Part II — Integrable Systems
Definitions

Based on lectures by A. Ashton

Notes taken by Dexter Chua

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Part IB Methods, and Complex Methods or Complex Analysis are essential; Part II Classical Dynamics is desirable.

Integrability of ordinary differential equations: Hamiltonian systems and the Arnol'd–Liouville Theorem (sketch of proof). Examples. [3]

Integrability of partial differential equations: The rich mathematical structure and the universality of the integrable nonlinear partial differential equations (Korteweg-de Vries, sine–Gordon). Backlund transformations and soliton solutions. [2]

The inverse scattering method: Lax pairs. The inverse scattering method for the KdV equation, and other integrable PDEs. Multi soliton solutions. Zero curvature representation. [6]

Hamiltonian formulation of soliton equations. [2]

Painleve equations and Lie symmetries: Symmetries of differential equations, the ODE reductions of certain integrable nonlinear PDEs, Painleve equations. [3]

Contents

0 Introduction

1 Integrability of ODE's

1.1 Vector fields and flow maps

Definition (Commutator)**.** For two vector fields $\mathbf{V}_1, \mathbf{V}_2 : \mathbb{R}^m \to \mathbb{R}^m$, we define a third vector field called the *commutator* by

$$[\mathbf{V}_1, \mathbf{V}_2] = \left(\mathbf{V}_1 \cdot \frac{\partial}{\partial \mathbf{x}}\right)\mathbf{V}_2 - \left(\mathbf{V}_2 \cdot \frac{\partial}{\partial \mathbf{x}}\right)\mathbf{V}_1,$$

where we write

$$\frac{\partial}{\partial \mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T.$$

More explicitly, the ith component is given by

$$[\mathbf{V}_1, \mathbf{V}_2]_i = \sum_{j=1}^m (\mathbf{V}_1)_j \frac{\partial}{\partial x_j}(\mathbf{V}_2)_i - (\mathbf{V}_2)_j \frac{\partial}{\partial x_j}(\mathbf{V}_1)_i$$

1.2 Hamiltonian dynamics

Definition (Poisson bracket)**.** For any two functions $f, g : M \to \mathbb{R}$, we define the *Poisson bracket* by

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{x}} J \frac{\partial g}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}}.$$

Definition (Hamilton's equation)**.** *Hamilton's equation* is an equation of the form

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \tag{$*$}$$

for some function $H : M \to \mathbb{R}$ called the *Hamiltonian*.

Definition (Hamiltonian vector field)**.** Given a Hamiltonian function H, the *Hamiltonian vector field* is given by

$$\mathbf{V}_H = J\frac{\partial H}{\partial \mathbf{x}}.$$

Definition (First integral)**.** Given a phase space M with a Hamiltonian H, we call $f : M \to \mathbb{R}$ a *first integral* of the Hamiltonian system if

$$\{f, H\} = 0.$$

Definition (Involution)**.** We say that two first integrals F, G are in *involution* if $\{F, G\} = 0$ (so F and G "*Poisson commute*").

Definition (Independent first integrals)**.** A collection of functions $f_i : M \to \mathbb{R}$ are independent if at each $\mathbf{x} \in M$, the vectors $\frac{\partial f_i}{\partial \mathbf{x}}$ for $i = 1, \cdots, n$ are independent.

Definition (Integrable system)**.** A $2n$-dimensional Hamiltonian system (M, H) is *integrable* if there exists n first integrals $\{f_i\}_{i=1}^n$ that are independent and in involution (i.e. $\{f_i, f_j\} = 0$ for all i, j).

1.3 Canonical transformations

Definition (Canonical transformation)**.** A coordinate change $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ is called *canonical* if it leaves Hamilton's equations invariant, i.e. the equations in the original coordinates

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{q}}, \quad \mathbf{p} = -\frac{\partial H}{\partial \mathbf{q}}.$$

is equivalent to

$$\dot{\mathbf{Q}} = \frac{\partial \tilde{H}}{\partial \mathbf{P}}, \quad \dot{\mathbf{P}} = -\frac{\partial \tilde{H}}{\partial \mathbf{Q}},$$

where $\tilde{H}(\mathbf{Q}, \mathbf{P}) = H(\mathbf{q}, \mathbf{p})$.

1.4 The Arnold-Liouville theorem

2 Partial Differential Equations

2.1 KdV equation

Definition (KdV equation)**.** The *KdV equation* is given by

$$u_t + u_{xxx} - 6uu_x = 0.$$

2.2 Sine–Gordon equation

Definition (Sine–Gordon equation)**.** The *sine–Gordon equation* is given by

$$u_{tt} - u_{xx} + \sin u = 0.$$

2.3 Bäcklund transformations

Definition (Bäcklund transformation)**.** A *Bäcklund transformation* is a system of equations that relate the solutions of some PDE's to

(i) A solution to some other PDE; or

(ii) Another solution to the same PDE.

In the second case, we call it an *auto-Bäcklund transformation*.

3 Inverse scattering transform

3.1 Forward scattering problem

3.1.1 Continuous spectrum

3.1.2 Discrete spacetime and bound states

3.1.3 Summary of forward scattering problem

3.2 Inverse scattering problem

3.3 Lax pairs

Definition (Lax pair)**.** Consider a time-dependent self-adjoint linear operator

$$L = a_m(x,t)\frac{\partial^m}{\partial x^m} + \cdots + a_1(x,t)\frac{\partial}{\partial x} + a_0(x,t),$$

where the $\{a_i\}$ (possibly matrix-valued) functions of (x,t). If there is a second operator A such that

$$L_t = LA - AL = [L, A],$$

where

$$L_t = \dot{a}_m \frac{\partial^m}{\partial x^m} + \cdots + \dot{a}_0,$$

denotes the derivative of L with respect to t, then we call (L, A) a *Lax pair*.

3.4 Evolution of scattering data

3.4.1 Continuous spectrum ($\lambda = k^2 > 0$)

3.4.2 Discrete spectrum ($\lambda = -\kappa^2 < 0$)

3.4.3 Summary of inverse scattering transform

3.5 Reflectionless potentials

Definition (Reflectionless potential)**.** A *reflectionless potential* is a potential $u(x,0)$ satisfying $R(k,0) = 0$.

3.6 Infinitely many first integrals

4 Structure of integrable PDEs

4.1 Infinite dimensional Hamiltonian system

Notation. For functions $u(x)$ and $v(x)$, we write

$$\langle u, v\rangle = \int_{\mathbb{R}} u(x)v(x)\,\mathrm{d}x.$$

If u, v are functions of time as well, then so is the inner product.

Definition (Functional)**.** A *functional* F is a real-valued function (on some function space) of the form

$$F[u] = \int_{\mathbb{R}} f(x, u, u_x, u_{xx}, \cdots)\,\mathrm{d}x.$$

Again, if u is a function of time as well, the $F[u]$ is a function of time.

Definition (Functional derivative/Euler-Lagrange derivative)**.** The *functional derivative* of $F = F[u]$ at u is the unique function δF satisfying

$$\langle \delta F, \eta\rangle = \lim_{\varepsilon \to 0} \frac{F[u + \varepsilon\eta] - F[u]}{\varepsilon}$$

for all smooth η with compact support.

Alternatively, we have

$$F[u + \varepsilon \eta] = F[u] + \varepsilon\langle \delta F, \eta\rangle + o(\varepsilon).$$

Note that δF is another function, depending on u.

Definition (Total derivative)**.** Consider a function $f(x, u, u_x, \cdots)$. For any given function $u(x)$, the total derivative with respect to x is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, u(x), u_x(x), \cdots) = \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx}\frac{\partial f}{\partial u_x} + \cdots$$

Definition (Poisson bracket for infinite-dimensional Hamiltonian systems)**.** We define the *Poisson bracket* for two functionals to be

$$\{F, G\} = \langle \delta F, \mathcal{J}\delta G\rangle = \int \delta F(x) \mathcal{J} \delta G(x)\,\mathrm{d}x.$$

Definition (Hamiltonian form)**.** An evolution equation for $u = u(x, t)$ is in *Hamiltonian form* if it can be written as

$$u_t = \mathcal{J}\frac{\delta H}{\delta u}.$$

for some functional $H = H[u]$ and some linear, antisymmetric $\mathcal{J}$ such that the Poisson bracket

$$\{F, G\} = \langle \delta F, \mathcal{J}\delta G\rangle$$

obeys the Jacobi identity.

Definition (Hamiltonian operator)**.** A *Hamiltonian operator* is linear antisymmetric function $\mathcal{J}$ on the space of functions such that the induced Poisson bracket obeys the Jacobi identity.

4.2 Bihamiltonian systems

Definition (Bihamiltonian system)**.** A PDE is *bihamiltonian* if it can be written in Hamiltonian form for different $\mathcal{J}$.

4.3 Zero curvature representation

4.4 From Lax pairs to zero curvature

5 Symmetry methods in PDEs

5.1 Lie groups and Lie algebras

Definition (Group)**.** A *group* is a set G with a binary operation

$$(g_1, g_2) \mapsto g_1 g_2$$

called "group multiplication", satisfying the axioms

(i) Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for all g_1, g_2, g_3

(ii) Existence of identity: there is a (unique) identity element $e \in G$ such that

$$ge = eg = g$$

for all $g \in G$

(iii) Inverses exist: for each $g \in G$, there is $g^{-1} \in G$ such that

$$gg^{-1} = g^{-1} g = e.$$

Definition (Group action)**.** A group G acts on a set X if there is a map $G \times X \to X$ sending $(g, x) \mapsto g(x)$ such that

$$g(h(x)) = (gh)(x), \quad e(x) = x$$

for all $g, h \in G$ and $x \in X$.

Definition (Lie group)**.** An *m-dimensional Lie group* is a group such that all the elements depend continuously on m parameters, in such a way that the maps $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ correspond to a smooth function of those parameters.

Definition (Lie algebra)**.** A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear, anti-symmetric map $[\,\cdot\,, \,\cdot\,]_L : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the *Jacobi identity*

$$[a, [b, c]_L]_L + [b, [c, a]_L]_L + [c, [a, b]_L]_L = 0.$$

This antisymmetric map is called the *Lie bracket*.

If $\dim \mathfrak{g} = m$, we say the Lie algebra has *dimension m*.

5.2 Vector fields and one-parameter groups of transformations

Definition (One-parameter group of transformations)**.** A smooth map $g^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is called a *one-parameter group of transformations* (1.p.g.t) if

$$g^0 = \mathrm{id}, \quad g^{\varepsilon_1} g^{\varepsilon_2} = g^{\varepsilon_1 + \varepsilon_2}.$$

We say such a one-parameter group of transformations is generated by the vector field

$$\mathbf{V}(\mathbf{x}) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(g^\varepsilon \mathbf{x})\right|_{\varepsilon=0}.$$

Conversely, every vector field $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^n$ generates a one-parameter group of transformations via solutions of

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\mathbf{x}} = \mathbf{V}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}}(0) = \mathbf{x}.$$

Notation. Consider a vector field $\mathbf{V} = (V_1, \cdots, V_n)^T : \mathbb{R}^n \to \mathbb{R}^n$. This vector field uniquely defines a differential operator

$$V = V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2} + \cdots + V_n \frac{\partial}{\partial x_n}.$$

Conversely, any linear differential operator gives us a vector field like that. We will confuse a vector field with the associated differential operator, and we think of the $\frac{\partial}{\partial x_i}$ as a basis for our vector field.

5.3 Symmetries of differential equations

5.4 Jets and prolongations

5.5 Painlevé test and integrability

Definition (Singularirty)**.** A *singularity* of a complex-valued function $w = w(z)$ is a place at which it loses analyticity.

Definition (Painlevé property)**.** We will say that an ODE of the form

$$\frac{\mathrm{d}^n w}{\mathrm{d}z^n} = F\left(\frac{\mathrm{d}^{n-1} w}{\mathrm{d}z^{n-1}}, \cdots, w, z\right)$$

has the *Painlevé property* if the movable singularities of its solutions are at worst poles.