Part II — Linear Analysis
Definitions

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Part IB Linear Algebra, Analysis II and Metric and Topological Spaces are essential

Normed and Banach spaces. Linear mappings, continuity, boundedness, and norms. Finite-dimensional normed spaces. [4]

The Baire category theorem. The principle of uniform boundedness, the closed graph theorem and the inversion theorem; other applications. [5]

The normality of compact Hausdorff spaces. Urysohn's lemma and Tiezte's extension theorem. Spaces of continuous functions. The Stone-Weierstrass theorem and applications. Equicontinuity: the Ascoli-Arzelà theorem. [5]

Inner product spaces and Hilbert spaces; examples and elementary properties. Orthonormal systems, and the orthogonalization process. Bessel's inequality, the Parseval equation, and the Riesz-Fischer theorem. Duality; the self duality of Hilbert space. [5]

Bounded linear operations, invariant subspaces, eigenvectors; the spectrum and resolvent set. Compact operators on Hilbert space; discreteness of spectrum. Spectral theorem for compact Hermitian operators. [5]

Contents

0 Introduction

1 Normed vector spaces

Definition (Normed vector space)**.** A *normed vector space* is a pair $(V, \|\cdot\|)$, where V is a vector space over a field $\mathbb{F}$ and $\|\cdot\|$ is a function $\|\cdot\| : V \mapsto \mathbb{R}$, known as the *norm*, satisfying

(i) $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$, with equality iff $\mathbf{v} = \mathbf{0}$.

(ii) $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$ for all $\lambda \in \mathbb{F}, \mathbf{v} \in V$.

(iii) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}, \mathbf{w} \in V$.

Definition (Topological vector space)**.** A *topological vector space* $(V, \mathcal{U})$ is a vector space V together with a topology $\mathcal{U}$ such that addition and scalar multiplication are continuous maps, and moreover singleton points $\{\mathbf{v}\}$ are closed sets.

Definition (Absolute convexity)**.** Let V be a vector space. Then $C \subseteq V$ is *absolutely convex* (or *balanced convex*) if for any $\lambda, \mu \in \mathbb{F}$ such that $|\lambda| + |\mu| \leq 1$, we have $\lambda C + \mu C \subseteq C$. In other words, if $\mathbf{c}_1, \mathbf{c}_2 \in C$, we have $\lambda\mathbf{c}_1 + \mu\mathbf{c}_2 \in C$.

Definition (Bounded subset)**.** Let V be a topological vector space. Then $B \subseteq V$ is *bounded* if for every open neighbourhood $U \subseteq V$ of $\mathbf{0}$, there is some $s > 0$ such that $B \subseteq tU$ for all $t > s$.

Definition (Banach spaces)**.** A normed vector space is a *Banach space* if it is complete as a metric space, i.e. every Cauchy sequence converges.

1.1 Bounded linear maps

Definition (Bounded linear map)**.** $T : X \to Y$ is a *bounded linear map* if there is a constant $C > 0$ such that $\|T\mathbf{x}\|_Y \leq C\|\mathbf{x}\|_X$ for all $\mathbf{x} \in X$. We write $\mathcal{B}(X, Y)$ for the set of bounded linear maps from X to Y.

Definition (Norm on $\mathcal{B}(X, Y)$)**.** Let $T : X \to Y$ be a bounded linear map. Define $\|T\|_{\mathcal{B}(X,Y)}$ by

$$\|T\|_{\mathcal{B}(X,Y)} = \sup_{\|x\| \leq 1} \|T\mathbf{x}\|_Y.$$

1.2 Dual spaces

Definition (Dual space)**.** Let V be a normed vector space. The *dual space* is

$$V^* = \mathcal{B}(V, \mathbb{F}).$$

We call the elements of V^* *functionals*. The *algebraic dual* of V is

$$V' = \mathcal{L}(V, \mathbb{F}),$$

where we do not require boundedness.

1.3 Adjoint

Definition (Adjoint)**.** Let X, Y be normal vector spaces. Given $T \in \mathcal{B}(X, Y)$, we define the *adjoint* of T, denoted T^*, as a map $T^* \in \mathcal{B}(Y^*, X^*)$ given by

$$T^*(g)(\mathbf{x}) = g(T(\mathbf{x}))$$

for $\mathbf{x} \in X$, $y \in Y^*$. Alternatively, we can write

$$T^*(g) = g \circ T.$$

1.4 The double dual

Definition (Double dual)**.** Let V be a normed vector space. Define $V^{**} = (V^*)^*$.

1.5 Isomorphism

Definition (Isomorphism)**.** Let X, Y be normed vector spaces. Then $T : X \to Y$ is an *isomorphism* if it is a bounded linear map with a bounded linear inverse (i.e. it is a homeomorphism).

We say X and Y are *isomorphic* if there is an isomorphism $T : X \to Y$.

We say that $T : X \to Y$ is an *isometric* isomorphism if T is an isomorphism and $\|T\mathbf{x}\|_Y = \|\mathbf{x}\|_X$ for all $\mathbf{x} \in X$.

X and Y are *isometrically isomorphic* if there is an isometric isomorphism between them.

1.6 Finite-dimensional normed vector spaces

Definition (Equivalent norms)**.** Let V be a vector space, and $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on V. We say that these are *equivalent* if there exists a constant $C > 0$ such that for any $\mathbf{v} \in V$, we have

$$C^{-1}\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_1 \leq C\|\mathbf{v}\|_2.$$

1.7 Hahn–Banach Theorem

Definition (Partial order)**.** A relation $\leq$ on a set X is a *partial order* if it satisfies

(i) $x \leq x$ (reflexivity)

(ii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry)

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity)

Definition (Total order)**.** Let $(S, \leq)$ be a partial order. $T \subseteq S$ is *totally ordered* if for all $x, y \in T$, either $x \leq y$ or $y \leq x$, i.e. every two things are related.

Definition (Upper bound)**.** Let $(S, \leq)$ be a partial order. $S' \subseteq S$ subset. We say $b \in S$ is an *upper bound* of this subset if $x \leq b$ for all $x \in S'$.

Definition (Maximal element)**.** Let $(S, \leq)$ be a partial order. Then $m \in S$ is a *maximal element* if $x \geq m$ implies $x = m$.

Definition (Reflexive)**.** We say V is *reflexive* if $\phi(V) = V^{**}$.

2 Baire category theorem

2.1 The Baire category theorem

Definition (Nowhere dense set)**.** Let X be a topological space. A subset $E \subseteq X$ is *nowhere dense* if $\bar{E}$ has empty interior.

Definition (First/second category, meagre and residual)**.** Let X be a topological space. We say that $Z \subseteq X$ is of *first category* or *meagre* if it is a countable union of nowhere dense sets.

A subset is of *second category* or *non-meagre* if it is not of first category.

A subset is *residual* if $X \setminus Z$ is meagre.

2.2 Some applications

3 The topology of $C(K)$

Definition (Hausdorff space). A topological space X is *Hausdorff* if for all distinct $p, q \in X$, there are $U_p, U_q \subseteq X$ that are open subsets of X such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \emptyset$.

Notation. $C(K)$ is the set of continuous functions $f : K \to \mathbb{R}$ with the norm

$$\|f\|_{C(K)} = \sup_{x \in K} |f(x)|.$$

3.1 Normality of compact Hausdorff spaces

Definition (Normal space). A topological space X is *normal* if for every disjoint pair of closed subsets C_1, C_2 of X, there exists $U_1, U_2 \subseteq X$ disjoint open such that $C_1 \subseteq U_1, C_2 \subseteq U_2$.

Definition (T_i space). A topological space X has the T_1 *property* if for all $x, y \in X$, where $x \neq y$, there exists $U \subseteq X$ open such that $x \in U$ and $y \notin U$.

A topological space X has the T_2 *property* if X is Hausdorff.

A topological space X has the T_3 *property* if for any $x \in X$, $C \subseteq X$ closed with $x \notin C$, then there are U_x, U_C disjoint open such that $x \in U_x, C \subseteq U_C$. These spaces are called *regular*.

A topological space X has the T_4 *property* if X is normal.

3.2 Tietze-Urysohn extension theorem

3.3 Arzelà-Ascoli theorem

Definition (Equicontinuous). Let K be a topological space, and $F \subseteq C(K)$. We say F is *equicontinuous at* $x \in K$ if for every ε, there is some U which is an open neighbourhood of x such that

$$(\forall f \in F)(\forall y \in U)\ |f(y) - f(x)| < \varepsilon.$$

We say F is *equicontinuous* if it is equicontinuous at x for all $x \in K$.

Definition (ε-net). Let X be a metric space, and let $E \subseteq X$. For $\varepsilon > 0$, we say that $N \subseteq X$ is an ε*-net* for E if and only if $\bigcup_{x \in \mathbb{N}} B(x, \varepsilon) \supseteq E$.

Definition (Totally bounded subset). Let X be a metric space, and $E \subseteq X$. We say that E is *totally bounded* for every ε, there is a finite ε-net N_ε for E.

3.4 Stone–Weierstrass theorem

Definition (Algebra). A vector space $(V, +)$ is called an *algebra* if there is an operation (called multiplication) $\cdot : V \to V$ such that $(V, +, \cdot)$ is a *rng* (i.e. ring not necessarily with multiplicative identity). Also, $\lambda(\mathbf{v} \cdot \mathbf{w}) = (\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w})$ for all $\lambda \in \mathbb{F}$, $\mathbf{v}, \mathbf{w} \in V$.

If V is in addition a normed vector space and

$$\|\mathbf{v} \cdot \mathbf{w}\|_V \leq \|\mathbf{v}\|_V \cdot \|\mathbf{w}\|_V$$

for all $\mathbf{v}, \mathbf{w} \in V$, then we say V is a *normed algebra*.

If V complete normed algebra, we say V is a *Banach algebra*.

If V is an algebra that is commutative as a rng, then we say V is a *commutative algebra*.

If V is an algebra with multiplicative identity, then V is a *unital algebra*.

4 Hilbert spaces

4.1 Inner product spaces

Definition (Inner product)**.** Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. We say $p : V \times V \to \mathbb{R}$ or $\mathbb{C}$ is an *inner product* on V it satisfies

(i) $p(\mathbf{v}, \mathbf{w}) = \overline{p(\mathbf{w}, \mathbf{v})}$ for all $\mathbf{v}, \mathbf{w} \in V$. (antisymmetry)

(ii) $p(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \mathbf{u}) = \lambda_1 p(\mathbf{v}_1, \mathbf{w}) + \lambda_2 p(\mathbf{v}_2, \mathbf{w})$. (linearity in first argument)

(iii) $p(\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v} \in V$ and equality holds iff $\mathbf{v} = \mathbf{0}$. (non-negativity)

We will often denote an inner product by $p(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. We call $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ an *inner product space.*

Definition (Orthogonality)**.** In an inner product space, $\mathbf{v}$ and $\mathbf{w}$ are *orthogonal* if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Definition (Euclidean space)**.** A normed vector space $(V, \|\cdot\|)$ is a *Euclidean space* if there exists an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ such that

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Definition (Hilbert space)**.** A Euclidean space $(E, \|\cdot\|)$ is a *Hilbert space* if it is complete.

Definition (Orthogonal space)**.** Let E be a Euclidean space and $S \subseteq E$ an arbitrary subset. Then the *orthogonal space* of S, denoted by $S^\perp$ is given by

$$S^\perp = \{\mathbf{v} \in E : \forall \mathbf{w} \in S, \langle \mathbf{v}, \mathbf{w} \rangle = 0\}.$$

4.2 Riesz representation theorem

4.3 Orthonormal systems and basis

Definition (Orthonormal system)**.** Let E be a Euclidean space. A set of unit vectors $\{\mathbf{e}_\alpha\}_{\alpha \in A}$ is called an *orthonormal system* if $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle = 0$ if $\alpha \neq \beta$.

Definition (Maximal orthonormal system)**.** Let E be a Euclidean space. An orthonormal space is called *maximal* if it cannot be extended to a strictly larger orthonormal system.

Definition (Hilbert space basis)**.** Let H be a Hilbert space. A maximal orthonormal system is called a *Hilbert space basis.*

4.4 The isomorphism with ℓ_2

4.5 Operators

Definition (Spectrum and resolvent set)**.** Let X be a Banach space and $T \in \mathcal{B}(X)$, we define the *spectrum* of T, denoted by $\sigma(T)$ by

$$\sigma(t) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

The *resolvent set*, denoted by $\rho(T)$, is

$$\rho(t) = \mathbb{C} \setminus \sigma(T).$$

Definition (Resolvent)**.** Let X be a Banach space. The *resolvent* is the map $R : \rho(T) \to \mathcal{B}(X)$ given by

$$\lambda \mapsto (T - \lambda I)^{-1}.$$

Definition (Eigenvalue)**.** We say λ is an *eigenvalue* of T if $\ker(T - \lambda I) \neq \{0\}$.

Definition (Point spectrum)**.** Let X be a Banach space. The *point spectrum* is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$

Definition (Approximate point spectrum)**.** Let X be a Banach space. The *approximate point spectrum* is defined as

$$\sigma_{ap}(X) = \{\lambda \in \mathbb{C} : \exists \{\mathbf{x}_n\} \subseteq X : \|\mathbf{x}_n\|_X = 1 \text{ and } \|(T - \lambda I)\mathbf{x}_n\|_X \to 0\}.$$

Definition (Compact operator)**.** Let X, Y be Banach spaces. We say $T \in \mathcal{L}(X, Y)$ is *compact* if for every bounded subset E of X, $T(E)$ is totally bounded.

We write $\mathcal{B}_0(X)$ for the set of all compact operators $T \in \mathcal{B}(X)$.

4.6 Self-adjoint operators

Definition (Self-adjoint operator)**.** Let H be a Hilbert space, $T \in \mathcal{B}(H)$. Then T is *self-adjoint* or *Hermitian* if for all $\mathbf{x}, \mathbf{y} \in H$, we have

$$\langle T\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, T\mathbf{y}\rangle.$$