$$

Then $\hat{\ell}_1$ is not a Banach space.

Proposition. There exists an $f \in C([0,1])$ which is nowhere differentiable.

Theorem (Banach-Steinhaus theorem/uniform boundedness principle)**.** Let V be a Banach space and W be a normed vector space. Suppose T_α is a collection of bounded linear maps $T_\alpha : V \to W$ such that for each fixed $\mathbf{v} \in V$,

$$\sup_\alpha \|T_\alpha(\mathbf{v})\|_W < \infty.$$

Then

$$\sup_\alpha \|T_\alpha\|_{\mathcal{B}(V,W)} < \infty.$$

Theorem (Osgood)**.** Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that for all $x \in [0,1]$

$$\sup_n |f_n(x)| < \infty$$

Then there are some a, b with $0 \leq a < b \leq 1$ such that

$$\sup_n \sup_{x \in [a,b]} |f_n(x)| < \infty.$$

Theorem (Open mapping theorem)**.** Let V and W be Banach spaces and $T : V \to W$ be a bounded surjective linear map. Then T is an open map, i.e. $T(U)$ is an open subset of W whenever U is an open subset of V.

Theorem (Inverse mapping theorem)**.** Let V, W be Banach spaces, and $T : V \to W$ be a bounded linear map which is both injective and surjective. Then T^{-1} exists and is a bounded linear map.

Theorem (Closed graph theorem)**.** Let V, W be Banach spaces, and $T : V \to W$ a linear map. If the graph of T is closed, i.e.

$$\Gamma(T) = \{(\mathbf{v}, T(\mathbf{v})) : \mathbf{v} \in V\} \subseteq V \times W$$

is a closed subset of the product space (using the norm $\|(\mathbf{v}, \mathbf{w})\|_{V \times W} = \max\{\|\mathbf{v}\|_V, \|\mathbf{w}\|_W\}$), then T is bounded.

3 The topology of $C(K)$

3.1 Normality of compact Hausdorff spaces

Theorem. Let X be a Hausdorff space. If $C_1, C_2 \subseteq X$ are *compact* disjoint subsets, then there are some $U_1, U_2 \subseteq X$ disjoint open such that $C_1 \subseteq U_1, C_2, \subseteq U_2$.

In particular, if X is a compact Hausdorff space, then X is normal (since closed subsets of compact spaces are compact).

3.2 Tietze-Urysohn extension theorem

Lemma (Urysohn's lemma)**.** Let X be normal and C_0, C_1 be disjoint closed subsets of X. Then there is a $f \in C(X)$ such that $f|_{C_0} = 0$ and $f|_{C_1} = 1$, and $0 \leq f(x) \leq 1$ for all X.

Theorem (Tietze-Urysohn extension theorem)**.** Let X be a normal topological space, and $C \subseteq X$ be a closed subset. Suppose $f : C \to \mathbb{R}$ is a continuous function. Then there exists an extension $\tilde{f} : X \to \mathbb{R}$ which is continuous and satisfies $\tilde{f}|_C = f$ and $\|\tilde{f}\|_{C(X)} = \|f\|_{C(C)}$.

3.3 Arzelà-Ascoli theorem

Theorem (Arzelà-Ascoli theorem)**.** Let K be a compact topological space. Then $F \subseteq C(K)$ is pre-compact, i.e. $\bar{F}$ is compact, if and only if F is bounded and equicontinuous.

Proposition. Let X be a complete metric space. Then $E \subseteq X$ is totally bounded if and only if for every sequence $\{y_i\}_{i=1}^\infty \subseteq E$, there is a subsequence which is Cauchy.

Corollary. Let X be a complete metric space. Then $E \subseteq X$ is totally bounded if and only if $\bar{E}$ is compact.

Theorem (Arzelà-Ascoli theorem)**.** Let K be a compact topological space. Then $F \subseteq C(K)$ is pre-compact, i.e. $\bar{F}$ is compact, if and only if F is bounded and equicontinuous.

Proposition. Let X be a (complete) metric space. Then $E \subseteq X$ is totally bounded if and only if for every sequence $\{y_i\}_{i=1}^\infty \subseteq E$, there is a subsequence which is Cauchy.

Theorem (Peano*)**.** Given f continuous, then there is some $\varepsilon > 0$ such that $x' = f(x)$ with boundary condition $x(0) = x_0 \in \mathbb{R}$ has a solution in $(-\varepsilon, \varepsilon)$.

3.4 Stone–Weierstrass theorem

Theorem (Weierstrass approximation theorem)**.** The set of polynomials are dense in $C([0,1])$.

Theorem (Stone-Weierstrass theorem)**.** Let K be compact, and $\mathcal{A} \subseteq C_{\mathbb{R}}(K)$ be a subalgebra (i.e. it is a subset that is closed under the operations) with the property that it separates points, i.e. for every $x, y \in K$ distinct, there exists

some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then either $\bar{\mathcal{A}} = C_{\mathbb{R}}(K)$ or there is some $x_0 \in K$ such that
$$\bar{\mathcal{A}} = \{f \in C_{\mathbb{R}(K)} : f(x_0) = 0\}.$$

Lemma. Let K compact, $\mathcal{L} \subseteq C_{\mathbb{R}}(K)$ be a subset which is closed under taking maximum and minimum, i.e. if $f, g \in \mathcal{L}$, then $\max\{f, g\} \in \mathcal{L}$ and $\min\{f, g\} \in \mathcal{L}$ (with $\max\{f, g\}$ defined as $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, and similarly for minimum).

Given $g \in C_{\mathbb{R}}(K)$, assume further that for any $\varepsilon > 0$ and $x, y \in K$, there exists $f_{x,y} \in \mathcal{L}$ such that
$$|f_{x,y}(x) - g(x)| < \varepsilon, \quad |f_{x,y}(y) - g(y)| < \varepsilon.$$
Then there exists some $f \in \mathcal{L}$ such that
$$\|f - g\|_{C_{\mathbb{R}}(K)} < \varepsilon,$$
i.e. $g \in \bar{\mathcal{L}}$.

Lemma. Let $\mathcal{A} \subseteq C_{\mathbb{R}}(K)$ be a subalgebra that is a closed subset in the topology of $C_{\mathbb{R}}(K)$. Then $\mathcal{A}$ is closed under taking maximum and minimum.

Theorem (Stone-Weierstrass theorem)**.** Let K be compact, and $\mathcal{A} \subseteq C_{\mathbb{R}}(K)$ be a subalgebra (i.e. it is a subset that is closed under the operations) with the property that it separates points, i.e. for every $x, y \in K$ distinct, there exists some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Then either $\bar{\mathcal{A}} = C_{\mathbb{R}}(K)$ or there is some $x_0 \in K$ such that
$$\bar{\mathcal{A}} = \{f \in C_{\mathbb{R}(K)} : f(x_0) = 0\}.$$

Theorem (Complex version of Stone-Weierstrass theorem)**.** Let K be compact and $\mathcal{A} \subseteq C_{\mathbb{C}}(K)$ be a subalgebra over $\mathbb{C}$ which separates points and is closed under complex conjugation (i.e. if $f \in \mathcal{A}$, then $\bar{f} = \mathcal{A}$). Then either $\bar{\mathcal{A}} = C_{\mathbb{C}}(K)$ or these is an x_0 such that $\bar{\mathcal{A}} = \{f \in C_{\mathbb{C}}(K) : f(x_0) = 0\}$.

4 Hilbert spaces

4.1 Inner product spaces

Proposition. Let $f \in C(S^1)$. Then

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2\;\mathrm{d}x = 0.$$

Proposition (Cauchy-Schwarz inequality)**.** Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ be an inner product space. Then for all $\mathbf{v}, \mathbf{w} \in V$,

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle},$$

with equality iff there is some $\lambda \in \mathbb{R}$ or $\mathbb{C}$ such that $\mathbf{v} = \lambda\mathbf{w}$ or $\mathbf{w} = \lambda\mathbf{v}$.

Proposition. Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ be an inner product space. Then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

defines a norm.

Proposition. Let $(E, \|\cdot\|)$ be a Euclidean space. Then there is a *unique* inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ such that $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$.

Proposition (Parallelogram law)**.** Let $(E, \|\cdot\|)$ be a Euclidean space. Then for $\mathbf{v}, \mathbf{w} \in E$, we have

$$\|\mathbf{v} - \mathbf{w}\|^2 + \|\mathbf{v} + \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2.$$

Proposition (Pythagoras theorem)**.** Let $(E, \|\cdot\|)$ be a Euclidean space, and let $\mathbf{v}, \mathbf{w} \in E$ be orthogonal. Then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$

Proposition. Let $(E, \|\cdot\|)$ be a Euclidean space. Then $\langle\,\cdot\,,\,\cdot\,\rangle : E \times E \to \mathbb{C}$ is continuous.

Proposition. Let $(E, \|\cdot\|)$ denote a Euclidean space, and $\bar{E}$ its completion. Then the inner product extends to an inner product on $\bar{E}$, turning $\bar{E}$ into a Hilbert space.

Proposition. Let E be a Euclidean space and $S \subseteq E$. Then $S^\perp$ is a closed subspace of E, and moreover

$$S^\perp = (\overline{\operatorname{span} S})^\perp.$$

Theorem. Let $(E, \|\cdot\|)$ be a Euclidean space, and $F \subseteq E$ a *complete* subspace. Then $F \oplus F^\perp = E$.

Hence, by definition of the direct sum, for $\mathbf{x} \in E$, we can write $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 \in F$ and $\mathbf{x}_2 \in F^\perp$. Moreover, $\mathbf{x}_1$ is uniquely characterized by

$$\|\mathbf{x}_1 - \mathbf{x}\| = \inf_{\mathbf{y}\in F} \|\mathbf{y} - \mathbf{x}\|.$$

Corollary. Let E be a Euclidean space and $F \subseteq E$ a complete subspace. Then there exists a projection map $P : E \to E$ defined by $P(\mathbf{x}) = \mathbf{x}_1$, where $\mathbf{x}_1 \in F$ is as defined in the theorem above. Moreover, P satisfies the following properties:

(i) $P(E) = F$ and $P(F^\perp) = \{0\}$, and $P^2 = P$. In other words, $F^\perp \leq \ker P$.

(ii) $(I - P)(E) = F^\perp$, $(I - P)(F) = \{0\}$, $(I - P)^2 = (I - P)$.

(iii) $\|P\|_{\mathcal{B}(E,E)} \leq 1$ and $\|I - P\|_{\mathcal{B}(E,E)} \leq 1$, with equality if and only if $F \neq \{0\}$ and $F^\perp \neq \{0\}$ respectively.

4.2 Riesz representation theorem

Proposition (Riesz representation theorem)**.** Let H be a Hilbert space. Then $\phi : H \to H^*$ defined by $\mathbf{v} \mapsto \langle \cdot, \mathbf{v}\rangle$ is an isometric anti-isomorphism, i.e. it is isometric, bijective and

$$\phi(\lambda \mathbf{v} + \mu \mathbf{w}) = \bar{\lambda}\phi(\mathbf{v}) + \bar{\mu}\phi(\mathbf{v}).$$

Proposition. For $f \in C(S^1)$, defined, for each $k \in \mathbb{Z}$,

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} f(x)\;\mathrm{d}x.$$

The partial sums are then defined as

$$S_N(f)(x) = \sum_{n=-N}^{N} e^{inx}\hat{f}(k).$$

Then we have

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2\;\mathrm{d}x = 0.$$

4.3 Orthonormal systems and basis

Proposition. Let H be a Hilbert space. Let S be a maximal orthonormal system. Then $\overline{\operatorname{span} S} = H$.

Proposition. Let E be Euclidean, and let S be an orthonormal system. If $\overline{\operatorname{span} S} = E$, then S is maximal.

Proposition. Let $\{\mathbf{x}_i\}_{i=1}^n$, $n \in \mathbb{N}$ be linearly independent. Then there exists $\{\mathbf{e}_i\}_{i=1}^n$ such that $\{\mathbf{e}_i\}_{i=1}^n$ is an orthonormal system and

$$\operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_j\} = \operatorname{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_j\}$$

for all $j \leq n$.

Proposition. Let H be separable, i.e. there is an infinite set $\{\mathbf{y}_i\}_{i\in\mathbb{N}}$ such that

$$\overline{\operatorname{span}\{\mathbf{y}_i\}} = H.$$

Then there exists a countable basis for $\operatorname{span}\{\mathbf{y}_i\}$.

4.4 The isomorphism with ℓ_2

Lemma (Bessel's inequality)**.** Let E be Euclidean and $\{\mathbf{e}_i\}_{i=1}^N$ with $N \in \mathbb{N}\cup\{\infty\}$ an orthonormal system. For any $\mathbf{x} \in E$, define $x_i = \langle \mathbf{x}, \mathbf{e}_i\rangle$. Then for any $j \leq N$, we have

$$\sum_{i=1}^{j} |x_i|^2 \leq \|\mathbf{x}\|^2.$$

Proposition. Let H be a separable Hilbert space, with a countable basis $\{\mathbf{e}_i\}_{i=1}^N$, where $N \in \mathbb{N} \cup \{\infty\}$. Let $\mathbf{x}, \mathbf{y} \in H$ and

$$x_i = \langle \mathbf{x}, \mathbf{e}_i\rangle, \quad y_i = \langle \mathbf{y}, \mathbf{e}_i\rangle.$$

Then

$$\mathbf{x} = \sum_{i=1}^{N} x_i \mathbf{e}_i, \quad \mathbf{y} = \sum_{i=1}^{N} y_i \mathbf{e}_i,$$

and

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{N} x_i \bar{y}_i.$$

Moreover, the sum converges absolutely.

Proposition. Let H be a separable Hilbert space with orthonormal basis $\{\mathbf{e}_i\}_{i\in\mathbb{N}}$. Let $\{a_i\}_{i\in\mathbb{N}} \in \ell_2(\mathbb{C})$. Then there exists an $\mathbf{x} \in H$ with $\langle \mathbf{x}, \mathbf{e}_i\rangle = a_i$. Moreover, this $\mathbf{x}$ is exactly

$$\mathbf{x} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i.$$

4.5 Operators

Theorem. Let X be a Banach space, $T \in B(X)$. Then $\sigma(T)$ is a non-empty, closed subset of

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|_{\mathcal{B}(X)}\}.$$

Lemma. Let X be a Banach space, $T \in \mathcal{B}(X)$ and $\|T\|_{\mathcal{B}(X)} < 1$. Then $I - T$ is invertible.

Lemma. Let X be a Banach space, $S_1 \in \mathcal{B}(X)$ be invertible. Then for all $S_2 \in \mathcal{B}(X)$ such that

$$\|S_1^{-1}\|_{\mathcal{B}(X)}\|S_1 - S_2\|_{\mathcal{B}(X)} < 1,$$

S_2 is invertible.

Theorem. Let X be a Banach space, $T \in \mathcal{B}(X)$. Then $\sigma(T)$ is a non-empty, closed subset of

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|_{\mathcal{B}(X)}\}.$$

Proposition (Liouville's theorem for Banach space-valued analytic function)**.** Let X be a Banach space, and $F : \mathbb{C} \to X$ be entire (in the sense that F is given by an absolutely convergent power series in some neighbourhood of any point) and norm bounded, i.e.

$$\sup_{z\in\mathbb{C}} \|F(z)\|_X < \infty.$$

Then F is constant.

Theorem. We have
$$\sigma_{ap}(T) \supseteq \partial\sigma(T),$$
where $\partial\sigma(T)$ is the boundary of $\sigma(T)$ in the topology of $\mathbb{C}$. In particular, $\sigma_{ap}(T) \neq \emptyset$.

Proposition. Let X, Y be Banach spaces. Then $T \in \mathcal{L}(X, Y)$ is compact if and only if $T(B(1))$ is totally bounded if and only if $\overline{T(B(1))}$ is compact.

Proposition. Let X be a Banach space. Then $\mathcal{B}_0(X)$ is a closed subspace of $\mathcal{B}(X)$. Moreover, if $T \in \mathcal{B}_0(X)$ and $S \in \mathcal{B}(X)$, then $TS, ST \in \mathcal{B}_0(X)$.

In a more algebraic language, this means $\mathcal{B}_0(X)$ is a closed ideal of the algebra $\mathcal{B}(X)$.

Theorem. Let X be an infinite-dimensional Banach space, and $T \in \mathcal{B}(X)$ be a compact operator. Then $\sigma_p(T) = \{\lambda_i\}$ is at most countable. If $\sigma_p(T)$ is infinite, then $\lambda_i \to 0$.

The spectrum is given by $\sigma(T) = \sigma_p(T) \cup \{0\}$. Moreover, for every non-zero $\lambda_i \in \sigma_p(T)$, the eigenspace has finite dimensions.

Proposition. Let H be a Hilbert space, and $T \in \mathcal{B}_0(H)$ a compact operator. Let $a > 0$. Then there are only finitely many linearly independent eigenvectors whose eigenvalue have magnitude $\geq a$.

Lemma. Let H be a Hilbert space, and $T \in \mathcal{B}(H)$ compact. Then $\operatorname{im}(I - T)$ is closed.

Proposition. Let H be a Hilbert space, $T \in \mathcal{B}(H)$ compact. If $\lambda \neq 0$ and $\lambda \in \sigma(T)$, then $\lambda \in \sigma_p(T)$.

Theorem. Let H be an infinite-dimensional Hilbert space, and $T \in \mathcal{B}(H)$ be a compact operator. Then $\sigma_p(T) = \{\lambda_i\}$ is at most countable. If $\sigma_p(T)$ is infinite, then $\lambda_i \to 0$.

The spectrum is given by $\sigma(T) = \sigma_p(T) \cup 0$. Moreover, for every non-zero $\lambda_i \in \sigma_p(T)$, the eigenspace has finite dimensions.

4.6 Self-adjoint operators

Theorem (Spectral theorem)**.** Let H be an infinite dimensional Hilbert space and $T : H \to H$ a compact self-adjoint operator.

(i) $\sigma_p(T) = \{\lambda_i\}_{i=1}^N$ is at most countable.

(ii) $\sigma_p(T) \subseteq \mathbb{R}$.

(iii) $\sigma(T) = \{0\} \cup \sigma_p(T)$.

(iv) If E_{λ_i} are the eigenspaces, then $\dim E_{\lambda_i}$ is finite if $\lambda_i \neq 0$.

(v) $E_{\lambda_i} \perp E_{\lambda_j}$ if $\lambda_i \neq \lambda_j$.

(vi) If $\{\lambda_i\}$ is infinite, then $\lambda_i \to 0$.

(vii)
$$T = \sum_{i=1}^N \lambda_i P_{E_{\lambda_i}}.$$

Proposition. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ self-adjoint. Then $\sigma_p(T) \subseteq \mathbb{R}$.

Proposition. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ self-adjoint. If $\lambda, \mu \in \sigma_p(T)$ and $\lambda \neq \mu$, then $E_\lambda \perp E_\mu$.

Proposition. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a *compact* self-adjoint operator. If $T \neq 0$, then T has a non-zero eigenvalue.

Lemma. Let H be a Hilbert space, and $T \in \mathcal{B}(H)$ a compact self-adjoint operator. Then

$$\|T\|_{\mathcal{B}(H)} = \sup_{\|\mathbf{x}\|_H = 1} |\langle \mathbf{x}, T\mathbf{x}\rangle|$$

Proposition. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a *compact* self-adjoint operator. If $T \neq 0$, then T has a non-zero eigenvalue.

Proposition. Let H be an infinite dimensional Hilbert space and $T : H \to H$ a compact self-adjoint operator. Then

$$T = \sum_{i=1}^{N} \lambda_i P_{E_{\lambda_i}}.$$