EXAMPLE SHEET 1: LINEAR ANALYSIS

(1) Prove the Hölder's inequality, i.e., for every $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ (using the convention that if $p = 1$, then $q = \infty$) and for every real-valued sequences $a = (a_1, ..., a_n, ...)$ and $b = (b_1, ..., b_n, ...)$ which are elements of ℓ_p and ℓ_q respectively, the following holds:

$$\|ab\|_{\ell_1} \leq \|a\|_{\ell_p} \|b\|_{\ell_q}.$$

(You may find it helpful to first show that for every $0 \leq t \leq 1$, $a \geq 0$ and $b \geq 0$, the following holds: $a^t b^{1-t} \leq ta + (1-t)b$.)

(2) Prove the Minkowski inequality, i.e., for every $p \in [1, \infty]$ and for every real-valued sequences $a = (a_1, ..., a_n, ...)$ and $b = (b_1, ..., b_n, ...)$ which are both elements of ℓ_p, the following holds:

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p.$$

(You may find Hölder's inequality helpful here.) Using this, show that ℓ_p is indeed a normed vector space.

(3) Show that ℓ_p is complete for every $p \in [1, \infty]$.

(4) Let $p, q \in [1, \infty]$. Show that $\ell_p \subset \ell_q$ if and only if $p \leq q$.

(5) Show that for any $p \in [1, \infty)$, ℓ_p^* is isometrically isomorphic to ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$ (using the convention that if $p = 1$, then $q = \infty$).

(6) Let c_0 be the space of all real-valued sequences converging to 0, endowed with the ℓ_∞ norm. Show that c_0^* is isometrically isomorphic to ℓ_1.

(7) Show that a normed vector space is complete if and only if all absolutely convergent series are convergent, i.e., the following statement holds: if $\lim_{N\to\infty} \sum_{n=1}^N \|x_n\| < \infty$, then $\sum_{n=1}^N x_n$ is convergent as $N \to \infty$.

(8) Let X be a normed vector space, and $T : X \to X$, $S : X \to X$ be bounded linear maps. Show that the composition $T \circ S$ is bounded and the following holds:

$$\|T \circ S\| \leq \|T\| \|S\|.$$

Does equality always hold?

(9) Let X and Y be normed vector spaces. Find conditions on X and Y which guarantee that $\mathcal{B}(X, Y)$ is also complete.

(10) Show that in a finite dimensional normed vector space, a subset is compact if and only if it is closed and bounded. (This should be an easy consequence of what was proven in lectures.) In general, let X be a Banach space. Prove that a subset $S \subset X$ is compact if and only if it is closed and totally bounded, i.e., for every $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ whose union contains S.

(11) As an application of the previous problem, show that for every sequence of real numbers x_n with $|x_n| \to 0$, the following subset of c_0:

$$\{y \in c_0 : |y_n| \leq |x_n|\}$$

is compact.

(12) We have shown in the lectures that all linear functionals on a finite dimensional normed space are bounded. Prove that this also characterizes finite dimensional normed vector spaces, i.e., given an infinite dimensional normed vector space X, show that there exists an unbounded linear functional. (You may use the axiom of choice here. More precisely, it can be shown using the Zorn's lemma that X has a *Hamel basis*, i.e. a subset of X: $\{x_\gamma \in X : \gamma \in \Gamma\}$ such that every $x \in X$ can be represented uniquely as a finite linear combination of x_γ.)

EXAMPLE SHEET 2: LINEAR ANALYSIS

(1) Let $p \in [1, \infty)$ and $x \in \ell_p$. Show that there is an $f \in \ell_p^*$ such that $f(x) = \|x\|_{\ell_p}$ and $\|f\|_{\ell_p^*} \leq 1$.

(2) Let X be a normed vector space over $\mathbb{F}$ and $f : X \to \mathbb{F}$ be a linear map. Prove that f is continuous if and only if $ker(f)$ is closed.

(3) Show that $\hat{L}_p([0,1]) := \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$ with the norm given by

$$\|f\|_{\hat{L}_p([0,1])} := (\int_0^1 |f|^p dx)^{\frac{1}{p}}$$

is not complete.

(4) Let V be the space of polynomials on $\mathbb{R}$. Does there exists a norm $\|\cdot\|$ on V such that $(V, \|\cdot\|)$ is complete?

(5) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$. If for every $t \in [0,1]$, $\sup_n |f_n(t)|$ is finite, show that there is an interval $[a,b]$ with $a < b$ such that $\sup_n \sup_{t \in [a,b]} |f_n(t)| < \infty$. (Osgood Theorem)

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $x > 0$, we have $f(nx) \to 0$ as $n \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

(7) Show that the set of all rational numbers $\mathbb{Q}$ is not a G_δ set, i.e., it is not a countable intersection of open subsets of $\mathbb{R}$.

(8) Does there exist a function $f : [0,1] \to \mathbb{R}$ which is continuous at every rational number and discontinuous at every irrational number? (Hint: You may find the previous problem useful.)

(9) In this problem, we study the Fourier series and its convergence. Define the operator $\hat{}: C(\mathbb{S}^1) \to \tilde{c}_0$ so that for every $k \in \mathbb{Z}$,

$$\hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Define also $S_n : C(\mathbb{S}^1) \to C(\mathbb{S}^1)$ so that $S_n(f)$ is the n-th partial sum of the Fourier series of f given by the following formula:

$$S_n(f)(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}.$$

Here $\tilde{c}_0$ is defined[1] as $\tilde{c}_0 := \{g : \mathbb{Z} \to \mathbb{C} : |g(n)| \to 0 \text{ as } n \to \pm\infty\}$ endowed with the sup norm, i.e.,

$$\|g\| := \sup_{n \in \mathbb{Z}} |\hat{g}(n)|.$$

A basic question that we investigate here is whether $S_n(f)$ converges as $n \to \infty$.

[1]Notice that this is slightly different from the usual c_0!

(a) Show that the image of $\hat{}$ indeed lies in $\tilde{c}_0$. In other words, show that for every $f \in C(\mathbb{S}^1)$, we have $|\hat{f}(k)| \to 0$. (You can use that continuously differentiable functions are dense in $C(\mathbb{S}^1)$. This will be proven later in the course.)

(b) Prove the following formula:

$$S_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt,$$

where D_n is defined by

$$D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}.$$

(c) Let $\phi_n \in C(\mathbb{S}^1)^*$ be defined as $\phi_n(f) := S_n(f)(0)$. Show that for every n, $\|\phi_n\|_{C(\mathbb{S}^1)^*} < \infty$, but $\sup_n \|\phi_n\|_{C(\mathbb{S}^1)^*} = \infty$.

(d) Deduce that there exists a function $f \in C(\mathbb{S}^1)$ whose Fourier series diverges at 0, i.e., $S_n(f)(0)$ does not have a finite limit as $n \to \infty$.

(10) This problem, which continues the discussions on Fourier series, is intended for students who have learnt measure theory.

(a) Show that we can in fact define $\hat{}: L^1(\mathbb{S}^1) \to \tilde{c}_0$, i.e., for every $f \in L^1(\mathbb{S}^1)$, we have $|\hat{f}(k)| \to 0$ as $k \to \pm\infty$. (Here, $L^1(\mathbb{S}^1)$ is defined to be the space of (equivalent classes[2] of) Lebesgue measurable functions with $\int_{-\pi}^{\pi} |f|(t)dt < \infty$, with the norm $\|f\|_{L^1(\mathbb{S}^1)} := \int_{-\pi}^{\pi} |f|(t)dt$. This makes $L^1(\mathbb{S}^1)$ into a Banach space.)

(b) Show that the map $\hat{}$ is bounded and injective.

(c) On the other hand, prove that $\hat{}$ is not surjective. (Hint: Prove that D_n defined in the previous problem has the property that $\|D_n\|_{L^1} \to \infty$ as $n \to \infty$ but $\|\hat{D}_n\|_{\tilde{c}_0} = 1$.)

(11) Let $f : [0,1] \to \mathbb{R}$ be a pointwise limit of a sequence of continuous functions. Show that f has a point of continuity.

[2] where two functions are equivalent if they agree except on a measure 0 set.

EXAMPLE SHEET 3: LINEAR ANALYSIS

(1) Using the Tietze-Urysohn extension theorem for real-valued functions proved in the lectures, show the following analogue for complex-valued functions: Let X be a normal space, $C \subset X$ be a closed subset and $f : C \to \mathbb{C}$ be a continuous function. Then there exists a continuous extension $\tilde{f} : X \to \mathbb{C}$ such that $\tilde{f} \restriction_C = f$ and $\|\tilde{f}\|_{C_{\mathbb{C}}(X)} = \|f\|_{C_{\mathbb{C}}(C)}$.

(2) Let K be a compact Hausdorff space and $f \in C_{\mathbb{R}}(K)$. Show that there exists $\phi \in (C_{\mathbb{R}}(K))^*$ such that $\phi(f) = \|f\|_{C_{\mathbb{R}}(K)}$ and $\|\phi\|_{(C_{\mathbb{R}}(K))^*} = 1$.

(3) Let K be a compact Hausdorff space. Show that $C_{\mathbb{R}}(K)$ is finite dimensional if and only if K is finite.

(4) Let X be a normal space and S be a subset of X. Show that there is a continuous function $f : X \to \mathbb{R}$ such that $S = f^{-1}(\{0\})$ if and only if S is a closed $\mathcal{G}_\delta$ set, i.e., S is a closed set which is also a countable intersection of open sets.

(5) Let $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a continuous non-negative function such that $g(x) \to 0$ as $x \to \pm\infty$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions which are equicontinuous and such that $|f_n(x)| \leq g(x)$ for every $x \in \mathbb{R}$. Show that there exists a subsequence of f_n which converges uniformly on $\mathbb{R}$.

(6) Let X and Y be compact Hausdorff spaces. Let $\mathcal{A}$ be the algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C_{\mathbb{R}}(X)$ and $h \in C_{\mathbb{R}}(Y)$. Show that $\mathcal{A}$ is dense in $C_{\mathbb{R}}(X \times Y)$.

(7) Given $f, g \in C_{\mathbb{R}}(\mathbb{T})$, define the convolution by $f \star g : \mathbb{T} \to \mathbb{R}$

$$(f \star g)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y)g(y)\, dy.$$

Show that this makes $C_{\mathbb{R}}(\mathbb{T})$ with the usual $\|\cdot\|_{C_{\mathbb{R}}(\mathbb{T})}$ norm a Banach algebra. Is it commutative? Is it unital?

(8) Prove the following statement that was used in the proof of the Stone-Weierstrass theorem: For every $\epsilon > 0$, the Taylor series of the function $\sqrt{\epsilon^2 + x}$ about $x = \frac{1}{2}$ converges uniformly for $x \in [0, 1]$.

(9) Show that a normed space E is Euclidean if and only if the parallelogram law holds, i.e., if for every $v, w \in E$,

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

(10) Let X be an inner product space and $T : X \to X$ be a linear map. Prove that $< Tx, Ty >=< x, y >$ for all $x, y \in X$ if and only if $\|Tx\| = \|x\|$ for all $x \in X$.

(11) Let X be a complex inner product space and $T : X \to X$ be a linear map. Prove that if $< Tx, x >= 0$ for every $x \in X$, then $T = 0$. Show by an example that the same statement does not hold if X is a real inner product space.

(12) Show that ℓ_p is a Hilbert space if and only if $p = 2$.

(13) Consider the complex-valued continuous functions $C_{\mathbb{C}}([0,1])$ and define an inner product by

$$< f, g > := \int_0^1 f(x)\overline{g}(x)\, dx.$$

Show that this indeed defines an inner product and moreover that it does **not** make $C_{\mathbb{C}}([0,1])$ into a Hilbert space.

(14) Construct a Euclidean space E and a closed subspace F such that $F + F^{\perp} \neq E$.

EXAMPLE SHEET 4: LINEAR ANALYSIS

(1) Let $L_2(\mathbb{S}^1)$ be the Hilbert space which is defined to be the completion of $C(\mathbb{S}^1)$ under the inner product

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\bar{g}(x)\, dx.$$

Show that $\{e^{ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{S}^1)$. Prove the Riemann-Lebesgue Lemma: Let $f \in L_2(\mathbb{S}^1)$. Then

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx = 0.$$

(2) Let Y be a closed subspace of a Hilbert space H. Show that $Y^{\perp\perp} = Y$. Deduce that if S is a subset of H, then $S^{\perp\perp} = \overline{span(S)}$. Show that $span(S)$ is dense in H if and only if $S^{\perp} = \{0\}$.

(3) Let X be a Banach space and $T \in \mathcal{B}(X)$. Define

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \text{there exists a sequence } \{x_n\}_{n=1}^{\infty} \subset X \text{ such that } \|x_n\|_X = 1 \text{ and } \lim_{n\to\infty} \|(T - \lambda I)x_n\|_X \to 0\}$$

and

$$\sigma_{com}(T) := \{\lambda \in \mathbb{C} : Im(T - \lambda I) \text{ is not dense in } X\}.$$

Prove that

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{com}(T).$$

(4) Let $\{a_n\}_{n=1}^{\infty} \in \ell_\infty$. Define $T : \ell_2 \to \ell_2$ by $T(x_1, x_2, ...) := (a_1x_1, a_2x_2, ...)$. Show that $T \in \mathcal{B}(H)$ and $\|T\|_{\mathcal{B}(H)} = \|(a_1, a_2, ...)\|_{\ell_\infty}$. Find $\sigma(T)$, $\sigma_{ap}(T)$ and $\sigma_p(T)$. Show that T is compact if and only if $\{a_n\}_{n=1}^{\infty} \in c_0$, i.e., $|a_n| \to 0$ as $n \to \infty$.

(5) Let K be a non-empty compact subset of $\mathbb{C}$. Show that there is a Hilbert space H and a bounded linear operator T such that $K = \sigma(T)$. (Hint: You may find the previous problem helpful.)

(6) Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Let r be a complex rational function with no poles in $\sigma(T)$. Show that $\sigma(r(T)) = \{r(\lambda) : \lambda \in \sigma(T)\}$.

(7) Let H be a Hilbert space and $\{e_i\}_{i=1}^{\infty} \subset H$ be an orthonormal basis. Let $T \in \mathcal{B}(H)$. Define the Hilbert-Schmidt norm of T to be

$$\|T\|_{HS} := \Big(\sum_{i=1}^{\infty} \|Te_i\|_H^2\Big)^{\frac{1}{2}}.$$

Show that if the Hilbert-Schmidt norm of T is finite, then T is compact.

(8) Let H be a Hilbert space. We say that an operator $T \in \mathcal{B}(H)$ is *normal* if $TT^* = T^*T$. Prove that if T is normal, then $\|Tx\|_H = \|T^*x\|_H$ for every $x \in H$. Conclude that $ker(T) = ker(T^*) = (im(T))^{\perp} = (im(T^*))^{\perp}$.

(9) Show that if T is normal, then $\sigma(T) = \sigma_{ap}(T)$.

(10) Let H be a Hilbert space and $\{e_i\}_{i\in\mathbb{N}}$ be a countable orthonormal basis. Define $T : H \to H$ by

$$T(e_i) = \frac{1}{i} e_{i+1}.$$

Show that T is compact but has no eigenvalues.

(11) Construct a bounded self-adjoint operator T on a non-zero Hilbert space such that T has no eigenvalues.

(12) Prove the Lax-Milgrim Theorem: Let H be a real Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear functional. Assume that there exists $C > 0$ such that the following two estimates hold for every $x \in H$ and $y \in H$:

$$|B(x, y)| \leq C\|x\|_H\|y\|_H$$

and

$$|B(x, y)| \geq C^{-1}\|x\|_H\|y\|_H.$$

Then for every $f \in H^*$, there exists $x \in H$ such that

$$f(y) = B(x, y) \quad \text{for every } y \in H.$$

(Hint: Define a map $A : H \to H$ such that for every $x \in H$, $B(x, y) =< y, Ax >$ for every y (why does this map exist?). Show that this map is bijective. Then use the Riesz representation theorem.)

(13) Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a compact self-adjoint operator. Prove the Fredholm alternative: Consider the equations (in x) for $\lambda \in \mathbb{R} \setminus \{0\}$ and $x_0 \in H$:

$$Tx = \lambda x \tag{1}$$

and

$$Tx = \lambda x + x_0. \tag{2}$$

Then exactly one of the following holds:

(a) (1) has no non-zero solutions and (2) has a unique solution.

(b) (1) has a finite dimensional space N_λ of solutions, where $dim(N_\lambda) \geq 1$ and (2) has a solution if and only if $x_0 \perp N_\lambda$. Moreover, the space of solutions to (2) has dimension $= dim(N_\lambda)$.