Part II — Integrable Systems
Theorems

Based on lectures by A. Ashton

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Part IB Methods, and Complex Methods or Complex Analysis are essential; Part II Classical Dynamics is desirable.

Integrability of ordinary differential equations: Hamiltonian systems and the Arnol'd–Liouville Theorem (sketch of proof). Examples. [3]

Integrability of partial differential equations: The rich mathematical structure and the universality of the integrable nonlinear partial differential equations (Korteweg-de Vries, sine–Gordon). Backlund transformations and soliton solutions. [2]

The inverse scattering method: Lax pairs. The inverse scattering method for the KdV equation, and other integrable PDEs. Multi soliton solutions. Zero curvature representation. [6]

Hamiltonian formulation of soliton equations. [2]

Painleve equations and Lie symmetries: Symmetries of differential equations, the ODE reductions of certain integrable nonlinear PDEs, Painleve equations. [3]

Contents

0 Introduction

1 Integrability of ODE's

1.1 Vector fields and flow maps

Proposition.

(i) $g^0 = \mathrm{id}$

(ii) $g^{t+s} = g^t g^s$

(iii) $(g^t)^{-1} = g^{-t}$

Proposition. Let $\mathbf{V}_1, \mathbf{V}_2$ be vector fields with flows g_1^t and g_2^s. Then we have

$$[\mathbf{V}_1, \mathbf{V}_2] = 0 \iff g_1^t g_2^s = g_2^s g_1^t.$$

1.2 Hamiltonian dynamics

Proposition.

(i) This is linear in each argument.

(ii) This is antisymmetric, i.e. $\{f, g\} = -\{g, f\}$.

(iii) This satisfies the Leibniz property:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

(iv) This satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

(v) We have

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.$$

Proposition. Let $f: M \to \mathbb{R}$ be a smooth function. If $\mathbf{x}(t)$ evolves according to Hamilton's equation, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}.$$

Proposition. We have

$$[\mathbf{V}_f, \mathbf{V}_g] = -\mathbf{V}_{\{f,g\}}.$$

1.3 Canonical transformations

Proposition. A map $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ is canonical iff $D\mathbf{y}$ is *symplectic*, i.e.

$$D\mathbf{y} J (D\mathbf{y})^T = J.$$

1.4 The Arnold-Liouville theorem

Theorem (Arnold-Liouville theorem)**.** We let (M, H) be an integrable $2n$-dimensional Hamiltonian system with independent, involutive first integrals $f_1, \cdots, f_n$, where $f_1 = H$. For any fixed $\mathbf{c} \in \mathbb{R}^n$, we set

$$M_{\mathbf{c}} = \{(\mathbf{q}, \mathbf{p}) \in M : f_i(\mathbf{q}, \mathbf{p}) = c_i, i = 1, \cdots, n\}.$$

Then

(i) $M_{\mathbf{c}}$ is a smooth n-dimensional surface in M. If $M_{\mathbf{c}}$ is compact and connected, then it is diffeomorphic to

$$T^n = S^1 \times \cdots \times S^1.$$

(ii) If $M_{\mathbf{c}}$ is compact and connected, then locally, there exists canonical coordinate transformations $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ called the *action-angle coordinates* such that the angles $\{\phi_k\}_{k=1}^n$ are coordinates on $M_{\mathbf{c}}$; the actions $\{I_k\}_{k=1}^n$ are first integrals, and $H(\mathbf{q}, \mathbf{p})$ does not depend on $\boldsymbol{\phi}$. In particular, Hamilton's equations

$$\dot{\mathbf{I}} = 0, \quad \dot{\boldsymbol{\phi}} = \frac{\partial \tilde{H}}{\partial \mathbf{I}} = \text{constant}.$$

2 Partial Differential Equations

2.1 KdV equation

2.2 Sine–Gordon equation

2.3 Bäcklund transformations

3 Inverse scattering transform

3.1 Forward scattering problem

3.1.1 Continuous spectrum

3.1.2 Discrete spacetime and bound states

3.1.3 Summary of forward scattering problem

3.2 Inverse scattering problem

Theorem (GLM inverse scattering theorem)**.** A potential $u = u(x)$ that decays rapidly to 0 as $|x| \to \infty$ is completely determined by its scattering data

$$S = \left\{\{\chi_n, c_n\}_{n=1}^N, R(k)\right\}.$$

Given such a scattering data, if we set

$$F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x} + \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx} R(k)\;\mathrm{d}k,$$

and define $k(x, y)$ to be the *unique* solution to

$$k(x, y) + F(x + y) + \int_x^\infty k(x, z) f(z + y)\;\mathrm{d}z = 0,$$

then

$$u(x) = -2\frac{\mathrm{d}}{\mathrm{d}x} k(x, x).$$

3.3 Lax pairs

Theorem (Isospectral flow theorem)**.** Let (L, A) be a Lax pair. Then the discrete eigenvalues of L are time-independent. Also, if $L\psi = \lambda\psi$, where λ is a discrete eigenvalue, then

$$L\tilde{\psi} = \lambda\tilde{\psi},$$

where

$$\tilde{\psi} = \psi_t + A\psi.$$

3.4 Evolution of scattering data

3.4.1 Continuous spectrum ($\lambda = k^2 > 0$)

3.4.2 Discrete spectrum ($\lambda = -\kappa^2 < 0$)

3.4.3 Summary of inverse scattering transform

3.5 Reflectionless potentials

3.6 Infinitely many first integrals

4 Structure of integrable PDEs

4.1 Infinite dimensional Hamiltonian system

Proposition. If $u_t = \mathcal{J}\delta H$ and $I = I[u]$, then

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \{I, H\}.$$

In particular $I[u]$ is a first integral of $u_t = \mathcal{J}\delta H$ iff $\{I, H\} = 0$.

4.2 Bihamiltonian systems

Theorem. Suppose a system is bi-Hamiltonian via $(\mathcal{J}_0, H_0)$ and $(\mathcal{J}_1, H_1)$. It is a fact that we can find a sequence $\{H_n\}_{n\geq 0}$ such that

$$\mathcal{J}_1 \delta H_{n+1} = \mathcal{J}_0 \delta H_n.$$

Under these definitions, $\{H_n\}$ are all first integrals of the system and are in involution, i.e.

$$\{H_n, H_m\} = 0$$

for all $n, m \geq 0$, where the Poisson bracket is taken with respect to $\mathcal{J}_1$.

4.3 Zero curvature representation

4.4 From Lax pairs to zero curvature

5 Symmetry methods in PDEs

5.1 Lie groups and Lie algebras

5.2 Vector fields and one-parameter groups of transformations

5.3 Symmetries of differential equations

5.4 Jets and prolongations

Proposition (Prolongation formula)**.** Let

$$V(x,u) = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}.$$

Then we have

$$\mathrm{pr}^{(n)}V = V + \sum_{k=1}^{n} \eta_k \frac{\partial}{\partial u^{(k)}},$$

where

$$\begin{aligned} \eta_0 &= \eta(x,u) \\ \eta_{k+1} &= \mathrm{D}_x \eta_k - u^{(k+1)} D_x \xi. \end{aligned}$$

5.5 Painlevé test and integrability