Mathematical Tripos Part II Michaelmas 2016
Integrable Systems, Example sheet 1 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Let g^t be the flow map generated by the vector field V, so that $\mathbf{x}(t) = g^t\mathbf{x}_0$ is the unique solution to the differential equation $\dot{\mathbf{x}} = V(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Show that

$$g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.$$

2. Let V_1, V_2 be vector fields that generate g_1^t and g_2^t respectively. By considering $\Delta(s,t) = g_1^t g_2^s \mathbf{x}_0 - g_2^s g_1^t \mathbf{x}_0$ and Taylor expanding upto and including second order, show that if g_1 and g_2 commute then $[V_1, V_2] = 0$.

3. Establish the Leibniz rule and the Jacobi identity for Poisson brackets

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \quad \{f, gh\} = g\{f,h\} + h\{f,g\}.$$

Deduce from the Jacobi identity that if $f = f(\mathbf{q},\mathbf{p})$ and $g = g(\mathbf{q},\mathbf{p})$ are two first integrals of a Hamiltonian system, then so is $h = \{f,g\}$.

4. Let $V_f = J\partial_{\mathbf{x}} f$ and $V_g = J\partial_{\mathbf{x}} g$ be two Hamiltonian vector fields corresponding to the functions f and g. By considering $[V_f, V_g]\cdot\partial_{\mathbf{x}} h$ for arbitrary h, show that $[V_f, V_g] = -V_{\{f,g\}}$. [*Hint*: Use Jacobi]

5. Let $\mathbf{x} = (x_1, x_2, x_3)$ be Cartesian coordinates for $\mathbf{R}^3$. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be the velocity of an incompressible fluid ($\operatorname{div}\mathbf{u} = 0$) with vorticity $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ and pressure $p = p(\mathbf{x},t)$. Starting from Euler's equation

$$\mathbf{u}_t + (\mathbf{u}\cdot\partial_{\mathbf{x}})\mathbf{u} + \partial_{\mathbf{x}} p = 0$$

show that $\boldsymbol{\omega}_t = [\boldsymbol{\omega}, \mathbf{u}]$. Deduce the flows generated by the velocity field and the vorticity field commute iff the vorticity is time independent. [*Hint*: use $\partial_{\mathbf{x}}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\partial_{\mathbf{x}})\mathbf{b} + (\mathbf{b}\cdot\partial_{\mathbf{x}})\mathbf{a} + \mathbf{a}\times\operatorname{curl}\mathbf{b} + \mathbf{b}\times\operatorname{curl}\mathbf{a}$]

6. Let $\mathbf{x} = (\mathbf{q},\mathbf{p})$ and $\mathbf{y} = (\mathbf{Q},\mathbf{P})$. Using Hamilton's equations in the form $\dot{\mathbf{x}} = J\partial_{\mathbf{x}} H(\mathbf{x})$ and the chain rule, show that a coordinate transformation $\mathbf{x}\mapsto\mathbf{y} = \mathbf{y}(\mathbf{x})$ is canonical if and only if the derivative $D\mathbf{y}(\mathbf{x})$ is symplectic, i.e. $(D\mathbf{y})J(D\mathbf{y})^t = J$. ($D\mathbf{y}$ denotes the Jacobian matrix with entries $(D\mathbf{y})_{ij} = \partial y_i/\partial x_j$).
Hence show the transformation $\mathbf{x}\mapsto\mathbf{y}(\mathbf{x})$ is canonical iff the inverse transformation $\mathbf{y}\mapsto\mathbf{x}(\mathbf{y})$ is canonical.

7. Consider the four-dimensional phase space M with coordinates $(\mathbf{q},\mathbf{p}) = (\phi, r, p_\phi, p_r)$ and Hamiltonian

$$H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}$$

where α is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that this system is integrable in the sense of the Arnold-Liouville theorem. Show that on the level set $M_{\mathbf{c}} = \{H = c_1, p_\phi = c_2\}$ the coordinate p_r can be written in the form

$$p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2}(r_1 - r)(r - r_2)$$

where you should compute the numbers r_1, r_2 explicitly.
The coordinates $(\phi, p_\phi) \equiv (\phi, I_\phi)$ look like an "action-angle" pair. Construct the remaining action-angle coordinates by considering

$$I_r = \frac{1}{2\pi}\oint_{\Gamma_r} \mathbf{p}\cdot \mathrm{d}\mathbf{q},$$

where Γ_r is the cycle on $M_{\mathbf{c}}$ on which $\phi = $ const. Conclude that

$$H(\phi, r, p_\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.$$

8. Let $\xi = x - ct$. By considering solutions of the form $u(x,t) = f(\xi)$ derive the 1-soliton solution to each of the following nonlinear (integrable) PDEs

$$\begin{aligned} \text{KdV:} \quad & u_t + u_{xxx} - 6uu_x = 0, \\ \text{Sine-Gordon:} \quad & u_{tt} - u_{xx} + \sin u = 0. \end{aligned}$$

In both cases you should assume that f and all its derivatives tend to zero as $\xi \to -\infty$. In each case you will find it useful to multiply an ODE by $f'(\xi)$. Plot the solutions for two differing values of t.

9. Assuming $u = u(x,t)$ is small, obtain the dispersion relation for the linearised Sine-Gordon equation. Deduce that small amplitude solutions to the Sine-Gordon equation are dispersive.

10. Let v be any solution of the equation $v_{xt} = 0$. Show that the two equations

$$u_x + v_x = \sqrt{2}\exp\left(\frac{u-v}{2}\right), \quad u_t - v_t = \sqrt{2}\exp\left(\frac{u+v}{2}\right)$$

are compatible only if u satisfies Liouville's equation: $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that

$$u(x,t) = 2\log\left(\frac{\sqrt{2}}{\text{const} - \int^x \exp[-f(\xi)]\,\mathrm{d}\xi - \int^t \exp[g(\tau)]\,\mathrm{d}\tau}\right) + g(t) - f(x)$$

solves Liouville's equation for arbitrary functions f and g.

Additional problems

These questions should not be attempted at the expense of earlier ones.

11. Let g^t be the flow associated with the Hamiltonian vector field $V_H = J\partial_{\mathbf{x}} H$. If $\mathbf{x}(0) = \mathbf{y}$, use Taylor's theorem to show that

$$g^t\mathbf{y} = \mathbf{y} + tV_H(\mathbf{y}) + o(t). \qquad (\star)$$

Let $D(t) = g^t D(0)$ be a region in M evolving via the Hamiltonian flow and let $\mathrm{Vol}(t)$ denote the volume of this region. By making the change of variables $\mathbf{x}(t) = g^t\mathbf{y}$, where $\mathbf{y} \in D(0)$, show that

$$\mathrm{Vol}(t) \equiv \int_{D(t)} \mathrm{d}^{2n}\mathbf{x} = \int_{D(0)} \det\left(\frac{\partial x_i}{\partial y_j}\right) \mathrm{d}^{2n}\mathbf{y}.$$

Using $(\star)$ and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon\,\mathrm{tr}(A) + o(\epsilon)$ for any matrix A, deduce that the derivative of $\mathrm{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume (this is known as Liouville's theorem – it's awesome).

12. Recall in the sketch proof of the Arnold-Liouville theorem we used the map $\varphi : \mathbf{R}^n/\mathrm{Stab}(\mathbf{x}) \to M_{\mathbf{c}}$ defined by $\varphi(\mathbf{t}) = g^{\mathbf{t}}\mathbf{x}$, where $g^{\mathbf{t}} = g_1^{t_1} \cdots g_n^{t_n}$ was the flow map associated with the independent, commutative Hamiltonian vector fields V_{f_i}, $i = 1, \ldots, n$. In this exercise we will show that φ is surjective.

(a) Show that $\varphi(\mathbf{t} + \epsilon\mathbf{e}_i) \equiv g_1^{t_1} \cdots g_i^{t_i+\epsilon} \cdots g_n^{t_n}\mathbf{x} = \varphi(\mathbf{t}) + \epsilon V_{f_i}(\varphi(\mathbf{t})) + o(\epsilon)$.

(b) Deduce that the Jacobian matrix $D\varphi(\mathbf{t})$ is of maximal rank for all $\mathbf{t} \in \mathbf{R}^n$.

(c) Using the inverse function theorem, conclude that φ is a local diffeomorphism.

(d) Let γ be a curve in $M_{\mathbf{c}}$ connecting $\mathbf{x}$ to an arbitrary point $\mathbf{y} \in M_{\mathbf{c}}$. By applying part (c) to small open sets covering the curve γ, show that φ is surjective.

13. Define $\mathbf{y}(s) = g_1^{-t} g_2^s g_1^t \mathbf{x}$ with $\mathbf{x}$ arbitrary, t fixed and g_1^t, g_2^s are the flow maps associated with the vector fields V_1 and V_2. Compute $\dot{\mathbf{y}}(s)$ and show that it is independent of t if $[V_1, V_2] = 0$. Deduce that $\dot{\mathbf{y}}(s) = V_2(\mathbf{y})$. Conclude that if two vector fields commute, then their corresponding flows commute.

Mathematical Tripos Part II Michaelmas 2016
Integrable Systems, Example sheet 2 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Using the method of characteristics show that the solution to the initial value problem $u_t = 6uu_x$, $u(x,0) = f(x)$ is given by $u(x,t) = f(\xi)$ where ξ is defined implicitly by $\xi = x + 6tf(\xi)$. Show that the slope u_x first becomes infinite when $t = \min_\xi [6f'(\xi)]^{-1}$.

2. Let L be a Schrödinger operator with potential u which decays rapidly at infinity. Show that if $L\psi = \lambda\psi$ and $L\psi' = \lambda\psi'$ then the Wronskian $W(\psi, \psi') \equiv \psi\psi'_x - \psi'\psi_x$ is constant. Using this fact establish the following results concerning the discrete and continuous parts of the spectrum of L respectively:

(i) Show that if ψ and ψ' are bound states corresponding to the same discrete eigenvalue then $\psi \propto \psi'$. Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.

(ii) Show that the reflection and transmission coefficients obey $|R(k)|^2 + |T(k)|^2 = 1$ for all k. [*Hint*: if $L\Phi = k^2\Phi$ then $L\Phi^* = k^2\Phi^*$ also, the star denoting complex conjugation.]

3. Let L be the Schrödinger operator associated with the Dirac potential $u(x) = 2\alpha\delta(x)$, $\alpha \neq 0$. Show that the reflection and transmission coefficients associated with the continuous spectrum are

$$R(k) = -\frac{\mathrm{i}\alpha}{k + \mathrm{i}\alpha}, \quad T(k) = \frac{k}{k + \mathrm{i}\alpha}.$$

Verify that $|T|^2 + |R|^2 = 1$. Show that there are no bound states if $\alpha > 0$ and one bound state if $\alpha < 0$.

4. Consider the family of self-adjoint operators $L(t)$ on some complex inner product space defined by

$$L(t) = U(t)L(0)U(t)^*$$

where $U(t)$ is a unitary operator, i.e. $U(t)U(t)^* = I$. Show that $L(t)$ and $L(0)$ have the same eigenvalues. Show that there exists a anti-symmetric operator A such that $U_t = -AU$ and $L_t = [L, A]$.

5. Define the linear operators

$$L = -\partial_x^2 + u(x,t), \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u.$$

Show that the KdV equation is equivalent to Lax's equation $L_t = [L, A]$.

Show that A is anti-symmetric $\langle \varphi, A\psi \rangle = -\langle A\varphi, \psi \rangle$ for any smooth, rapidly decaying functions ψ and φ. If $\|\psi\| = 1$ and $\psi' = \psi_t + A\psi$, show that ψ and ψ' are orthogonal, i.e. $\langle \psi', \psi \rangle = 0$. Conclude that if u satisfies the KdV equation and ψ is a bound state for L then $\psi_t + A\psi = 0$. [*Hint*: use question 2(i).]

6. Let $L(t)$ and $A(t)$ be $n \times n$ matrices such that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, A].$$

Show that $\mathrm{tr}(L^p)$, $p \in \mathbf{Z}$, does not depend on t.

7. Show that for any non-singular square matrix $A = A(x)$

$$\frac{1}{\det A} \frac{\mathrm{d}}{\mathrm{d}x} \det A = \mathrm{tr}\left(A^{-1} \frac{\mathrm{d}A}{\mathrm{d}x} \right).$$

8. Suppose a potential $u = u(x)$ is *reflectionless*, i.e. $R(k) = 0$ in the scattering data for the associated Schrödinger operator L. By writing the GLM equation in the form

$$K(x,y) = -F(x+y) - \int_x^\infty K(x,z)F(z+y)\,\mathrm{d}z, \quad \text{where} \quad F(x) = \sum_{n=1}^{N} c_n^2 e^{-\chi_n x}$$

show that the unknown function K *must* have the form $K(x,y) = \sum_{n=1}^{N} K_n(x)e^{-\chi_n y}$ for some unknown functions $\{K_n\}$. Without looking at your notes, construct an equation of the form $A\mathbf{K} = \mathbf{b}$ where $\mathbf{K} = (K_1, \ldots, K_N)^t$ and $\mathbf{b}$ is a vector you should determine. Deduce that $u(x) = -2\log[\det A]''(x)$.

9. The $N = 2$ soliton solution to the KdV is given by $(\chi_1 > \chi_2)$

$$u(x,t) = -8\left[\frac{(\chi_1^2 e^{\eta_1} + \chi_2^2 e^{\eta_2}) + 2(\chi_1 - \chi_2)^2 e^{\eta_1+\eta_2} + \alpha_{12}(\chi_1^2 e^{\eta_1+2\eta_2} + \chi_2^2 e^{2\eta_1+\eta_2})}{(1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12}e^{\eta_1+\eta_2})^2}\right]$$

where $\eta_i(x,t) = 2\chi_i x - 8\chi_i^3 t + \beta_i$ for $i = 1,2$ and $\alpha_{12} = (\chi_1 - \chi_2)^2(\chi_1 + \chi_2)^{-2}$. By setting $\eta_1 = \text{const}$ and taking the limit $t \to \infty$ show that in a frame of reference travelling at speed $4\chi_1^2$ the 2-soliton reduces to a one soliton solution

$$u(x,t) = -2\chi_1^2 \operatorname{sech}^2[\chi_1(x - 4\chi_1^2 t) + \phi_\infty]$$

where you should determine the constant ϕ_∞. By instead taking the limit $t \to -\infty$, calculate the phase shift $\Delta\phi = \phi_\infty - \phi_{-\infty}$ induced by the soliton interaction.

10. Suppose $u = u(x,t)$ satisfies the Hamiltonian evolution equation $u_t = \mathcal{J}\delta H$. Show that if $I = I[u]$ then $I_t = \{I, H\}$, where $\{F, G\} = \langle \delta F, \mathcal{J}\delta G\rangle$ is a Poisson bracket on the space of functionals. Deduce that if I_1 and I_2 are conserved, then so is $I_3 = \{I_1, I_2\}$.

11. Show that KdV $u_t + u_{xxx} - 6uu_x = 0$ can be written in Hamiltonian form in two distinct ways

$$H_0[u] = \int \tfrac{1}{2}u^2\,\mathrm{d}x, \quad \mathcal{J}_0 = -\partial_x^3 + 4u\partial_x + 2u_x \quad \text{and} \quad H_1[u] = \int \left(\tfrac{1}{2}u_x^2 + u^3\right)\mathrm{d}x, \quad \mathcal{J}_1 = \partial_x.$$

In both cases check that the operator $\mathcal{J}$ is anti-symmetric.

Additional problems

These questions should not be attempted at the expense of earlier ones.

12. Let $f_0(x) = e^{-\mathrm{i}kx}$ where $k \in \mathbf{R} \setminus \{0\}$ is fixed. Define the sequence $\{f_n\}_{n\geq 0}$ by $f_{n+1} = Kf_n$, where

$$(Kf_n)(x) \equiv \frac{1}{k}\int_{-\infty}^{x} \sin[k(x-y)]u(y)f_n(y)\,\mathrm{d}y,$$

and u, the potential associated with a Schrödinger operator L, has compact support. Prove by induction

$$|f_n(x)| \leq \frac{\mathcal{E}(x)^n}{k^n n!}, \quad \text{where} \quad \mathcal{E}(x) = \int_{-\infty}^{x} |u(y)|\,\mathrm{d}y.$$

Deduce that the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly. Conclude that the function $\varphi = \sum_{n=0}^{\infty} K^n(e^{-\mathrm{i}kx})$ satisfies $L\varphi = k^2\varphi$ and $\varphi = e^{-\mathrm{i}kx}$ for sufficiently negative x. What if u is only assumed to be integrable?

13. Let φ be as in the previous question, so that $L\varphi = k^2\varphi$ and

$$\varphi(x,k) = \begin{cases} e^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ a(k)e^{-\mathrm{i}kx} + b(k)e^{\mathrm{i}kx} & \text{as } x \to +\infty. \end{cases}$$

By considering the equation $(I - K)\varphi(x,k) = e^{-\mathrm{i}kx}$, with K defined in the previous question, show that

$$a(k) = 1 - \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} e^{\mathrm{i}ky}u(y)\varphi(y,k)\,\mathrm{d}y, \quad b(k) = \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} e^{-\mathrm{i}ky}u(y)\varphi(y,k)\,\mathrm{d}y.$$

Mathematical Tripos Part II Michaelmas 2016
Integrable Systems, Example sheet 3 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Suppose an evolution equation is bi-Hamiltonian: $u_t = \mathcal{E}\delta K_1 = \mathcal{J}\delta K_0$ for some functionals K_0, K_1 and Hamiltonian operators $\mathcal{J}$ and $\mathcal{E}$. Assuming the recurrence relation $\mathcal{E}\delta K_{n+1} = \mathcal{J}\delta K_n$ can always be solved, show that bi-Hamiltonian systems have infinitely many first integrals in involution. [*Hint*: follow the argument from lectures in which we proved KdV has infinitely many first integrals in involution.]

2. Let $g = g(x,t)$ be a non-singular matrix. Show that if the matrices (U, V) are solutions to the zero curvature equations $U_t - V_x + [U, V] = 0$ then so are

$$\tilde{U} = gUg^{-1} + \frac{\partial g}{\partial x}g^{-1}, \quad \tilde{V} = gVg^{-1} + \frac{\partial g}{\partial t}g^{-1}.$$

What is the relationship between the associated linear problems?

3. Let $v = v(x,t)$ be a complex valued function. Show that the matrices (U, V) defined by

$$U(\lambda) = \mathrm{i}\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \mathrm{i}\begin{bmatrix} 0 & \bar{v} \\ v & 0 \end{bmatrix}, \quad V(\lambda) = 2\mathrm{i}\lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2\mathrm{i}\lambda \begin{bmatrix} 0 & \bar{v} \\ v & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{v}_x \\ -v_x & 0 \end{bmatrix} - \mathrm{i}\begin{bmatrix} |v|^2 & 0 \\ 0 & -|v|^2 \end{bmatrix}$$

satisfy the zero curvature equations iff v satisfies the *nonlinear Schrödinger* (NLS) equation

$$\mathrm{i}v_t + v_{xx} + 2|v|^2 v = 0.$$

This is another integrable equation which arises in optics and the mathematical theory of water waves. Show that the small amplitude solutions to the NLS equation are dispersive.

4. Verify that the following maps define 1-parameter groups of transformations and find the vector fields V_1, V_2, V_3 which generate them:

$$x \mapsto g_1^\epsilon x = x + \epsilon, \quad x \mapsto g_2^\epsilon x = e^\epsilon x, \quad x \mapsto g_3^\epsilon x = \frac{x}{1 - \epsilon x}.$$

Deduce that these vector fields generate a three-parameter group of transformations of the form

$$x \mapsto \frac{ax + b}{cx + d}, \quad ad - bc = 1.$$

Compute the structure constants $\{f_{ij}^k\}_{i,j,k=1}^3$ defined by $[V_i, V_j] = \sum_{k=1}^3 f_{ij}^k V_k$.

5. Compute the 1-parameter group of transformations generated by

$$V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Find new coordinates (X, Y) with $X = X(x,y)$ and $Y = Y(x,y)$ such that $V(X) = 1$ and $V(Y) = 0$. Use your results to integrate the ODE

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = F(xy),$$

where F is an arbitrary function of one variable.

6. Let g_1^ϵ and g_2^ϵ be *commutative* 1-parameter groups of transformations generated by the vector fields V_1 and V_2 respectively. Show that $g^\epsilon = g_1^\epsilon g_2^\epsilon$ also defines a 1-parameter group of transformations and show that it is generated by $V = V_1 + V_2$.

Conversely, show that if a 1-parameter group of transformations g^ϵ is generated by $V = V_1 + V_2$ where $[V_1, V_2] = 0$, then $g^\epsilon = g_1^\epsilon g_2^\epsilon$ where the g_1^ϵ and g_2^ϵ are generated by V_1 and V_2 as before.

7. Write each of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

$$g_1^\epsilon(x,t,u) = (x+\epsilon, t+2\epsilon, u+3\epsilon), \quad g_2^\epsilon(x,t,u) = (e^\epsilon x, e^\epsilon t, u-\epsilon), \quad g_3^\epsilon(x,t,u) = (e^\epsilon x, t+u\epsilon, u).$$

Hence *write down* the vector fields which generate these transformations. Check your answers are correct by showing the relevant ODEs are satisfied. Show that

$$g^\epsilon(x,t,u) = (x\cosh\epsilon + t\sinh\epsilon, x\sinh\epsilon + t\cosh\epsilon, u)$$

defines a 1-parameter group of transformations. Does the previous method fail in this case? Find the generator of g^ϵ and comment on the aforementioned failure.

8. Let $\tilde{\mathbf{x}} = g^\epsilon\mathbf{x}$ be a new set of coordinates born of a 1-parameter group of transformations g^ϵ with generator V. Use Taylor's theorem to show (formally) that for nice functions f

$$f(\tilde{\mathbf{x}}) = f(\mathbf{x}) + \epsilon Vf(\mathbf{x}) + \frac{\epsilon^2}{2!}V(Vf)(\mathbf{x}) + \frac{\epsilon^3}{3!}V(V(Vf))(\mathbf{x}) + \cdots = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}(V^n f)(\mathbf{x}).$$

Deduce that, at least formally, $g^\epsilon \equiv \exp(\epsilon V)$. Show that $\exp(\epsilon\partial_x)x = x+\epsilon$ and $\exp(\epsilon x\partial_x)x = e^\epsilon x$.

9. Let g^ϵ be a 1-parameter group of transformations generated by V. A function $F = F(\mathbf{x})$ is said to be an *invariant* of g^ϵ if $F(g^\epsilon\mathbf{x}) = F(\mathbf{x})$ for all $\mathbf{x}$. Show that F is an invariant if and only if $VF(\mathbf{x}) = 0$.

10. Compute the 1-parameter groups of transformations associated with the vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t\frac{\partial}{\partial x}, \quad V_4 = \beta x\frac{\partial}{\partial x} + \gamma t\frac{\partial}{\partial t} + \delta u\frac{\partial}{\partial u}.$$

Find the constants $(\alpha,\beta,\gamma,\delta)$ for which these vector fields generate symmetries of the KdV equation. Determine the structure constants in the corresponding 4-dimensional Lie algebra of vector fields.

11. Let $u = u(x)$. Calculate the first prolongation of the following 1-parameter groups of transformations

$$g_1^\epsilon(x,u) = (x+\epsilon, u), \quad g_2^\epsilon(x,u) = (e^\epsilon x, u+\epsilon), \quad g_3^\epsilon(x,u) = (x\cos\epsilon - u\sin\epsilon, x\sin\epsilon + u\cos\epsilon).$$

Let V_1, V_2, V_3 be the corresponding generators. Using your answers to the previous part, show that

$$\mathrm{pr}^{(1)} V_1 = V_1, \quad \mathrm{pr}^{(1)} V_2 = V_2 - u_x\frac{\partial}{\partial u_x}, \quad \mathrm{pr}^{(1)}V_3 = V_3 + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

Without looking at your notes, derive the first prolongation formula and verify these are correct.

12. Let $u = u(x,t)$. The vector field $V = \xi\partial_x + \phi\partial_t + \eta\partial_u$ generates a 1-parameter group of transformations

$$(x,t,u) \mapsto (\tilde{x},\tilde{t},\tilde{u}) = (x+\epsilon\xi(x,t,u), t+\epsilon\phi(x,t,u), u+\epsilon\eta(x,t,u)) + o(\epsilon).$$

By considering the contact condition $\mathrm{d}\tilde{u} = \tilde{u}_{\tilde{t}}\mathrm{d}\tilde{t} + \tilde{u}_{\tilde{x}}\mathrm{d}\tilde{x}$ show that $\mathrm{pr}^{(1)}V = V + \eta^x\partial_{u_x} + \eta^t\partial_{u_t}$ where

$$\eta^t = \mathrm{D}_t\eta - u_t\mathrm{D}_t\phi - u_x\mathrm{D}_t\xi, \quad \eta^x = \mathrm{D}_x\eta - u_x\mathrm{D}_x\xi - u_t\mathrm{D}_x\phi,$$

where D_x and D_t are total derivatives.

13. The modified KdV equation is $v_t + v_{xxx} - 6v^2v_x = 0$. Find a Lie-point symmetry of the form

$$g^\epsilon(x,t,v) = (e^{\alpha\epsilon}x, e^{\beta\epsilon}t, e^{\gamma\epsilon}v)$$

for appropriate numbers (α,β,γ). Consider the group invariant solution $v(x,t) = (3t)^{-1/3}w(z)$, where $z = x(3t)^{-1/3}$, and construct a 3rd order differential equation for w. Integrate this equation once to show that w satisfies Painlevé II.

Additional problems

These questions should not be attempted at the expense of earlier ones.

14. Let $\tilde{\mathbf{x}} = g^\epsilon\mathbf{x}$ be a 1-parameter group of coordinate transformations generated by V. Show that g^ϵ is a Lie-Point symmetry of the equation $\Delta[\mathbf{x}] = 0$ iff $V(\Delta) = 0$ on solutions to $\Delta[\mathbf{x}] = 0$.

15. Let $u = u(x)$ and $V = \xi\partial_x + \eta\partial_u$. Calculate $\mathrm{pr}^{(2)}V$. Show that the equation $u_{xx} = 0$ admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?